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Subdivision Algorithm for the Generation

of Parametric Surfaces

by

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Department of Mathematices and Statistics,

Brunel University, Uxbridge, Middx., UB8 3PH, Britain

Abstract

In this paper, an interpolatory subdivision algorithm for surfaces over arbitrary triangulations is introduced and its properties over uniform triangulations studied. The Butterfly Scheme, which is introduced by Dyn, Gregory and Levin is a special case of this algorithm. In our analysis, the matrix approach is employed and the idea of "Cross Difference of Directional Divided Difference" analysis is presented. This method is a generalization of the technique used by Dyn, Gregory and Levin etc. to analyse univariate subdivision algorithms. It is proved that the algorithm produces smooth surfaces provided the shape parameters are kept within an appropriate range.

§1. Introduction

Recursive Subdivision Algorithms consist of a class of numerically stable, highly efficient, easily manipulated and implemented algorithms for the generation of parametric curves and surfaces. These algorithms use the idea that the desired curves and surfaces are to be generated from some finite points, the so called control points, by some iterative methods consisting predominantly of simple local weighting processes. One of the advantages of using these algorithms is that they are eminently suited for the generation of smooth curves and surfaces in interactive computer systems. Interpolatory Subdivision algorithms play a very important role in these applications.

Although subdivision algorithms have been being studied intensively for about fifty years, they have been used for scientists and technicians since long ago. For example, the so called Carpenter's Technique is a very simple algorithm of this type. While the de Rahm's "Trisection Algorithm" (1947), the de Casteljau's Algorithm (1959) for the Bernstein-Bézier curves and the Chaikin's Algorithm (1974) for curves are subdivision algorithms which contribute much to the rapid development and investigation of this type of algorithms. Recently, a lot of work has been done in this area to study subdivision algorithms systematically. This includes the works by Dyn, Gregory, Levin, Dahmen,

Micchelli, Cavaretta, Daubechies and Largarias ... etc. Our work in this area is to investigate explicit conditions under which a subdivision algorithm could produce smooth surfaces with certain prescribed properties such as interpolatory and monotonicity. That is, we try to generalize the Dyn-Gregory-Levin's uniform analysis (cf. [11]) for univariate subdivision algorithms to the surface case.

In this paper, we report briefly some of our work on this subject. We firstly introduce a general interpolatory subdivision algorithm for surfaces over arbitrary triangulations and then present an analysis of a 10-point Interpolatory Scheme for surfaces over uniform triangulations. The Butterfly Scheme, which is a 8-point interpolatory scheme and was introduced by Dyn, Gregory and Levin (cf. [12,14]), is a special case of the algorithm. In the analysis, we use the matrix approach and hence the idea of "Cross Difference of Directional Divided Difference" analysis is introduced. This method is a generalization of the "Diadic Parametrization" technique used by Dyn, Gregory and Levin (cf. [11,13,14]). which was firstly used to analyse uniform subdivision algorithms for curves. It is proved that the algorithm produces smooth surfaces provided the shape parameters are kept within an appropriate range and an explicit condition for this is also provided. More details about the analysis can be found in [18]. Other analyses of uniform subdivision algorithms can also be found in [2,5,6,16,17,etc.].

This algorithm has wide practical applications. For example, it can be used to solve interpolatory-type surface fitting problem, or reversely, it can be employed to simplify problems like data reduction. It is also hoped that subdivision algorithms could be applied successfully in some optimization problems such as optimized data-transmission and wavelets processing etc.

§2. Mathematical Description of the Scheme and its Properties

The construction of the scheme is, originally, motivated by the ideas described in papers by Dubec (cf. [10]), Dyn, Gregory and Levin (cf. [11,12]). The scheme is formulated in order to solve such problems as high accuracy surface fitting and fast surface representation. Thus, the aim is to generalize the "4-point interpolatory subdivision scheme" escribed in [10,11] for surfaces. The scheme is so constructed that it preserves the advantages of the "4-point scheme". The main property of the scheme, in addition to the properties of general uniform subdivision schemes, is its generation of smooth interpolatory surfaces and the reproductivity of cubic parametric polynomial surfaces provided that the shape parameters are chosen within an appropriate range.

