Spatially Continuous and Discontinuous Galerkin Finite Element Approximations for Dynamic Viscoelastic Problems

A thesis submitted for the degree of Doctor of Philosophy
by

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Declaration

I hereby declare that the thesis is based on my original work, except for quotations and citations which have been duly acknowledged. I also declare that it has not been previously or concurrently submitted for any other degree at Brunel University London or other institutions.

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Abstract

Viscoelastic behaviour is modelled as a Volterra integral equation of the second kind in classical continuum mechanics. Dynamic viscoelastic problems can be represented by wave equations with hereditary integral terms. For example, a constitutive relation between stress and strain is given by

$$\underline{\boldsymbol{\sigma}}(t) = \underline{\boldsymbol{D}}(0)\underline{\boldsymbol{\varepsilon}}(t) - \int_0^t \underline{\boldsymbol{D}}_s(t-s)\underline{\boldsymbol{\varepsilon}}(s)ds,$$

or

$$\underline{\boldsymbol{\sigma}}(t) = \underline{\boldsymbol{D}}(t)\underline{\boldsymbol{\varepsilon}}(0) + \int_0^t \underline{\boldsymbol{D}}(t-s)\dot{\underline{\boldsymbol{\varepsilon}}}(s)ds,$$

where $\underline{\boldsymbol{\sigma}}(t)$ is the stress tensor, $\underline{\boldsymbol{\varepsilon}}(t)$ is the strain tensor, $\underline{\boldsymbol{D}}(t)$ is a symmetric positive definite fourth order tensor and $\underline{\boldsymbol{D}}_s(t-s) = \frac{\partial}{\partial s}\underline{\boldsymbol{D}}(t-s)$. The kernel in Volterra integral can be defined by

$$\underline{\boldsymbol{D}}(t) = \underline{\boldsymbol{D}}_0\varphi(t),$$

where $\underline{\boldsymbol{D}}_0$ is a piecewise constant tensor and $\varphi(t)$ is called a stress relaxation function. One can be defined by a sum of decaying exponentials $\varphi(t) = \varphi_0 + \sum_{q=1}^{N_\varphi} \varphi_q e^{-t/\tau_q}$, namely Dirichlet-Prony series, based on Maxwell or Zener model. This allows us to introduce two types of internal variables. On the other hand, power law gives another choice of kernels such that $\varphi(t) = t^{-\alpha}$ where $0 < \alpha < 1$, which models fractional order viscoelastic problems. This weakly singular kernel forces us to be more careful. This thesis deals with these two types of integro-differential equations.

Many people in engineering and mathematics take into account these type of problems in analytical and numerical ways. In this thesis, we aim to solve the dynamic viscoelastic problems with spatial finite element methods, as well as finite difference methods for time. We present variational formulations of our model problems with Continuous Galerkin Finite Element Method (CGFEM) and Discontinuous Galerkin Finite Element Method (DGFEM). Also, Crank-Nicolson finite difference scheme is applied for time discretisation and therefore, we are able to formulate fully discrete problems. We state and prove stability and error estimates. Typical approach of *a priori* estimates uses Grönwall's inequality for time integral, but we avoid using it for better stability and error bounds for long time integration. Not only are theoretical results presented but also various numerical experiments. All numerical simulations are carried out based on FEniCS.

Acknowledgement

I would like to acknowledge Vice Chancellor's scholarship in Brunel University London for financial support during my PhD. Also, I would like to thank College of Engineering, Design and Physical Sciences and Department of Mathematics for funds for research travels.

I would like to express my deep gratitude to Dr Simon Shaw, Dr Mike Warby and Dr Shashank Virmani, my supervisory team, for their guidance, enthusiasm, encouragement and inspiration. Especially, my principal supervisor, Simon, gave me so much motivation to be a researcher. Without his priceless help, it is impossible to complete this PhD journey.

My grateful thanks are extended to Professor Sergey Mikhailov and Dr Matthias Maischak for their advice in annual review meetings. I would also like to thank Dr Seungil Kim in Kyunghee University and Dr Robert Nürnberg in Imperial College London for their help to find PhD opportunity. Special thanks to my Brunel friends, Matthew, Arianna, Phillip and Daniela for enjoyable PhD life.

Last but not the least, I would like to express my great appreciation to my family and, the one and only Esther for precious support. I wish to thank my late father for his dedicated love and I believe he would be proud.

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$\boldsymbol{u}$ Displacement vector

$\dot{\boldsymbol{u}}, \boldsymbol{w}$ Velocity vector

$\ddot{\boldsymbol{u}}$ Acceleration vector

$\underline{\boldsymbol{\sigma}}$ Strain tensor

$\underline{\boldsymbol{\varepsilon}}$ Strain tensor

$\underline{\boldsymbol{D}}$ Hooke's tensor or stress relaxation tensor

φ Stress relaxation function

$\boldsymbol{f}$ Body force

$\boldsymbol{g}_N$ Traction on Neumann boundary Γ_N

ρ Density

$\boldsymbol{\psi}_q$ Internal variable of the displacement form

$\boldsymbol{\zeta}_q$ Internal variable of the velocity form

N_φ The number of internal variables

V^h Finite element space

N_{V^h} The dimension of V^h or the number of global basis functions

$\boldsymbol{u}_h$ Semidiscrete solution of displacement vector

$\boldsymbol{\psi}_{hq}, \boldsymbol{\zeta}_{hq}$ Semidiscrete solution of internal variables

$\mathfrak{u}_i$ A degree of freedom for displacement with respect to the basis $\boldsymbol{\Phi}_i$

$\mathfrak{w}_i$ A degree of freedom for velocity with respect to the basis $\boldsymbol{\Phi}_i$

$\boldsymbol{U}_h^n$ Fully discrete solution of displacement

$\boldsymbol{W}_h^n$ Fully discrete solution of velocity

$\boldsymbol{\Psi}_{hq}^n$ Fully discrete solution of internal variable of the displacement form

$\boldsymbol{\mathcal{S}}_{hq}^n$ Fully discrete solution of internal variable of the velocity form

$\boldsymbol{R}$ Elliptic projection operator

$\boldsymbol{R}_\epsilon$ DG Elliptic projection operator, $\epsilon = 1, -1, 0$

$a(\cdot,\cdot)$ CG bilinear form

$a_\epsilon(\cdot,\cdot)$ DG bilinear form, $\epsilon = 1, -1, 0$

$J_0^{\alpha_0,\beta_0}(\cdot,\cdot)$ Jump penalty operator

$F_d(\cdot)$,$F_v(\cdot)$ Linear forms with respect to form of internal variables

h Spatial mesh size

Δt Time step size

$\boldsymbol{q}_n(\cdot)$ Numerical integration of a fractional order e.g. see Chapter 5

† Similar symbols are introduced in scalar-valued problems (e.g. non-bold expressions)

Chapter 1

Introduction

A viscoelastic problem is modelled with a Volterra type integro-differential equation [1, 2, 3, 4]. The kernel in the integral can be either exponentially decaying or of weakly singular type. In this thesis, we describe these two viscoelastic model problems with spatially finite element methods and finite difference schemes in time. Our aim in the thesis is to investigate the performance of numerical approximations to viscoelastic models. We provide theorems regarding stability analysis and *a priori* error estimates. The thesis shows not only theoretical works, but also numerical experiments for important evidences of theories.

The classical theory of continuum mechanics is presented in [5] and references therein. The fundamental principle of linear elasticity and Newtonian fluids, to describe material response mathematically, is able to construct mathematical framework of viscoelasticity [4, 2, 3]. It enforces engineers and mathematicians to solve boundary value problem of integro-differential equation [1].

Analytic and numerical methods for Volterra equation were studied in [6, 7] and see references therein for more details. The analytic solution is given as a series form [6] and it naturally introduces semi-group approach (spectral methods) to deal with regularity of solution and stability analysis for fractional order viscoelasticity [8, 9, 10].

Finite element analysis is widely used for a large number of problems of engineering and mathematical models due to the several advantages such as capture of local effects, easy representation of solution, inclusion of material properties and complex geometry [11]. Hrennikoff [12] and Courant [13] firstly developed finite element method originated from the need to solve complex elasticity and structural analysis problems in civil and aeronautical engineering. In recent years, advanced finite element methods have been employed for linear elasticity problems, see e.g. [11, 14, 15, 16, 17].

In 1970s, Discontinuous Galerkin methods (DG) which combine features of the finite element and the finite volume framework were first proposed by Babuška [18], Baker [19], and Wheeler [20]. More recently, nonsymmetric interior penalty methods have been introduced by Rivière, Wheeler and Girault [21, 22], and by Houston, Schwab, and Süli [23]. For more references and applications of DG see [24] and references therein.

In case of viscoelasticity problem, it is necessary to deal with constitutive relation given as a Volterra integral equation, which means hereditary terms appear more than linear elasticity problem. In a classical manner of continuum mechanics, rheology models describe linear viscoelasticity models as Volterra integral equations of exponentially decaying kernels [1]. While the semidiscrete formulations were derived by finite element methods with respect to the space domain, a typical way of numerical integration, a quadrature rule, and standard finite difference method were used to formulate fully discrete schemes in [17]. On the other hand, Johnson [25] proposed introduction of internal variables for understanding of local constitutive equations. It allowed to replace memory terms by internal variables, see e.g. [26]. It is beneficial to get better errors and cheaper memory requirements of computing, on account of absence of quadrature errors and need for all history. However, it is only applicable for linear viscoelasticity with exponentially decaying kernels.

If the number of internal variables (i.e. the total number of Maxwell elements) tends to infinity, the viscoelastic behaviour describes fractional order evolution constitutive equation, e.g. see [1, 4, 10, 27]. We can choose another type kernels called *power law* types, imposing use of numerical methods for fractional order integration. McLean and Thomée presented numerical solution of a fractional order evolution equation, e.g. see [8] and improved version [9]. By fundamental solutions of fractional order differential equations [28, 29, 30, 31], Mittag-Leffler type kernels have also introduced to describe fractional order viscoelastic models by Adolfsson, Enelund and Olsson [32], also see [32, 33, 34, 10].