A mathematical description of a uniform subdivision scheme over uniform triangulations, which is also called Binary Subdivision Algorithm, is as follows. Suppose that the initial "control points" of a uniform triangular net work are denoted by $\mathbf{P}_\alpha^0, \alpha \in \mathbf{Z}^2$, then, the refined control points $\mathbf{P}_\alpha^{k+1}, \alpha \in \mathbf{Z}^2, k \geq 0$, are obtained from $\mathbf{P}_\alpha^k, \alpha \in \mathbf{Z}^2$ recursively by the following formula ("Mask"):

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$$\mathbf{P}_{\alpha}^{k+1} = \sum_{\beta \in \mathbf{Z}^2} a_{\alpha-2\beta} \mathbf{P}_{\beta}^k, \propto \, \in \mathbf{Z}^2. \tag{2.1}$$

An equivalent form of this expression is

$$\mathbf{P}_{\gamma+2\alpha}^{k+1} = \sum_{\beta \in \mathbf{Z}^2} a_{\gamma-2\beta} \mathbf{P}_{\alpha+\beta}^k, \alpha \in \mathbf{Z}^2 \tag{2.2}$$

where, $\gamma =: (\gamma_1, \gamma_2)$ and $\gamma_i = 0$ or 1, $i = 1,2$. Thus, the scheme is interpolatory if and only if

$$a_{\alpha} = \delta_{a,0,} \forall \alpha \in \mathbf{Z}^2. \tag{2.3}$$

Equation (2.2) shows clearly that the scheme is a 4-step subdivision scheme which can be described by the following

$$\begin{cases} \mathbf{P}_{2i,2j}^{k+1} = \sum_{m,n} a_{m,n} \mathbf{P}_{i+m,j+n}^k \\[2mm] \mathbf{P}_{2i+1,2j}^{k+1} = \sum_{m,n} b_{m,n} \mathbf{P}_{i+m,j+n}^k \\[2mm] \mathbf{P}_{2i,2j+1}^{k+1} = \sum_{m,n} c_{m,n} \mathbf{P}_{i+m,j+n}^k \\[2mm] \mathbf{P}_{2i+1,2j+1}^{k+1} = \sum_{m,n} d_{m,n} \mathbf{P}_{i+m,j+n}^k. \end{cases} \tag{2.4}$$

The 10-point scheme is given by the following choice of the coefficients in (2.4):

$$\begin{cases} a_{0,0} = 1, b_{-1,0} = b_{2,0} = w_3, \, b_{0,-1} = b_{1,1} = w_2, \\ b_{0,0} = b_{1,0} = \frac{1}{2} - 2w_1 - w_2 - w_3, \, b_{-1,-1} = b_{1,-1} = b_{0,1} = b_{2,1} = w_1, \\ c_{0,-1} = c_{0,2} = w_3, c_{-1,0} = c_{1,1} = w_2, \\ c_{0,0} = c_{0,1} = \frac{1}{2} - 2w_1 - w_2 - w_3, \, c_{-1,-1} = c_{-1,1} = c_{1,0} = c_{1,2} = w_1, \\ d_{-1,-1} = d_{2,2} = w_3, \, d_{1,0} = d_{0,1} = w_2, \\ d_{0,0} = d_{1,1} = \frac{1}{2} - 2w_1 - w_2 - w_3, \, d_{0,-1} = d_{-1,0} = d_{1,2} = d_{2,1} = w_1 \end{cases} \tag{2.5}$$

where, $w_i, i = \mathbf{1,2,3},$ are three (tension) parameters. This special choice of the coefficients comes from the 3-D symmetric structure of the scheme. In fact, there is a simpler way to describe the scheme which uses only a single formula (only one 'Mask') to characterize the scheme. The formula is given below (cf. Figure 1). This is due to the 3-direction-symmetry property of the scheme. Since the scheme is interpolatory, only the inserted values are to be evaluated. The formula for an inserted point, P_0, is given by