We consider the viscoelastic model problems by numerical approaches based on spatial finite element methods (e.g. see [11, 14, 35, 36, 37, 38] for CGFEM and [24, 11, 15, 39, 21, 22, 17, 23, 16] for DGFEM) as well as Crank-Nicolson scheme (e.g. see [40]). Using various novel papers in finite element theory and continuum mechanics, we shall develop numerical approximations to viscoelastic models with appropriate theoretical and computational works. Earlier work in [26] has provided DGFEM for dynamic linear solid viscoelasticity problems. We are going to extend their work to improve stability and error analysis, and introduce some equivalent form. Furthermore, according to [1], we can consider fractional order viscoelastic models. For instance, elastomer 3M-467 exhibits a fractional order constitutive relation between stress and strain such that the stress is proportional to 0.56 order derivative of the strain [41]. In a similar way with the linear Maxwell solid, we present and demonstrate numerical schemes with proofs of stability and error bounds. In addition, we implement bespoke codes as necessary in FEniCS environment (see [42, 43] and `https://fenicsproject.org` for details).

Rivière, Shaw and Whiteman presented the application of the DGFEM to dynamic linear solid viscoelasticity problems with internal stress representation [26]. The authors showed *a priori* error estimate (energy error estimate) by using Grönwall's inequality. However, the Grönwall constants increase exponentially in time hence the constants of stability bounds and error bounds are significantly large for long time period of viscoelastic response. On the other hand, in this thesis, we consider L_∞ norm (max norm) in time rather than use of Grönwall inequality. Also, we present L_2 error estimates as well as energy error estimates for semidiscrete and fully discrete cases. Whereas local constitutive relation were used as internal variables (second order tensor-valued functions) in [26], we introduce different type of internal variables (vector-valued functions) in two forms, displacement form and vector form. It has an advantage of saving memory requirements for numerical experiments. To be more precisely, each internal variable needs the memory to store a vector of length dimension d, instead a second order tensor of $d \times d$. We first study scalar-valued wave equation of viscoelastic functions, also known as antiplane strain problem, e.g. see [44, 45, 46], by CGFEM and DGFEM, respectively. In the same way, we deal with linear dynamic viscoelasticity problem of the generalised Maxwell solid.

On the one hand, McLean and Thomée developed numerical analysis of fractional order evolution equations [8, 9]. The evolution equations is a scalar analogue of a fractional order viscoelasticity problem of power law type and the authors proved *a priori* error estimates for the homogeneous Dirichlet boundary problem by using semi-group approach and Laplace transformation of kernels. On the other hand, we study the fractional order viscoelastic model problems, i.e. vector-valued problems, for mixed boundary conditions. While we consider regularity of solutions via Laplace transformation, we present suboptimal and optimal error estimates by using the same manner of *a priori* estimates as in the linear viscoelasticity models of exponentially decaying kernels.

A plan for this thesis is organised as follows:

- Chapter 1 introduces preliminary work in terms of notations, and general definitions and/or theorems in functional analysis. Also, we present basic concept of continuum mechanics in elastic and linear viscoelastic behaviours. Then we can put forward a linear viscoelastic solid model with internal variables. At the same time, we give some background of finite element methods. Here, we can observe useful tools to prove our claims.

- As following the linear viscoelastic model in Chapter 1, we can also reduce it to scalar-valued to simplify the model problems. In Chapter 2, we solve the simplified scalar wave equation with memory terms with CGFEM. Use of typical approach for *a priori* estimates and L_∞ norm estimates in time leads us to show well-posedness and error estimates.

- In Chapter 3, we describe DG approximations to the scalar wave equations and prove stability and error estimates as in CGFEM. In case of DGFEM, we consider also non-symmetric bilinear forms so that proofs in DG may slightly differ from CG formulations. We compare CGFEM and DGFEM as well.

- Turning back to vector-valued problems, we solve linear viscoelastic problems with internal variables in CG and DG. As shown in the scalar problems, we can derive and show the exactness and uniqueness of solution as well as *a priori* error estimates. However, details in proofs are changed, for example, it is necessary to use Korn's inequality. Numerical experiments for vector-valued problems have also been carried out.

- In Chapter 5, we introduce a kernel of power law type to model fractional order viscoelastic problems. We give some preliminary introduction to fractional calculus at the beginning. The spatial discretisation is the same as before but we have to be more careful in time integration due to fractional calculus. Even if we use second order finite difference schemes, we cannot take the full advantage of order of accuracy. However, we can resolve weak singularity with some requirements so that we get optimal error with respect to time. In numerical results, we can compare suboptimal and optimal cases.

1.1 Preliminary

Definition Let $v : \mathbb{R}^d \mapsto \mathbb{R}$ be a scalar function and $\underline{w} : \mathbb{R}^d \mapsto \mathbb{R}^d$ be a vector-valued function where d denotes the number of dimensions. Then the gradient of v and the divergence of $\underline{w}$ are defined as

$$\nabla v = \left(\frac{\partial v}{\partial x_i}\right)_{1\le i\le d}, \qquad \nabla \cdot \underline{w} = \sum_{i=1}^{d} \frac{\partial w_i}{\partial x_i}.$$

Definition Let V be a inner product space. Then its inner product is denoted by

$$u, v \in V, \qquad (u, v)_V \mapsto \mathbb{R},$$

and the induced norm is defined by

$$\|u\|_V = \sqrt{(u,u)_V}\ .$$

For instance, let $u, v \in L_2(\Omega)$ then

$$(u,v)_{L_2(\Omega)} = \int_\Omega uv \; d\Omega \text{ and } \|u\|_{L_2(\Omega)} = \left(\int_\Omega u^2 \; d\Omega\right)^{1/2}.$$

Moreover, in the vector-valued case, for $\underline{q}, \underline{r} \in L_2(\Omega)^d$

$$(\underline{q},\underline{r})_{L_2(\Omega)} = \int_\Omega \underline{q}\cdot\underline{r} \; d\Omega \text{ and } \|\underline{q}\|_{L_2(\Omega)} = \left(\int_\Omega |\underline{q}|^2 \; d\Omega\right)^{1/2}.$$

Definition A topological dual space to a Hilbert space V is denoted by V' which means V' is the set of linear functionals on V

$$V' = \{\phi : V \mapsto \mathbb{R} \mid \phi \text{ is a linear functional}\}.$$

For $\Omega \subset \mathbb{R}^d$ bounded, we introduce Hilbert space and Sobolev space as following. Let $\mathcal{D}(\Omega)$ be the space of C^∞ functions with compact support in Ω. Its dual space $\mathcal{D}'(\Omega)$ is the space of distributions then we define the partial differential operator $D^\alpha v \in \mathcal{D}'(\Omega)$ by

$$D^\alpha v(\phi) = (-1)^{|\alpha|} \int_\Omega v \frac{\partial^{|\alpha|}\phi}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}\, d\Omega, \quad \forall \phi \in \mathcal{D}(\Omega)$$

where the multi-index $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ for $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $|\alpha| = \sum_{i=1}^{d} \alpha_i$. If v is smooth enough,

$$D^\alpha v = \frac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}} \,.$$

With this notation, let us define the Sobolev space $H^s(\Omega)$ for non-negative integer s as

$$H^s(\Omega) = \{v \in L_2(\Omega) \mid D^\alpha v \in L_2(\Omega),\ \forall\ |\alpha| = 0, \ldots, s\}$$

also the Sobolev norm and semi-norm are given as

$$\|v\|_{H^s(\Omega)} = \left(\sum_{0 \le |\alpha| \le s} \|D^\alpha v\|^2_{L_2(\Omega)} \right)^{1/2} \quad \text{and} \quad |v|_{H^s(\Omega)} = \left(\sum_{|\alpha| = s} \|D^\alpha v\|^2_{L_2(\Omega)} \right)^{1/2} .$$

Note that for $s \ge 1$, $H^s(\Omega) \subset H^{s-1}(\Omega)$. More generally, we have a Sobolev space $W_p^k(\Omega)$ such that

$$W_p^k(\Omega) = \left\{ u \in L_p(\Omega) \;\middle|\; D^\alpha u \in L_p(\Omega),\ \forall |\alpha| \le k \right\}$$

for $k \in \mathbb{N}$ and $1 \le p \le \infty$.

<u>**Remark**</u> Let $1 \le p \le \infty$. For $v \in L_p(\Omega)$,

$$\|v\|_{L_p(\Omega)} = \begin{cases} \left(\int_\Omega |v|^p d\Omega\right)^{1/p}, & \text{if } 1 \le p < \infty, \\ \operatorname{ess\,sup} |v|, & \text{if } p = \infty. \end{cases}$$

Theorem 1.1. Trace Inequality [11]
Suppose that Ω has a Lipschitz boundary. Then there is a positive constant C, such that

$$\|v\|_{L_2(\partial\Omega)} \le C \|v\|^{1/2}_{L_2(\Omega)} \|v\|^{1/2}_{H^1(\Omega)}, \ \forall v \in H^1(\Omega).$$

Trace theorem and its applications can be found in [11, Chapter 1.6].