$$\begin{aligned} \mathbf{P}_0 = \frac{1}{2}\left(\mathbf{P}_e + \mathbf{P}_f\right) + \omega_1 \left(\mathbf{P}_a + \mathbf{P}_c + \mathbf{P}_h + \mathbf{P}_j - 2\mathbf{P}_e - 2\mathbf{P}_f\right) \\ + \omega_2 \left(\mathbf{P}_b + \mathbf{P}_i - \mathbf{P}_e - \mathbf{P}_f\right) + \omega_3 \, \left(\mathbf{p}_d + \mathbf{p}_g - \mathbf{p}_e - \mathbf{p}_f\right) \end{aligned} \tag{2.6}$$

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where, *o* is the midpoint of the edge joining the vertices *e* and *f*, see Figure 1. From this onstruction, it is obvious that the scheme can be used (possibly, with some modification at those so called Extraordinary Points) to produce surfaces over arbitrary triangulations (cf. [18]).

In the uniform subdivision process, formula (2.6) is used to evaluate all "midpoint" values to produce a refined uniform triangular control net in which the triangulation of the refined control nets is formed by the "standard 3-D meshing rule" which will be explained later in our convergence analysis. Repeated applications of this process will therefore result in finer and finer control nets. Moreover, further studies show that if the shape parameters $\{\omega_i\}$ are chosen appropriately, the scheme will produce smooth interpolatory surfaces. This will be discussed in the next section.

It can be shown that the scheme has the following properties.
1. The scheme is interpolatory.
2. The parameters $\{\omega_i\}$ work as tension controls along the three mesh directions respectively.
3. The scheme reproduces linear surfaces for all $\{\omega_i\}$.
4. The scheme reproduces bivariate cubic parametric surfaces if $\{\omega_i\}$ satisfy

$$w_1 = t - \frac{9}{16}, w_2 = -2t - \frac{18}{16} = -2w_1, w_3 = \frac{1}{2} - t \tag{2.7}$$

where, t is any real number.

5. The scheme reduces to the Butterfly Scheme [cf. 11] if the parameters satisfy

$$w_1 =: w, w_2 = -2w, w_3 = 0. \tag{2.8}$$

6. The scheme has certain data-dependent shape preserving properties.
7. The scheme produces smooth surfaces if the shape control parameters are chosen

properly. This will be discussed later.

§3. C^0 Covergence Analysis of the Scheme

To study the convergence property of the subdivision algorithm and the property of the surfaces produced by it, a definition of convergence of subdivision algorithms and a parametrization of the surfaces as well should be be introduced. By contrast to the univariate case, uniform convergence and the "diadic parametrization" are natural choices. The "diadic parametrization" means that the control points $\mathbf{P}_\alpha^k, \alpha \in \mathbf{Z}^2, k \geq 0,$ are parametrized at the "diadic points": $2^{-k}\alpha, \alpha \in \mathbf{Z}^2$ in the parameter plane, e.g., the u-v plane. So, if we define

$$\left(u^k, v^k\right) := 2^{-k}\alpha, \forall \alpha \in \mathbf{Z}^2, \tag{3.1}$$

then, the control net, which is defined by $\mathbf{P}_\alpha^k, \alpha \in \mathbf{Z}^2, k \geq 0,$ can be regarded as the unique piecewise linear interpolant $\mathbf{P}^k(u,v)$ from the uniform 3-D meshed u-v plane, which is produced by mesh directions $(0,1)$, $(1,0)$ and $(1,1)$, to $\mathbf{R}^3$ satisfying

$$\mathbf{P}^k\left(u^k, v^k\right) = \mathbf{P}_\alpha^k, \alpha \in \mathbf{Z}^2. \tag{3.2}$$

Hence, the convergence of the scheme can be defined as the convergence of the continuous surface sequence $\{\mathbf{P}^k(u,v)\}$. So we say the scheme is convergent if for any initial dada, there is a continuous surface $\mathbf{P}(u,v)$ such that

$$\lim_{k \to \infty} \mathbf{P}^k(u,v) = \mathbf{P}(u,v), \forall u, v \in \mathbf{R}. \tag{3.3}$$

To facilitate our analysis, we assume here that the initial data are just real numbers and that they are function values on the uniform integer grid $(i,j), i, j \in \mathbf{Z}$ in the u-v plane. Then at lever k, the control point values $P_\alpha^k, \alpha \in \mathbf{Z}^2,$ will be the function values at a refined grid $2^{-k}(i,j), (i,j) \in \mathbf{Z}^2$ since the diadic parametrization is assumed. By meshing the control nets $P_{i,j}^k$ in the same way as the uniform grid $2^{-k}(i,j)$ in the u-v plane, the 10-point scheme can then be written in the following compact form:

$$\begin{cases} P_{2i+2j}^{k+1} = P_{i,j}^k \\ P_{2i+1,2j}^{k+1} = \left(\dfrac{1}{2} - 4\omega_1 - 2\omega_2 - 3\omega_3\right)\left(P_{i,j}^k + P_{i+1,j}^k\right) \\ \qquad + \omega_1\left(P_{i-1,j-1}^k + P_{i+1,j-1}^k + P_{i,j+1}^k + P_{i+2,j+1}^k\right) \\ \qquad \left(P_{i,j-1}^k + P_{i+1,j+1}^k\right) + \omega_3\left(P_{i-1,j}^k + P_{i+2,j}^k\right) \end{cases} \tag{3.4}$$

with $P_{2i,2j+1}^{k+1}$ **and** being duals of the second equation. Now the forward difference operators $\Delta_i, i = 1,2,3,$ along the mesh directions can be defined:

$$\begin{cases} \Delta_1 = P_{i+1,j}^k - P_{i,j}^k \\ \Delta_2 = P_{i,j+1}^k - P_{i,j}^k \\ \Delta_3 = P_{i+1,j+1}^k - P_{i,j}^k. \end{cases} \tag{3.5}$$

From the above discussions, the following C^0 convergence result can be obtained:

Theorem 1. The scheme produces C^0 surfaces if the parameters $\{w_i\}$ satisfy

$$\begin{cases} \left|\dfrac{1}{2} - 2w_1 - w_2\right| + 2|w_2| + 2|w_3| + |w_1 - w_3| < 1 \\ 4|w_1| + 2|w_2| + 2|w_3| < \dfrac{1}{2}. \end{cases} \tag{3.6}$$

A simple symmetric solution to (3.6) is given by

$$5|w_1| + 3|w_2| + 3|w_3| < \dfrac{1}{2}. \tag{3.7}$$

Proof: The proof comes from a direct estimate of the differences of two control nets at adjacent levels. It can be shown that

$$\left\| P^{k+1}(u,v) - P^k(u,v) \right\| \infty \le (4|w_1| + 2|w_2| + 2|w_3|)E^k, \tag{3.8}$$

where E^k is defined by

$$E^k =: \max_{i,j,n} \left| \Delta_n p_{i,j}^k \right|. \tag{3.9}$$

A study of the sequence $\{E^k\}$ gives

$$E^{k+1} \le M(w_1, w_2, w_3)E^k \tag{3.10}$$

where

$$M(w_1, w_2, w_3) =: \max\left(\left| \frac{1}{2} - 2w_1 - w_2 \right| + 2|w_2| + 2|w_3| + |w_1 - w_3|, \right.$$
$$\left. 4|w_1| + 2|w_2| + 2|w_3| + \frac{1}{2} \right). \tag{3.11}$$

Hence, the control net sequence $\{P^k(u,v)\}$ forms a Cauchy sequence and therefore converges to a continuous surface if (3.6) holds. This completes the proof. ■

Remark 1. *For the cubic precision scheme, (3.6) becomes*

$$\frac{1}{2} < t < \frac{37}{64}. \tag{3.12}$$

Remark 2. *Other conditions for C^o convergence can also be obtained (cf. [18]).*

§4. C^1 Covergence Analysis of the Scheme

In order to prove that the scheme produces C^1 surfaces, the *Cross Differences of the Directional Divided Differences, CDD*, of the control nets are introduced and studied. This process is similar to the *Divided Difference Analysis* of univariate subdivision schemes described in [2,3,10,11 ,etc.] .