Theorem 1.2. Hölder Inequality
Let $p, q \in [1, \infty]$ with $1/p + 1/q = 1$. Then, if $f \in L_p(\Omega)$ and $g \in L_q(\Omega)$,

$$\|fg\|_{L_1(\Omega)} \le \|f\|_{L_p(\Omega)} \|g\|_{L_q(\Omega)}.$$

Here, q is called the Hölder conjugate of p. In case of $p = 2$, we call it Cauchy-Schwarz inequality,

$$(f, g)_{L_2(\Omega)} \le \|fg\|_{L_1(\Omega)} \le \|f\|_{L_2(\Omega)} \|g\|_{L_2(\Omega)} .$$

Theorem 1.3. Young's Inequality
For any positive a, b and ϵ,

$$ab \leq \frac{\epsilon}{2}a^2 + \frac{1}{2\epsilon}b^2.$$

Theorem 1.4. Property of L_∞ norm
We will use the following property of supremum or maximum later. Let f, g be non-negative bounded functions on A. For any $t \in A$, if

$$f(t) + g(t) \leq C, \qquad \text{for some positive } C,$$

then

$$\|f\|_{L_\infty(A)} + \|g\|_{L_\infty(A)} \leq 2C.$$

In a discrete sense, let $(f_n), (g_n)$ be non-negative sequences for $n \in \{0, \ldots, N\}$. If

$$f_n + g_n \leq C, \qquad \text{for some positive } C,$$

for any n, then

$$\max_{0\leq n\leq N} f_n + \max_{0\leq n\leq N} g_n \leq 2C.$$

Theorem 1.5. Poincaré-Friedrichs Inequality [11]
For any $v \in H^1(\Omega)$,

$$\|v\|_{L_2(\Omega)} \leq C\left(\|\nabla v\|_{L_2(\Omega)} + \left|\int_{\partial\Omega} v \; d\Gamma\right|\right),$$

where C is a positive constant that depends only on Ω and $\partial\Omega$.

Hereafter, let us introduce the time variable. For a function $z(\boldsymbol{x}, t)$, $\boldsymbol{x}$ is in the space domain Ω and t is in the time interval $[0, T]$. Then we define

$$L_s(0, T; V) = \left\{ z : (0, T) \mapsto V \text{ measurable and } \int_0^T \|z(t)\|_V^s dt < \infty \right\},$$

where V is a normed space equipped with the norm $\|\cdot\|_V$ and $1 \leq s \leq \infty$. Then the norm of $L_s(0, T; V)$ is defined as

$$\|z\|_{L_s(0,T;V)} = \begin{cases} \left(\int_0^T \|z(t)\|_V^s dt\right)^{1/s}, & 1 \leq s < \infty, \\ \operatorname*{ess\ sup}_{0\leq t\leq T} \|z(t)\|_V, & s = \infty. \end{cases}$$

Theorem 1.6. Grönwall's Inequality [47, 48]
Assume f, g are piecewise continuous functions on (a, b) and g is non-decreasing and non-negative. If there exists a positive constant C such that

$$f(t) \leq g(t) + C\int_a^t f(t')dt', \;\; \forall t \in (a, b),$$

then
$$f(t) \le e^{C(t-a)} g(t),$$
for any t. Furthermore, for a non-negative, non-decreasing and piecewise continuous function h, if
$$\forall t \in (a,b),\ f(t) + h(t) \le g(t) + C\int_a^t f(t')dt',$$
then
$$\begin{aligned} F(t) := f(t) + h(t) &\le g(t) + C\int_a^t f(t')dt', \\ &= g(t) + C\int_a^t (F(t') - h(t'))dt', \\ &\le g(t) + C\int_a^t F(t')dt' \\ &\le e^{C(t-a)} g(t), \end{aligned}$$
and hence
$$f(t) + h(t) \le e^{C(t-a)} g(t),\ \forall\ t \in (a,b).$$
On the other hand, as considering discrete Grönwall's inequality, let $\Delta t, B, C > 0$ and $(a_n), (b_n), (c_n), (d_n)$ be sequences of non-negative numbers such that
$$\forall n \in \mathbb{N}_0,\ a_n + \Delta t \sum_{i=0}^{n} b_i \le B + C\Delta t \sum_{i=0}^{n} a_i + \Delta t \sum_{i=0}^{n} c_i.$$
For $C\Delta t < 1$,
$$\forall n \in \mathbb{N}_0,\ a_n + \Delta t \sum_{i=0}^{n} b_i \le e^{C(n+1)\Delta t} \left(B + \Delta t \sum_{i=0}^{n} c_i \right).$$
Furthermore, more general version of discrete Grönwall's inequality is given in [48] *as follows. For non-negative sequences $(a_n), (b_n)$ and (g_n), if for $n \in \mathbb{N}$*
$$a_n \le b_n + \sum_{i=0}^{n-1} g_i a_i,$$
then
$$a_n \le b_n + \sum_{i=0}^{n-1} g_i b_i \exp\left(\sum_{j=i}^{n-1} g_j \right).$$
Theorem 1.7. Leibniz's Integral Rule
For a differentiable function $f(x,t)$ with respect x,
$$\frac{d}{dx}\int_a^b f(x,t)dt = \int_a^b \frac{\partial}{\partial x} f(x,t)dt.$$

In general, we have

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)dt = f(x,b(x))\frac{d}{dx}b(x) - f(x,a(x))\frac{d}{dx}a(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x}f(x,t)dt.$$

Moreover, in higher dimension

$$\nabla \int_a^b f(\boldsymbol{x},t)dt = \int_a^b \nabla f(\boldsymbol{x},t)dt.$$

Theorem 1.8. Crank-Nicolson Method [40]
Let y be a class of C^3 in time. Define $\Delta t = T/N > 0$ for finite time T and $N \in \mathbb{N}$, and $t_n = n\Delta t$. Taylor's expansion leads us to have

$$\frac{\dot{y}(\boldsymbol{x},t_{n+1}) + \dot{y}(\boldsymbol{x},t_n)}{2} = \frac{y(\boldsymbol{x},t_{n+1}) - y(\boldsymbol{x},t_n)}{\Delta t} + O(\Delta t^2).$$

This yields Crank-Nicolson finite difference scheme when we consider $\dot{y}(t) = F(y;t)$ where F is linear and smooth,

$$\frac{y(t_{n+1}) - y(t_n)}{\Delta t} \approx \bar{F}^n,$$

where bar notation denotes average by

$$\bar{F}^n := \frac{F(y;t_{n+1}) + F(y;t_n)}{2}.$$

We need to use this bar notation to express average values for the sake of Crank-Nicolson scheme. Note that Crank-Nicolson scheme is unconditionally stable with second order accuracy.

To make it clear, before considering our model problems, we introduce the following definitions and notations.

Notation

- Kronecker delta: Define the Kronecker delta by

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

- Einstein notation: We are going to use the convention notation that obeys the following rules [49]:

  1. Repeated indices are implicitly summed over.
  2. Each index can appear at most twice in any term.

For instance, let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$. Then we can write

$$\boldsymbol{x} \cdot \boldsymbol{y} = \sum_{i=1}^{d} x_i y_i \equiv x_i y_i.$$

In a similar way, we can also have matrix-vector multiplication by

$$(A\boldsymbol{x})_i = \sum_{j=1}^{d} A_{ij} x_j \equiv A_{ij} x_j \text{ for } i = 1, \ldots, d$$

where $A \in \mathbb{R}^{d \times d}$, $\boldsymbol{x} \in \mathbb{R}^d$.

- Standard basis of $\mathbb{R}^d$: $\{\underline{e}_i\}_{i=1}^d$. For example, if $d = 2$ the standard basis is

$$\left\{ \underline{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \underline{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

- For $f \in H^1(\Omega)$ and $\boldsymbol{v} \in [H^1(\Omega)]^d$,

$$f_{,j} := \frac{\partial}{\partial x_j} f, \qquad \text{and } v_{i,j} := \frac{\partial v_i}{\partial x_j} \text{ for } i, j = 1, \ldots, d$$

where $\Omega \subset \mathbb{R}^d$ and $\boldsymbol{v} = (v_i)_{i=1}^d$.

- Tensor divergence: Suppose $\underline{A}$ is a second order tensor in $\mathbb{R}^d$. Then its divergence is denoted by

$$\begin{aligned} \nabla \cdot \underline{A} = & \sum_{j=1}^{d} \left(\sum_{i=1}^{d} \frac{\partial A_{ij}}{\partial x_i} \right) \underline{e}_j = \sum_{j=1}^{d} \sum_{i=1}^{d} A_{ij,i} \underline{e}_j \\ = & A_{ij,i} \underline{e}_j \end{aligned}$$

with Einstein notation for $A_{ij,i} := \sum_{i=1}^{d} \frac{\partial A_{ij}}{\partial x_i}$.

- Tensor inner product: Let $\underline{A}$ and $\underline{B}$ be second order tensors in $\mathbb{R}^d$. Then we have

$$\underline{A} : \underline{B} = \sum_{j=1}^{d} \sum_{i=1}^{d} A_{ij} B_{ij} = A_{ij} B_{ij}.$$

By the tensor inner product, let us define L_2 norm in tensor by

$$(\underline{A}, \underline{B})_{L_2(\Omega)} = \int_{\Omega} \underline{A} : \underline{B} \; d\Omega.$$

In a similar way, we could also define the Sobolev norm and others.