The *CDD* at lever k along mesh direction m and n, $m,n =$1,2,3, $m \ne n$, is defined as follows:

$$C_{i,j,m,n}^k =: 2^k \Delta_m \Delta_n P_{i,j,}^k \ \forall i, j \in \mathbf{Z}. \tag{4.1}$$

Since the scheme is symmetric, we only need to study one type of *CDD*. Hence, without loss of generality, we assume that in (4.1), $m = 1$, $n = 2$ and the subscripts m and n will be omitted in our future discussion.

From the subdivision process (3.4) and definition (4.1), one can show that if

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$$w_2 = -2w_1, \tag{4.2}$$

then all these *CDD* terms will satisfy the following refinement equations:

$$
\begin{cases}
C_{2i,2j}^{k+1} & = 2w_1 C_{i-1,j}^k - (4w_1 - 2w_3)C_{i-1,j-1}^k + 2w_3 C_{i,j+1}^k \\
& \quad + (1+8w_1)C_{i,j}^k + 2w_1 C_{i,j-1}^k \\
& \quad + 2w_3 C_{i+1,j+1}^k + 2w_3 C_{i+1,j}^k \\
C_{2i+1,2j}^{k+1} & = (2w_1 - 2w_3)C_{i-1,j-1}^k - 8w_1 C_{i,j}^k \\
& \quad - (2w_1 - 2w_3)C_{i,j+1}^k + (2w_1 - 2w_3)C_{i+1,j+1}^k \\
& \quad - (2w_1 - 2w_3)C_{i+1,j}^k \\
C_{2i,2j+1}^{k+1} & = -(2w_1 + 2w_3)C_{i-1,j}^k + (2w_1 - 2w_3)C_{i-1,j-1}^k \\
& \quad - (2w_1 + 2w_3)C_{i,j+1}^k - 8w_1 C_{i,j}^k \\
& \quad + (2w_1 - 2w_3)C_{i+1,j+1}^k \\
C_{2i+1,2j+1}^{k+1} & = 2w_3 C_{i-1,j}^k + 2w_3 C_{i-1,j-1}^k + 2w_1 C_{i,j+1}^k \\
& \quad + (1+8w_1)C_{i,j}^k + 2w_3 C_{i,j+1}^k \\
& \quad -(4w_1 - 2w_3)C_{i+1,j+1}^k + 2w_1 C_{i+1,j}^k
\end{cases}
\tag{4.3}
$$

By applying this recursive relation repeatedly, we obtain the following result:

Theorem 2. *There exists a constant $B(w_1, w_3)$, which is a piecewise quadratic function of w_1 and w_3 such that*

$$C_d^{k+2} \le B(w_1, w_3)C_d^k, \forall k \ge 0, \tag{4.4}$$

where

$$C_d^k =: \max_{i,j,m,n,m \ne n} \left| C_{i,j,m,n}^k \right| \tag{4.5}$$

and $B(w_1, w_3) < 1$ provided that the shape parameters w_1 and w_2 satisfy

$$
\begin{cases}
w_2 + 2w_1 & = 0 \\
w_1 & \ne 0 \\
w_1 + 7w_3 & \le 0 \\
8(w_1 + 0.07) - 3(w_3 - 0.01) & \ge 0 \\
(w_1 + 0.10) + (w_3 + 0.07) & \ge 0 \\
10w_1 - 7w_3 & \le 0
\end{cases}
\tag{4.6}
$$

Proof: The details of the proof can be found in [18]. ∎

Remark 3. *Condition (4.6) means that the parameters $\{w_i\}$ should lie in a polygonal region Ω in the plane $\Pi : w_2 = -2w_1$. The region Ω is depicted in Figure 2.*

To prove that the limit surfaces are smooth, the following lemmas are required. A proof of them can be found in [18].

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Lemma 3. *If the directional divided difference function sequence* $\left\{D_\alpha^k(u,v)\right\}$ *converges uniformly to a continuous function* $D\alpha(u,v)$, *then, the control net sequence* $\left\{P^k(u,v)\right\}$ *converges uniformly to a* C^1 *function* $P(u,v)$. *Furthermore, we have*

$$\frac{\partial P(u,v)}{\partial \gamma} = D_\gamma(u,v) \tag{4.7}$$

where, $\gamma =: (\gamma_1, \gamma_2)$ *and* $\gamma_i = 0$ *or* 1, *i* $= 1,2$, $\gamma \neq 0$, *indicates the direction of the difference.* ∎

Remark 4. *The the directional divided difference function* $D_\gamma^k(u,v)$ *is defined as the piecewise linear interpolant to the triangulated directional divided difference data. For example, for* $\gamma = (0,1)$, *we have*

$$D_{(0,1)}^k(u^k,v^k) = 2^k \Delta_1 P_\alpha^k, \forall k \geq 0, \alpha \in Z^2. ∎ \tag{4.8}$$

Lemma 4. *If the directional divided difference function sequence* $\left\{D_\gamma^k(u,v)\right\}$ *has the property*

$$\left\|D_\gamma^{k+1}(u,v) - D_\gamma^k(u,v)\right\|\infty \leq (1 + 16|w_1| + 16|w_3|)C_d^k. ∎ \tag{4.9}$$

From **Theorem 2, Lemma 3** and **Lemma 4,** we now can conclude

Theorem 5. *The 10-point scheme produces* C^1 *surfaces if the shape parameters satisfy (4.6).* ∎

Corollary 6. *The cubic precision scheme produces smooth surfaces if the tension parameter t satisfy*

$$\frac{49}{100} \leq t \leq \frac{54}{100}. ∎ \tag{4.10}$$

Remark 5. *Condition (4.6) is a simple one. Other* C^1 *convergence conditions may also be obtained (cf. [18]).*

§5. Graphic Examples

Two graphic examples are given here to show the smoothing process of the scheme. Figure 3 shows the initial data and the surface produced by the butterfly scheme from the cardinal data $P_\alpha^0 = \delta_{\alpha,o}$ after the third subdivision. Figure 4 shows the surface produced by the cubic precision scheme from a uniform control net which is also shown. Since the software generates surfaces on rectangular grids, the triangulation along the (1,1) direction is unfortunately not displayed.