1.2 Continuum Mechanics: The Linear Elastic Problem

Let $\mathfrak{B}$ be a compressible solid body which occupies $\Omega \subset \mathbb{R}^d$ with density ρ. Its surface $\partial\Omega$ is separated by Γ_D and Γ_N, which is $\Gamma_D \cap \Gamma_N = \emptyset$ and a positive measure of Γ_D. Let $\boldsymbol{u} = (u_i)_{i=1}^d$ be a displacement vector. We define a body force $\boldsymbol{f}$ on Ω by $\boldsymbol{f}(\boldsymbol{x}, t) := (f_i(\boldsymbol{x}, t))_{i=1}^d$ and a surface traction $\boldsymbol{g}$ onto Γ_N by $\boldsymbol{g}(\boldsymbol{x}, t) := (g_i(\boldsymbol{x}, t))_{i=1}^d$. In a classical physics, a particle motion obeys the Newton's second law($\boldsymbol{F} = m\boldsymbol{a}$ where $\boldsymbol{F}$ is the force, m is the mass and $\boldsymbol{a}$ indicates the acceleration). Here, the acceleration is equivalent to a second order time derivative of the displacement vector, that is $\ddot{\boldsymbol{u}}$. Hence the equation of motion for an elastic model can be governed by the Newton's second law expressed as

$$\nabla \cdot \underline{\boldsymbol{\sigma}} + \boldsymbol{f} = \rho \ddot{\boldsymbol{u}} \tag{1.2.1}$$

where $\underline{\boldsymbol{\sigma}} := (\sigma_{ij})_{i,j=1}^d$ is a symmetric stress tensor. In a classical theory of elasticity, a strain tensor $\underline{\boldsymbol{\varepsilon}}$ associated with the displacement vector is defined by

$$\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{1.2.2}$$

where $\underline{\boldsymbol{\varepsilon}} := (\varepsilon_{ij})_{i,j=1}^d$, called Cauchy infinitesimal tensor. Furthermore, the constitutive relation between the stress and the strain follows Hooke's law and the constitutive equation is given by

$$\underline{\boldsymbol{\sigma}} = \underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}} \tag{1.2.3}$$

where $\underline{\boldsymbol{D}} := (D_{ijkl})_{i,j,k,l=1}^d$ is a positive definite fourth order stiffness tensor(also called Hooke's tensor) satisfying

$$D_{ijkl} = D_{jikl}, \; D_{ijkl} = D_{ikjl}, \text{ and } D_{ijkl} = D_{ijlk}.$$

Hence (1.2.3) is also written as

$$\sigma_{ij} = D_{ijkl}\varepsilon_{kl} \qquad \text{for } i, j = 1, \ldots, d.$$

On the other hand, the surface Γ_N is subject to the surface traction $\boldsymbol{g}$ which provides a boundary condition for the system

$$\sigma_{ij} n_j = g_i \qquad \text{for } i = 1, \ldots, d \tag{1.2.4}$$

where $\boldsymbol{n} := (n_i)_{i=1}^d$ is the outward unit normal vector to Γ_N.

Example (e.g. see [25] for detail) Let $d = 3$ and $\mathfrak{B}$ be a homogeneous isotropic elastic body. We assume the equation of motion is on an equilibrium state in other words $\ddot{\boldsymbol{u}} = 0$. Hence the equilibrium equation is given by

$$-\sigma_{ij,j} = f_i \qquad \text{in } \Omega, \text{ for } i = 1, 2, 3.$$

For an isotropic material, the constitutive equation is given by

$$\sigma_{ij} = \lambda \text{div}\boldsymbol{u}\delta_{ij} + \mu\varepsilon_{ij}(\boldsymbol{u}) \qquad \text{in } \Omega$$

for $i, j = 1, 2, 3$ where $\text{div}\boldsymbol{u} = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i}$, a constant $\lambda \in \mathbb{R}$ and μ is a positive constant. In addition, the boundary conditions allow us to have (1.2.4) and

$$\boldsymbol{u} = 0 \qquad \text{on } \Gamma_D.$$

Since the stress tensor is symmetric, use of Einstein notation gives

$$\boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\boldsymbol{v}) = \sigma_{ij}\varepsilon_{ij}(\boldsymbol{v}) = \frac{1}{2}(\sigma_{ij}v_{i,j} + \sigma_{ij}v_{j,i}) = \frac{1}{2}(\sigma_{ij}v_{i,j} + \sigma_{ji}v_{j,i}) = \sigma_{ij}v_{i,j}$$

$\forall \boldsymbol{v} \in [H^1(\Omega)]^3$. This result also holds for general spatial dimension d. It will lead us to derive a variational formulation for viscoelasticity later.

Theorem 1.9. Korn's Inequality [50, 51, 52, 11]
For any $\boldsymbol{v} \in [H^1(\Omega)]^d$ for $\Omega \subset \mathbb{R}^d$ bounded, there exists a positive constant C such that

$$\|\boldsymbol{v}\|^2_{H^1(\Omega)} \leq C \left(\|\boldsymbol{v}\|_{L_2(\Omega)} + \int_\Omega \varepsilon_{ij}(\boldsymbol{v})\varepsilon_{ij}(\boldsymbol{v}) d\Omega \right). \tag{1.2.5}$$

Furthermore, if Ω is Lipschitz and $\boldsymbol{v} \in [H^1(\Omega)]^d$ such that $\boldsymbol{v}|_{\Gamma_D} = \mathbf{0}$ with positive measure Γ_D then

$$C \|\boldsymbol{v}\|^2_{H^1(\Omega)} \leq \int_\Omega \varepsilon_{ij}(\boldsymbol{v})\varepsilon_{ij}(\boldsymbol{v}) d\Omega \tag{1.2.6}$$

for some positive C independent of $\boldsymbol{v}$.

These Korn's inequalities would be significantly important to prove stability bounds and error bounds for finite element approximations. Korn's inequalities are used for *a priori* estimates for a linear elastic problem. Also, it implies that there exists a positive constant c such that

$$c \|\boldsymbol{v}\|^2_{L_2(\Omega)} \leq \int_\Omega \varepsilon_{ij}(\boldsymbol{v})\varepsilon_{ij}(\boldsymbol{v}) d\Omega, \tag{1.2.7}$$

where $\boldsymbol{v} \in [H^1(\Omega)]^d$ and $\boldsymbol{v}|_{\Gamma_D} = \mathbf{0}$ with positive measure Γ_D.

1.3 Linear Viscoelasticity

In this section, we describe the theory of viscoelasticity and rheology model problems. Viscoelasticity is the property of materials equipped with viscosity and elasticity. It can be shown in dispersive media such as amorphous polymers, semi-crystalline polymers, bio tissue and metals at high temperature [5, 3]. Before we consider viscoelasticity, let us begin with elasticity and viscosity. In a physical experiment, a spring and a dashpot represent elasticity and viscosity, respectively. These two elements compose viscoelastic models which constructed by in series and parallel (see e.g. [3, 1, 2]). Based on rheological theory, Hooke's law and Newton's law provide constitutive equations for a spring and a dashpot, respectively as

$$\sigma = E\varepsilon \text{ (for a linear spring)}$$

and

$$\sigma = \eta\dot{\varepsilon} \text{ (for a linear dashpot)}$$

where σ is stress, ε is strain, E is the Young's modulus and η is the Newtonian viscosity. Combinations of springs and dashpots in series or in parallel would construct viscoelastic models satisfied by Hooke's law and Newton's law. In addition, the constitutive equations of viscoelastic behaviours would be given by the following principal rules [3]:

- For elements connected in series, their stresses coincide, and the total strain equals a sum of strains in individual elements.
- For elements connected in parallel, their strains coincide, and the total stress equals a sum of stresses in individual elements.

Under the above laws, we can consider rheological models of viscoelasticity.

1.3.1 Maxwell Model

One of the simplest viscoelastic models is constructed by one spring and one dashpot in series as seen in Figure 1.1. We denote the stress and strain of the spring by σ_S and ε_S, respectively. In a similar way, we define σ_D and ε_D for the dashpot, and σ and ε for total amount. By the rule for series, we have

$$\sigma_S = \sigma_D = \sigma \text{ and } \varepsilon = \varepsilon_S + \varepsilon_D.$$

On the other hand, Hookes' law and Newton's law give

$$\sigma_S = E\varepsilon_S \text{ and } \sigma_D = \eta\dot{\varepsilon}_D.$$

If we suppose strains are smooth enough in time and we differentiate it,

$$\dot{\varepsilon} = \dot{\varepsilon}_S + \dot{\varepsilon}_D.$$

Also, we assume the stresses are differentiable then $\dot{\sigma}_S = E\dot{\varepsilon}_S$. As a result, we can obtain

$$\dot{\varepsilon} = \frac{\dot{\sigma}_S}{E} + \dot{\varepsilon}_D = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta}.$$

This ODE can be solved by integration factor method which is a typical way to solve simple ODEs. When we multiply $e^{t/\tau}$ on the ODE where $\tau = \eta/E$, we have

$$e^{t/\tau}\dot{\sigma} + e^{t/\tau}\frac{\sigma}{\tau} = \frac{d}{dt}(e^{t/\tau}\sigma) = Ee^{t/\tau}\dot{\varepsilon}.$$

Integration yields

$$e^{t/\tau}\sigma(t) - \sigma(0) = E\int_0^t e^{s/\tau}\dot{\varepsilon}(s)ds.$$

With the initial condition $\sigma(0) = E\varepsilon(0)$, we can obtain

$$\sigma(t) = Ee^{-t/\tau}\varepsilon(0) + E\int_0^t e^{-(t-s)/\tau}\dot{\varepsilon}(s)ds. \tag{1.3.1}$$

(1.3.1) can be rewritten by

$$\sigma(t) = E\varepsilon(t) - E\int_0^t \frac{1}{\tau}e^{-(t-s)/\tau}\varepsilon(s)ds \tag{1.3.2}$$

with integration by parts.

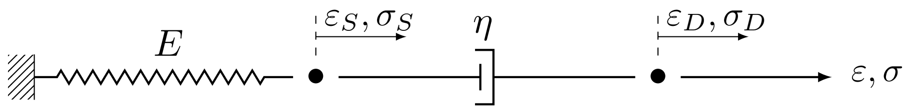

Figure 1.1: Maxwell model

1.3.2 Voigt Model

Voigt model is constructed by a spring and a dashpot in parallel as shown in Figure 1.2. The constitutive equation for Voigt model can be derived by the principle rule for parallel, Hooke's law and Newton law as shown in Maxwell model. First of all, we have

$$\varepsilon = \varepsilon_S = \varepsilon_D \text{ and } \sigma = \sigma_S + \sigma_D.$$

Then Hooke's law and Newton law give

$$\sigma_S = E\varepsilon_S \text{ and } \sigma_D = \eta\dot{\varepsilon}_D.$$

In a similar way with the Maxwell model, combining the result leads

$$\sigma = E\varepsilon + \eta\dot{\varepsilon}.$$

Then we can solve it with $e^{t/\tau}$ by

$$\int_0^t e^{s/\tau}\frac{\sigma(s)}{E}ds = \tau e^{t/\tau}\varepsilon(t) - \tau\varepsilon(0)$$

where $\tau = \eta/E$. Hence it yields

$$\varepsilon(t) = \frac{1}{E}e^{-t/\tau}\sigma(0) + \frac{1}{\eta}\int_0^t e^{-(t-s)/\tau}\sigma(s)ds \tag{1.3.3}$$

with $\sigma(0) = E\varepsilon(0)$. This can be also written as

$$\varepsilon(t) = \frac{1}{E}\sigma(t) - \frac{1}{E}\int_0^t e^{-(t-s)/\tau}\dot{\sigma}(s)ds. \tag{1.3.4}$$

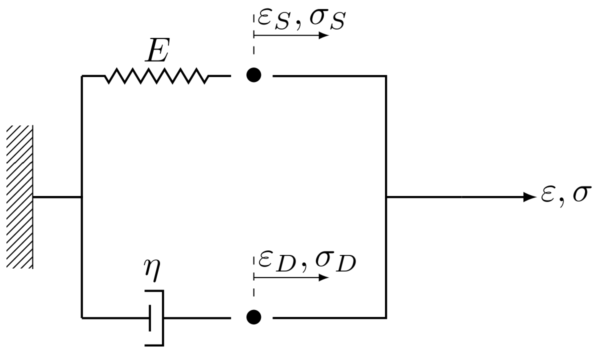

Figure 1.2: Voigt model

1.3.3 Maxwell Solid

Maxwell Solid consists of one spring and a series of a spring and a dashpot in parallel as shown in Figure 1.3. Using the principle rules, Hooke's law and Newton's law, the constitutive equation can be derived. We can observe the relations between stresses and strains, respectively as

$$\sigma^* = \sigma_D,\ \sigma = \sigma^* + \sigma_S$$

and

$$\varepsilon = \varepsilon_S = \varepsilon^* + \varepsilon_D.$$

In addition,

$$\sigma_S = E_0\varepsilon_S,\ \sigma^* = E_1\varepsilon^* \text{ and } \sigma_D = \eta\dot{\varepsilon}_D.$$

Since $\sigma^* = \sigma_D = E_1\varepsilon^*$,

$$E_1\varepsilon^* = \eta\dot{\varepsilon}_D = \eta(\dot{\varepsilon} - \dot{\varepsilon}^*).$$

When we solve this ODE, it yields

$$\varepsilon^*(t) = e^{-t/\tau}\varepsilon(0) + \int_0^t e^{-(t-s)/\tau}\dot{\varepsilon}(s)ds$$

where $\tau = \eta/E_1$ with $\varepsilon^*(0) = \varepsilon(0)$. Now, let us define the stress relaxation function $E(t)$ by

$$E(t) = E_0 + E_1 e^{-t/\tau}.$$

Recall the relations of stresses and strains. Using Hooke's law,

$$\begin{aligned}\sigma(t) =& E_0\varepsilon_S(t) + E_1\varepsilon^*(t)\\ =& E_0\varepsilon(t) + E_1 e^{-t/\tau}\varepsilon(0) + E_1\int_0^t e^{-(t-s)/\tau}\dot{\varepsilon}(s)ds\\ =& E_0\varepsilon(t) + (-E_0\varepsilon(0) + E_0\varepsilon(0)) + E_1 e^{-t/\tau}\varepsilon(0) + E_1\int_0^t e^{-(t-s)/\tau}\dot{\varepsilon}(s)ds\end{aligned}$$

$$
\begin{aligned}
&= E_0 \int_0^t \dot{\varepsilon}(s)ds + E_0\varepsilon(0) + E_1 e^{-t/\tau}\varepsilon(0) + E_1 \int_0^t e^{-(t-s)/\tau}\dot{\varepsilon}(s)ds \\
&= (E_0 + E_1 e^{-t/\tau})\varepsilon(0) + \int_0^t (E_0 + E_1 e^{-(t-s)/\tau})\dot{\varepsilon}(s)ds
\end{aligned}
$$

so

$$
\sigma(t) = E(t)\varepsilon(0) + \int_0^t E(t-s)\dot{\varepsilon}(s)ds. \tag{1.3.5}
$$

With integration by parts, we also gain

$$
\sigma(t) = E(0)\varepsilon(t) - \int_0^t E_s(t-s)\varepsilon(s)ds \tag{1.3.6}
$$

where $E_s(t-s) := \frac{\partial}{\partial s}E(t-s)$.

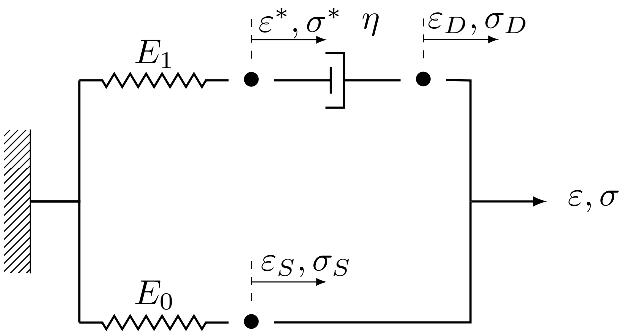

Figure 1.3: Maxwell solid

1.3.4 Internal Variables

As shown in the Maxwell solid, a pair of spring and dashpot, called a *Maxwell element*, follows the principal rule of continuum mechanics to lead a constitutive equation. In parallel construction of Maxwell elements, local strains are same each other and the total stress is equal to a sum of local stresses. It is a key idea of internal variables that we set local constitutive equations as new variables. According to [25], when we consider the generalised Maxwell solid, we will introduce internal variables for the stress relaxation functions. To be specific, locally constitutive relations are dealt with in order to derive total constitutive equation between total stress and total strain. In this thesis, each internal variable will be defined as a vector-valued function rather than a tensor-valued function [26].

In addition, we can define *stress relaxation function* and it enables us to solve the model problem with Laplacian transformation (convolution form with exponential kernels) in the integral form in time [8].

The generalised Maxwell solid would be constructed by a linear spring connected in parallel to a sequence of N_φ spring-dashpot pairs as shown in Figure 1.4. Hooke's law and the principal rule for parallel lead us to have

$$\varepsilon_0 = \varepsilon,\ \sigma_0 = E_0\varepsilon,\ \text{and}\ \sigma_q^* = E_q\varepsilon_q^*$$

for each $q \in \{1, \ldots, N_\varphi\}$. For each spring-dashpot pair, we can derive ODEs such that

$$\dot{\varepsilon}_q^* + \frac{\varepsilon_q^*}{\tau_q} = \dot{\varepsilon} \ \Rightarrow\ \varepsilon_q^*(t) = e^{-t/\tau_q}\varepsilon(0) + \int_0^t e^{-(t-s)/\tau_q}\dot{\varepsilon}(s)ds$$

where $\tau_q = E_q/\eta_q$ for each $q \in \{1, \ldots, N_\varphi\}$. Since the total stress equals a sum of stresses and $\sigma_q = \sigma_q^*$ for each q, we can derive

$$\begin{aligned}\sigma(t) =&\sigma_0(t) + \sigma_1(t) + \cdots + \sigma_{N_\varphi}(t)\\ =&E_0\varepsilon(t) + E_1\varepsilon_1^*(t) + \cdots + E_{N_\varphi}\varepsilon_{N_\varphi}^*(t)\\ =&E_0\varepsilon(t) + \sum_{q=1}^{N_\varphi}\left(E_q e^{-t/\tau_q}\varepsilon(0) + \int_0^t E_q e^{-(t-s)/\tau_q}\dot{\varepsilon}(s)ds\right).\end{aligned}$$

If we define

$$E(t) := E_0 + \sum_{q=1}^{N_\varphi} E_q e^{-t/\tau_q}, \tag{1.3.7}$$

then the total stress can be expressed as

$$\sigma(t) =E(0)\varepsilon(t) - \int_0^t E_s(t-s)\varepsilon(s)ds \tag{1.3.8}$$

or

$$\sigma(t) =E(t)\varepsilon(0) + \int_0^t E(t-s)\dot{\varepsilon}(s)ds \tag{1.3.9}$$

by integration by parts. We can expand the constitutive relation with respect to the displacement vector $\boldsymbol{u}$ by

$$\underline{\boldsymbol{\sigma}}(\boldsymbol{u};\boldsymbol{x},t) =\underline{\boldsymbol{D}}(0)\underline{\boldsymbol{\varepsilon}}(\boldsymbol{u};\boldsymbol{x},t) - \int_0^t \underline{\boldsymbol{D}}_s(t-s)\underline{\boldsymbol{\varepsilon}}(\boldsymbol{u};\boldsymbol{x},s)ds \tag{1.3.10}$$

or

$$\underline{\boldsymbol{\sigma}}(\boldsymbol{u};\boldsymbol{x},t) =\underline{\boldsymbol{D}}(t)\underline{\boldsymbol{\varepsilon}}(\boldsymbol{u};\boldsymbol{x},0) + \int_0^t \underline{\boldsymbol{D}}(t-s)\dot{\underline{\boldsymbol{\varepsilon}}}(\boldsymbol{u};\boldsymbol{x},s)ds \tag{1.3.11}$$

where $\underline{\boldsymbol{D}}_s(t-s) := \frac{\partial}{\partial s}\underline{\boldsymbol{D}}_s(t-s)$, $\underline{\boldsymbol{D}}$ is a positive definite fourth order tensor as seen in (1.2.3). Then (1.3.8) and (1.3.9) are scalar analogues of (1.3.10) and (1.3.11), respectively. Moreover, when we define a generic stress relaxation function by

$$\varphi(t) = \varphi_0 + \sum_{q=1}^{N_\varphi} \varphi_q e^{-t/\tau_q}$$

with $N_\varphi \in \mathbb{N}$, $\varphi(0) = 1$, positive coefficients $\{\varphi_q\}_{q=0}^{N_\varphi}$ and $\{\tau_q\}_{q=1}^{N_\varphi}$, we have

$$\underline{\boldsymbol{D}}(t) = \underline{\boldsymbol{D}}(0)\varphi(t) \tag{1.3.12}$$

from [1]. Then we can express $\underline{\boldsymbol{D}}_s(t-s)$ by

$$\underline{\boldsymbol{D}}_s(t-s) = \underline{\boldsymbol{D}}(0)\varphi_s(t-s)$$

where

$$\varphi_s(t-s) = \sum_{q=1}^{N_\varphi} \frac{\varphi_q}{\tau_q} e^{-(t-s)/\tau_q}.$$