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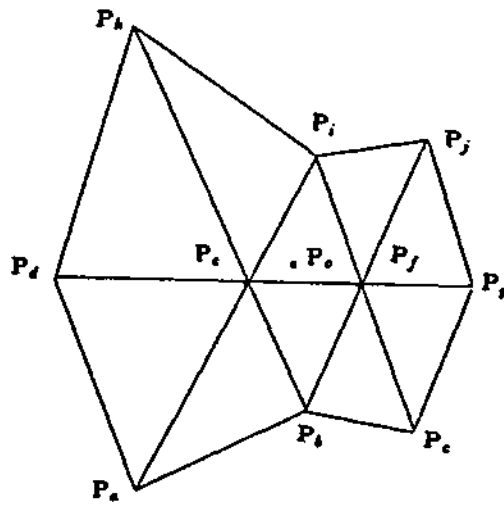
Figure 1. Geometric construction of the scheme
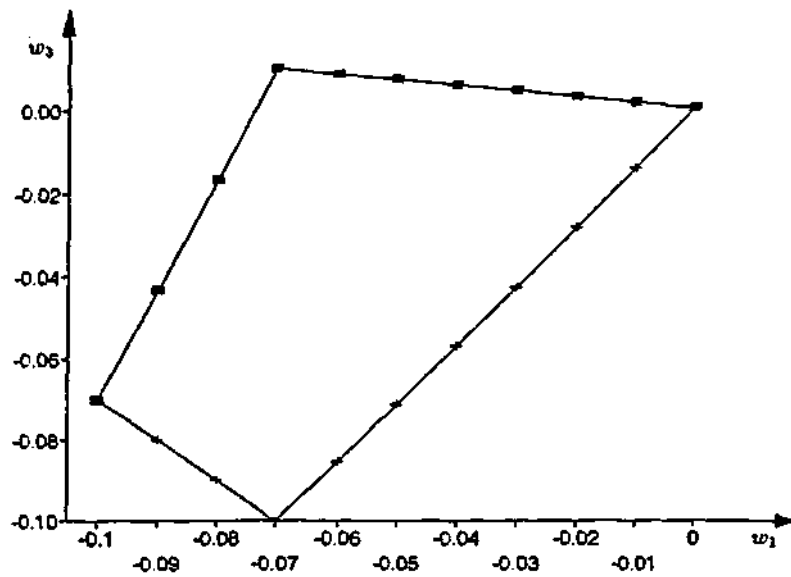

Figure 2. The C^1 convergence region Ω of the scheme

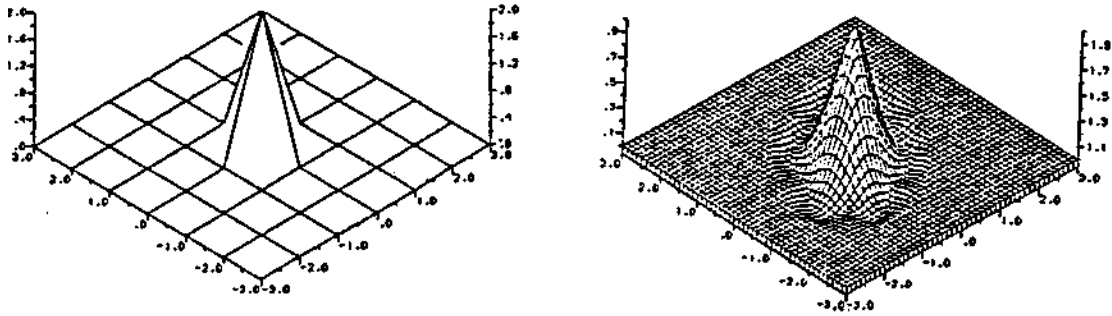*Subdivision Algorithm Jot Parametric Surfaces*

Figure 3. The cardinal function, $k = 3, w_1 = -\dfrac{1}{16}, \; w_2 = -2w_1, w_3 = 0.$
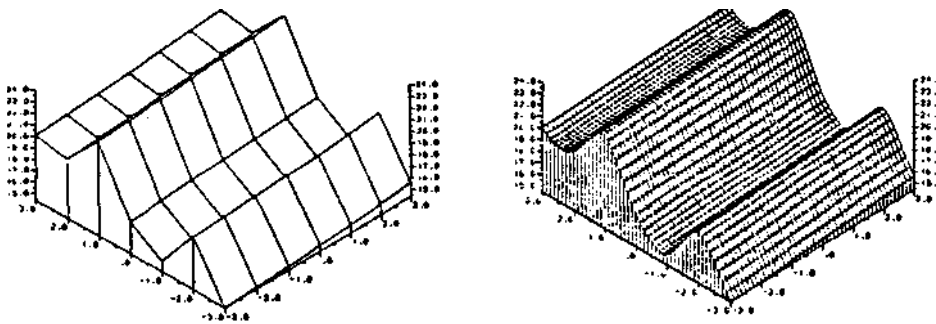

Figure 4. The cubic precision scheme, $k = 3, t = 0.52,$
e.g., $w_1 = -0.0425, w_2 = -2w_1, w_3 = -0.02.$