Let us define internal variables $\boldsymbol{\psi}_q$ by

$$\boldsymbol{\psi}_q(t) = \int_0^t \frac{\varphi_q}{\tau_q} e^{-(t-s)/\tau_q} \boldsymbol{u}(t')ds \tag{1.3.13}$$

for $q = 1, \ldots, N_\varphi$. Hence we can replace (1.3.10) with $\{\boldsymbol{\psi}_q\}_{q=1}^{N_\varphi}$ by

$$\underline{\boldsymbol{\sigma}}(\boldsymbol{u}(t)) = \underline{\boldsymbol{D}}(0)\underline{\varepsilon}\big(\boldsymbol{u}(t) - \sum_{q=1}^{N_\varphi} \boldsymbol{\psi}_q(t)\big). \tag{1.3.14}$$

On the other hand, (1.3.12) also yields

$$\begin{aligned}
\int_0^t \underline{\boldsymbol{D}}(t-s)\dot{\underline{\varepsilon}}(\boldsymbol{u};\boldsymbol{x},s)ds &= \int_0^t \varphi_0 \underline{\boldsymbol{D}}(0)\dot{\underline{\varepsilon}}(\boldsymbol{u};\boldsymbol{x},s)ds \\
&\quad + \sum_{q=1}^{N_\varphi} \int_0^t \varphi_q e^{-(t-s)/\tau_q} \underline{\boldsymbol{D}}(0)\dot{\underline{\varepsilon}}(\boldsymbol{u};\boldsymbol{x},s)ds \\
&= \varphi_0 \underline{\boldsymbol{D}}(0)\underline{\varepsilon}(\boldsymbol{u};\boldsymbol{x},t) - \varphi_0 \underline{\boldsymbol{D}}(0)\underline{\varepsilon}(\boldsymbol{u};\boldsymbol{x},0) \\
&\quad + \sum_{q=1}^{N_\varphi} \int_0^t \varphi_q e^{-(t-s)/\tau_q} \underline{\boldsymbol{D}}(0)\dot{\underline{\varepsilon}}(\boldsymbol{u};\boldsymbol{x},s)ds
\end{aligned}$$

and

$$\underline{\boldsymbol{D}}(t)\underline{\varepsilon}(\boldsymbol{u};\boldsymbol{x},0) = \varphi_0 \underline{\boldsymbol{D}}(0)\underline{\varepsilon}(\boldsymbol{u};\boldsymbol{x},0) + \sum_{q=1}^{N_\varphi} \varphi_q e^{-t/\tau_q} \underline{\boldsymbol{D}}(0)\underline{\varepsilon}(\boldsymbol{u};\boldsymbol{x},0).$$

When we define $\boldsymbol{\zeta}_q$ by

$$\boldsymbol{\zeta}_q(t) = \int_0^t \varphi_q e^{-(t-s)/\tau_q} \dot{\boldsymbol{u}}(s) ds \tag{1.3.15}$$

for $q = 1, \ldots, N_\varphi$ we can write (1.3.11) as

$$\underline{\boldsymbol{\sigma}}(\boldsymbol{u}(t)) = \underline{\boldsymbol{D}}(0)\underline{\boldsymbol{\varepsilon}}\big(\varphi_0 \boldsymbol{u}(t) + \sum_{q=1}^{N_\varphi} \boldsymbol{\zeta}_q(t)\big) + \sum_{q=1}^{N_\varphi} \varphi_q e^{-t/\tau_q} \underline{\boldsymbol{D}}(0)\underline{\boldsymbol{\varepsilon}}(\boldsymbol{u}_0) \tag{1.3.16}$$

when $\boldsymbol{u}_0 = \boldsymbol{u}(0)$. We will call $\{\boldsymbol{\psi}_q\}_{q=1}^{N_\varphi}$ and $\{\boldsymbol{\zeta}_q\}_{q=1}^{N_\varphi}$ the internal variables for the displacement form and the velocity form, respectively.

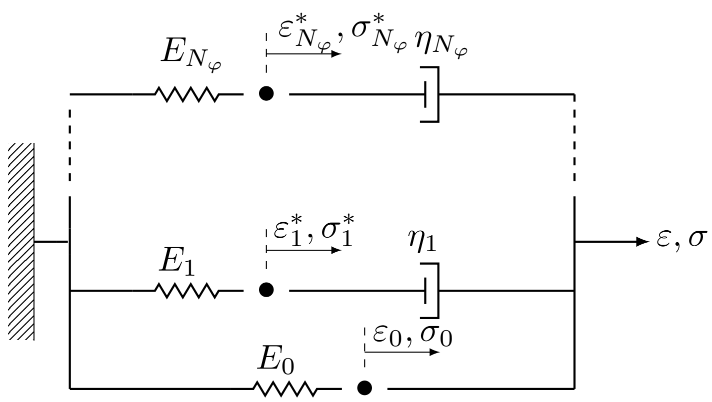

Figure 1.4: Generalised Maxwell solid

1.3.5 Primal Model Problem

In the same sense in the elastic theory, Newton's second law gives the equation of motion for a viscoelastic model. Recall (1.2.1) and (1.2.2). Thus our primal model problem is given as

$$\rho \ddot{\boldsymbol{u}} - \nabla \cdot \underline{\boldsymbol{\sigma}} = \boldsymbol{f} \quad \text{in} \quad (0,T] \times \Omega \tag{1.3.17}$$
$$\boldsymbol{u} = \boldsymbol{g}_D \quad \text{on} \quad [0,T] \times \Gamma_D \tag{1.3.18}$$
$$\underline{\boldsymbol{\sigma}} \cdot \boldsymbol{n} = \boldsymbol{g}_N \quad \text{on} \quad [0,T] \times \Gamma_N \tag{1.3.19}$$
$$\boldsymbol{u} = \boldsymbol{u}_0 \quad \text{on} \quad \{0\} \times \Omega \tag{1.3.20}$$
$$\dot{\boldsymbol{u}} = \boldsymbol{w}_0 \quad \text{on} \quad \{0\} \times \Omega \tag{1.3.21}$$

where Ω is a viscoelastic material domain in $\mathbb{R}^d$, Γ_D and Γ_N represent Dirichlet boundary and Neumann boundary, respectively, and $[0,T]$ is the time domain. Γ_D and Γ_N are disjoint and the measure of Γ_D is non-zero. Since we have the equivalent constitutive equations (1.3.10) and (1.3.11), (1.3.17) can be rewritten as

$$\rho \ddot{u}_i(t) - \left(D_{ijkl}(0)\varepsilon_{kl}(\boldsymbol{u}(t))\right)_{,j} + \int_0^t \left(\frac{\partial D_{ijkl}(t-s)}{\partial s}\varepsilon_{kl}(\boldsymbol{u}(s))\right)_{,j} ds = f_i(t), \qquad (1.3.22)$$

and

$$\rho \ddot{u}_i(t) + \int_0^t \left(D_{ijkl}(t-s)\varepsilon_{kl}(\dot{\boldsymbol{u}}(s))\right)_{,j} ds = f_i(t) + \left(D_{ijkl}(t)\varepsilon_{kl}(\boldsymbol{u}_0)\right)_{,j}, \qquad (1.3.23)$$

respectively, for $i = 1, \ldots, d$. Moreover, when we use internal variables for the constitutive equation (1.3.14) and (1.3.16), our primal model problem is given by

(Displacement form)

$$\rho \ddot{\boldsymbol{u}} - \nabla \cdot \left(\underline{\boldsymbol{D}}(0)\underline{\boldsymbol{\varepsilon}}\big(\boldsymbol{u} - \sum_{q=1}^{N_\varphi} \boldsymbol{\psi}_q\big) \right) = \boldsymbol{f} \qquad \text{in } (0,T] \times \Omega \qquad (1.3.24)$$

$$\tau_q \dot{\boldsymbol{\psi}}_q + \boldsymbol{\psi}_q = \varphi_q \boldsymbol{u} \qquad \text{for } q = 1\ldots, N_\varphi \text{ in } [0,T] \times \Omega \qquad (1.3.25)$$

and

(Velocity form)

$$\rho \ddot{\boldsymbol{u}} - \nabla \cdot \left(\varphi_0 \underline{\boldsymbol{D}}(0)\underline{\boldsymbol{\varepsilon}}\big(\boldsymbol{u} + \sum_{q=1}^{N_\varphi} \boldsymbol{\zeta}_q\big) \right) = \boldsymbol{f} + \nabla \cdot \left(\sum_{q=1}^{N_\varphi} \varphi_q e^{-t/\tau_q} \underline{\boldsymbol{D}}(0)\underline{\boldsymbol{\varepsilon}}(\boldsymbol{u}_0) \right)$$

$$\text{in } (0,T] \times \Omega \qquad (1.3.26)$$

$$\tau_q \dot{\boldsymbol{\zeta}}_q + \boldsymbol{\zeta}_q = \tau_q \varphi_q \dot{\boldsymbol{u}} \qquad \text{for } q = 1\ldots, N_\varphi \text{ in } [0,T] \times \Omega \qquad (1.3.27)$$

(1.3.25) and (1.3.27) are governed by first time derivatives of (1.3.13) and (1.3.15), respectively. Note that $\boldsymbol{\psi}_q(0) = \mathbf{0}$ and $\boldsymbol{\zeta}_q(0) = \mathbf{0}$ for each q by definitions, (1.3.13) and (1.3.15).

1.4 Finite Element Methods

Finite element methods are approximate ways to solve PDEs with variational formulations. First of all, choose a grid for a given domain and define a test space. Then we should derive the variational form and find the approximate solution in the test space satisfying the variational form. More detailed information is given in [11, 24] and references therein. In our case, we are going to use Continuous Galerkin Finite Element Method (CGFEM) and Discontinuous Galerkin Finite Element Method (DGFEM) for spatial discretisation. In this section, we present the background of CGFEM and DGFEM as well as some elliptic projection properties for error estimates later.

1.4.1 Continuous Galerkin Finite Element Method (CGFEM)

Let $\Omega \subset \mathbb{R}^d$ for $d \in \mathbb{N}$ be a bounded polytope domain. Suppose $\partial\Omega = \Gamma_D \cup \Gamma_N$ with $\Gamma_D \cap \Gamma_N = \emptyset$ and Γ_D is of positive measure. Consider the following the elliptic problem

$$-\nabla \cdot D\nabla u = f \qquad \text{in } \Omega, \tag{1.4.1}$$
$$u = 0 \qquad \text{on } \Gamma_D, \tag{1.4.2}$$
$$\nabla u \cdot \underline{n} = g \qquad \text{on } \Gamma_N \tag{1.4.3}$$

where D is a positive constant and $\underline{n}$ is the outward unit normal vector. Let

$$V = \{v \in H^1(\Omega) \mid v|_{\Gamma_D} = 0\}.$$

Then for any $v \in V$ it is true that

$$(D\nabla u, \nabla v)_{L_2(\Omega)} = (f, v)_{L_2(\Omega)} + (g, v)_{L_2(\Gamma_N)}$$

by integration by parts where $(g, v)_{L_2(\Gamma_N)} = \int_{\Gamma_N} gv \, d\Gamma$. When we define a bilinear form and a linear form by

$$a(v, w) = (D\nabla v, \nabla w)_{L_2(\Omega)} \text{ and } F(v) = (f, v)_{L_2(\Omega)} + (g, v)_{L_2(\Gamma_N)}$$

for $v, w \in V$. Then the elliptic problem of (1.4.1)-(1.4.3) generates the variational form such that find $u \in V$ satisfying

$$a\,(u, v) = F(v) \tag{1.4.4}$$

for any $v \in V$.

<u>Definition</u> Let $(V, (\cdot,\cdot)_V)$ be a Hilbert space. Then we have the induced norm defined by

$$\|v\|_V = ((v, v)_V)^{1/2}, \ \forall v \in V.$$

Suppose $a(\cdot,\cdot)$ is a bilinear form on $V \times V$ and $F \in V'$.

(i) $a(\cdot,\cdot)$ is coercive on V if $a(v, v) \geq \kappa\|v\|_V^2$ $\forall v \in V$ for some positive constant κ.
(ii) $a(\cdot,\cdot)$ is continuous on V if $|a(w, v)| \leq C\|w\|_V\|v\|_V$ $\forall v, w \in V$ for some positive constant C.
(iii) F is continuous if $|F(v)| \leq K\|v\|_V$ $\forall v \in V$ for some positive constant K

Here, κ, K and C are independent of any v, w.

Theorem 1.10. Lax-Milgram Theorem [11, 53]
Given a Hilbert space $(V, (\cdot,\cdot)_V)$, if there exist a coercive continuous bilinear form $a(\cdot,\cdot)$ and continuous linear functional $F \in V'$,

$$\exists! u \in V \text{ such that } a(u,v) = F(v) \qquad \forall v \in V.$$

Lemma 1.1. *The given bilinear form $a(\cdot,\cdot)$ is coercive. That is*

$$\kappa \|v\|^2_{H^1(\Omega)} \le a(v,v), \ \forall\ v \in V = \left\{ v \in H^1(\Omega) \ \middle|\ v|_{\Gamma_D} = 0 \right\},$$

for some positive κ.

Proof. The proof is shown in [54]. □

Let us define the energy norm by

$$\|v\|^2_V = a(v,v), \ \forall v \in V.$$

Then Lemma 1.1 allows us to have the norm equivalence between H^1 norm and the energy norm by $\forall v \in V$

$$\kappa \|v\|^2_{H^1(\Omega)} \le \|v\|^2_V \le D \|v\|^2_{H^1(\Omega)}.$$

Also, we can observe

$$|a(v,w)| \le D \|v\|_{H^1(\Omega)} \|w\|_{H^1(\Omega)}$$

for any $v, w \in V$ by Cauchy-Schwarz inequality and the definition of H^1 norm. Hence the bilinear form is continuous. Moreover, we assume $f \in L_2(\Omega)$ and $g \in L_2(\Gamma_N)$ so that we can have continuity of the linear form F. Therefore, we can solve the variational problem (1.4.4) uniquely.

From now on, we are going to approximate the solution of variational problem in practice. We follow [11, Chapter 3] in order to consider the construction of finite elements. Let $\mathcal{E}_h$ be a set of non-degenerate subdivisions of the domain Ω. Then for $E \in \mathcal{E}_h$, E is a sub-interval in $d = 1$, a triangle in $d = 2$ or a tetrahedron in $d = 3$. By using Lagrange finite element, we can construct $V^h \subset V$ such that is the set of continuous piecewise polynomials (e.g. see [11, 55]). We denote

$$V^h = \text{span}\,\{\ \Phi_i \mid i = 1, \ldots, N_{V^h}\ \} \cap V$$

where N_{V^h} is the number of global functions and Φ_i is a global basis function such that is a piecewise polynomial of degree $k \in \mathbb{N}$ for $i = 1, \ldots, N_{V^h}$. In other words, for $E \in \mathcal{E}_h$, $\Phi_i|_E$ is either a polynomial of degree k or a constant 0 but continuous in the domain, $\forall i = 1, \ldots, N_{V^h}$. Additionally, we can express for any $v \in V^h$

$$v(\boldsymbol{x}) = \sum_{i=1}^{N_{V^h}} v_i \Phi_i(\boldsymbol{x})$$

for $v_i \in \mathbb{R},\ \forall i \in \{1, \ldots, N_{V^h}\}$. Now, we consider the variational problem such that find $u_h \in V^h$ satisfying

$$a\,(u_h, v) = F(v) \tag{1.4.5}$$

for any $v \in V^h$. Since $u_h \in V^h$, u_h can be expressed with a linear combination of $\{\Phi_i\}_{i=1}^{N_{V^h}}$ and the degree of freedoms can be computed by substitution of global basis functions in (1.4.5).

In [11, 38, 55], the error estimates for elliptic problems have been introduced. If $u \in H^s(\Omega) \cap V$ for $s \in \mathbb{N}$, we have the following results

$$\|u - u_h\|_{H^1(\Omega)} + \|u - u_h\|_{L_2(\Omega)} \leq Ch^{r-1}|u|_{H^r(\Omega)} \tag{1.4.6}$$

where $r = \min(k+1, s)$, h represents a mesh size, and C is a positive constant independent of u and h. If the domain is convex or has a smooth boundary, elliptic regularity will be provided so that we have

$$\|u - u_h\|_{L_2(\Omega)} \leq Ch^r|u|_{H^r(\Omega)} \tag{1.4.7}$$

(e.g. see [11, Chapter 5.5] for the elliptic regularity and its condition).

Let us define an elliptic projection operator R such that for $w \in V$

$$R : V \mapsto V^h \text{ by } a(Rw, v) = a(w, v),\ \forall v \in V^h.$$

Remark By the definition of elliptic projection, it is true that for any $w \in V$

$$a(w - Rw, v) = a(w, v) - a(Rw, v) = 0,\ \forall v \in V^h.$$

We call this property Galerkin orthogonality.

Remark (e.g. see [38] in detail)
For any $w \in V \cap H^s(\Omega)$,

$$\|w - Rw\|_{H^1(\Omega)} + \|w - Rw\|_{L_2(\Omega)} \leq Ch^{r-1} \tag{1.4.8}$$

for some positive C independent of h and $r = \min(k + 1, s)$. Furthermore, if elliptic regularity is provided, it satisfies

$$\|w - Rw\|_{L_2(\Omega)} \leq Ch^r. \tag{1.4.9}$$

1.4.2 Discontinuous Galerkin Finite Element Method(DGFEM)

As part of framework for DGFEM, we will define a subdivision as following the definitions in [24]. Let E be a bounded polytope domain with diameter $h_E := \sup_{x,y \in E} \|x - y\|$ where $\|\cdot\|$ is the Euclidean norm. $|E|$ denotes the measure of E. In a similar way, let us define $|e|$ where e is the edge of E. The main concept of DG scheme is that when a variational problem is dealt, the test functions are defined as piecewise continuous functions on each element but it could be discontinuous on edges. In other words, our finite dimensional test space consists of the piecewise polynomials but does not need to be continuous on whole domain. Define the space of polynomials of degree less than or equal to k on E for $E \subset \mathbb{R}^d$ by

$$\mathcal{P}_k(E) = \text{span}\{x_1^{i_1} \cdots x_d^{i_d} \mid i_1 + \cdots + i_d \leq k,\ \boldsymbol{x} \in E\}.$$

Suppose $\mathcal{E}_h = \{E_i : i \in I\}$ where the measure of $E_i \cap E_j$ is zero for any $i, j \in I$ with $i \neq j$ where I is an index set. Then let us define

$$\mathcal{D}_k(\mathcal{E}_h) = \{v \mid v|_{E_i} \in \mathcal{P}_k(E_i) \text{ for each } i \in I\}.$$

Assume Ω is a polytopic domain in $\mathbb{R}^2$ or $\mathbb{R}^3$ which is subdivided into elements E, where E is a triangle in 2D or a tetrahedron in 3D and the intersection of elements is either a vertex, an edge, or a face. Let h be a maximum diameter of elements then we define the set $\mathcal{E}_h$ of the elements. Then

$$\forall e \subset \partial E,\ \forall E \in \mathcal{E}_h,\ |e| \leq h_E^{d-1} \leq h^{d-1}.$$

Also, we suppose that the subdivision is quasi-uniform, which means there exists a positive constant C such that

$$h \leq C h_E,\ \forall E \in \mathcal{E}_h.$$

In the end, we can introduce the broken Sobolev space

$$H^s(\mathcal{E}_h) = \{v \in L_2(\Omega) \mid \forall E \in \mathcal{E}_h,\ v|_E \in H^s(\Omega)\}$$

with the broken Sobolev norm $|||\cdot|||_{H^s(\mathcal{E}_h)}$ by

$$|||v|||_{H^s(\mathcal{E}_h)} = \left(\sum_{E \in \mathcal{E}_h} \|v\|_{H^s(E)} \right)^{1/2}.$$

As a result, we have the following facts

$$H^s(\Omega) \subset H^s(\mathcal{E}_h) \text{ and } H^{s+1}(\mathcal{E}_h) \subset H^s(\mathcal{E}_h)$$

Let Γ_h be the set of interior edges(2D) or faces(3D) of subdivision $\mathcal{E}_h$. Then for each edge or face element e, we have a unit normal vector $\underline{n}_e$. If $e \subset \partial\Omega$, $\underline{n}_e$ is the outward unit normal vector.

Definition Suppose two elements E_1^e and E_2^e share the common edge e and there is a function v on E_1^e and E_2^e. Then we define an average and a jump for v by

$$\{v\} = \frac{(v|_{E_1^e}) + (v|_{E_2^e})}{2}, \qquad [v] = (v|_{E_1^e}) - (v|_{E_2^e})$$

where the normal vector $\underline{n}_e$ is oriented from E_1^e to E_2^e. On the other hand, if $e \subset \partial\Omega$ and $e \subset \partial E_1^e$

$$\{v\} = [v] = (v|_{E_1^e}).$$

As following the above definition, we will introduce the jump operators of the function values and the derivatives values by

$$J_0^{\alpha_0,\beta_0}(v,w) = \sum_{e\subset\Gamma_h\cup\Gamma_D} \frac{\alpha_0}{|e|^{\beta_0}} \int_e [v][w]de,$$

where $\alpha_0 \in \mathbb{R}$ and β_0 is a positive constant depending on the dimension d.

Theorem 1.11. Inverse Polynomial Trace Theorem [39]
This theorem is an analogue of trace theorem with respect to each element E for polynomials. The trace inequalities are given by

$$\begin{aligned}
&\forall v \in \mathcal{P}_k(E),\ \forall e \subset \partial E,\ \|v\|_{L_2(e)} \leq C|e|^{1/2}|E|^{-1/2}\|v\|_{L_2(E)},\\
&\forall v \in \mathcal{P}_k(E),\ \forall e \subset \partial E,\ \|v\|_{L_2(e)} \leq Ch_E^{-1/2}\|v\|_{L_2(E)},\\
&\forall v \in \mathcal{P}_k(E),\ \forall e \subset \partial E,\ \|\nabla v\cdot \underline{n}_e\|_{L_2(e)} \leq C|e|^{1/2}|E|^{-1/2}\|\nabla v\|_{L_2(E)},\\
&\forall v \in \mathcal{P}_k(E),\ \forall e \subset \partial E,\ \|\nabla v\cdot \underline{n}_e\|_{L_2(e)} \leq Ch_E^{-1/2}\|\nabla v\|_{L_2(E)},
\end{aligned}$$

where C is a positive constant and is independent of h_E but depending on the polynomials degree k. It enables us to estimate trace norm of boundary values and boundary normal derivatives for polynomials with the measures of edge and element or the diameter. We shall use them to prove coercivity and continuity (with measures of edge and element) and stability/error analysis (with diameters).

Theorem 1.12. Poincaré's Inequality [35, 24]
In Theorem 1.5, *Poincaré-Freidrichs inequality is introduced for $H^1(\Omega)$. For piecewise H^1 functions, Poincaré-Freidrichs inequalities are given in* [35]. *Also, we can expand this inequality onto the broken Sobolev space $H^1(\mathcal{E}_h)$. A generalisation of the inequality is given by*

$$\forall v \in H^1(\mathcal{E}_h),\ \|v\|_{L_2(\Omega)} \leq C\left(|||\nabla v|||^2_{H^0(\mathcal{E}_h)} + \sum_{e\subset\Gamma_h\cup\Gamma_D}\frac{1}{|e|^{1/(d-1)}}\|[v]\|^2_{L_2(e)}\right)^{1/2}$$

for some positive C. If $\beta_0(d-1)\geq 1$ and $|e|\leq 1$ for $e\subset\Gamma_h\cup\Gamma_D$,

$$\forall v \in H^1(\mathcal{E}_h),\ \|v\|_{L_2(\Omega)} \leq C\left(|||\nabla v|||^2_{H^0(\mathcal{E}_h)} + \sum_{e\subset\Gamma_h\cup\Gamma_D}\frac{1}{|e|^{\beta_0}}\|[v]\|^2_{L_2(e)}\right)^{1/2}. \quad (1.4.10)$$

Theorem 1.13. Inverse Inequality(or Markov Inequality) [24, 56]
For any $E \in \mathcal{E}_h$, there is a positive constant C such that

$$\forall v \in \mathcal{P}_k(E),\ \left\|\nabla^j v\right\|_{L_2(E)} \leq Ch_E^{-j}\|v\|_{L_2(E)},\ \forall 0\leq j\leq k. \quad (1.4.11)$$

Remark We stated framework of DG and useful inequalities for *a priori* estimates. We did not give any example of DGFEM at the moment. DGFEM for elliptic problems has been studied in [24, 20]. In case of elasticity problems, DGFEM has been developed, see e.g. [16, 15]. More applications of DGFEM are seen in [24] and references therein.

Summary

Chapter 1 provides preliminary works, for example some notations, mathematical backgrounds, continuum mechanics, model problems and fundamental theory in finite element methods. We follow the given introduction as well as many previous research results in papers of finite element methods and/or viscoelasticity to solve linear viscoelastic problems in two ways; one is CGFEM and the other is DGFEM.

Chapter 2

CGFEM to Simplified Scalar Wave Equation with Memory

2.1 Model Problem

Hereafter, we will consider a simpler analogue of (1.3.17)-(1.3.21). Instead of dealing with the vector-valued problem, the scalar wave equation with memory is our topic in this chapter. Hence the model problem is given by
find $u : [0,T] \times \Omega \mapsto \mathbb{R}$ such that

$$\rho\ddot{u} - \nabla \cdot \underset{\sim}{\sigma} = f \quad \text{in} \quad (0,T] \times \Omega, \tag{2.1.1}$$

$$u = 0 \quad \text{on} \quad [0,T] \times \Gamma_D, \tag{2.1.2}$$

$$\underset{\sim}{\sigma} \cdot \underset{\sim}{n} = g_N \quad \text{on} \quad [0,T] \times \Gamma_N, \tag{2.1.3}$$

$$u = u_0 \quad \text{on} \quad \{0\} \times \Omega, \tag{2.1.4}$$

$$\dot{u} = w_0 \quad \text{on} \quad \{0\} \times \Omega, \tag{2.1.5}$$

where the domain and its boundary follow as before we assume that Ω is bounded and open, $\partial\Omega = \Gamma_D \cup \Gamma_N$, the measure of Γ_D is of positive, and Γ_D and Γ_N are disjoint. Furthermore, we suppose that ρ is a positive constant and $f \in C(0,T; L_2(\Omega))$ and $g_N \in C^1(0,T; L_2(\Gamma_N))$. Here, the strain tensor $\underset{\sim}{\varepsilon}$ becomes a gradient operator ∇. At last, the definition of $\underset{\sim}{\sigma}$ determines either the displacement form or the velocity form as shown in (1.3.8) and (1.3.9).

In fact, this scalar analogue represents the viscoelastic materials subjected to antiplane shear problem. Antiplane viscoelastic models in 3D reduce the vector-valued problems to scalar wave equations in 2D (see e.g. [44, 45, 46]). To be specific, a strain tensor is defined by Cauchy infinitesimal tensor as in (1.2.1). However, in case of antiplane problems, antiplane shear deformation leads the displacement vector to be defined by $\boldsymbol{u} = (0, 0, u)$ so that we have

$$\underset{\sim}{\varepsilon}(\boldsymbol{u}) = \begin{pmatrix} 0 & 0 & \frac{1}{2}u_{,1} \\ 0 & 0 & \frac{1}{2}u_{,2} \\ \frac{1}{2}u_{,1} & \frac{1}{2}u_{,2} & 0 \end{pmatrix}.$$

Thus, for any symmetric positive definite fourth order tensor $\underline{\boldsymbol{D}}$, we can express $\underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{u})$ by $D\nabla u$ where D is a matrix. In this manner, we can derive the above scalar model problem.

2.1.1 Displacement Form

Recall (1.3.7), $E(t) = E_0 + \sum_{i=1}^{N} E_i e^{-t/\tau_i}$. Let us define

$$E(t) = D\left(\varphi_0 + \sum_{q=1}^{N_\varphi} \varphi_q e^{-t/\tau_q}\right) = D\varphi(t)$$

where $D = \sum_{q=0}^{N_\varphi} E_q > 0$, $\varphi_q = E_q/D > 0$ for $q = 0, \ldots, N_\varphi$ and $\tau_q > 0$ for $q = 1, \ldots, N_\varphi$. Then we have

$$\varphi(0) = \varphi_0 + \sum_{q=1}^{N_\varphi} \varphi_q = 1, \;\; \varphi_0 = 1 - \sum_{q=1}^{N_\varphi} \varphi_q > 0,$$

and

$$\varphi_s(t-s) = \frac{\partial}{\partial s}\varphi(t-s) = \sum_{q=1}^{N_\varphi} \frac{\varphi_q}{\tau_q} e^{-(t-s)/\tau_q}.$$

Thus, (1.3.8) can be written as

$$\underline{\sigma}(t) = D\nabla u(t) - D\nabla \int_0^t \varphi_s(t-s)u(s)\;ds. \tag{2.1.6}$$

Set $\psi_q(t) := \frac{\varphi_q}{\tau_q}\int_0^t e^{-(t-s)/\tau_q}u(s)\;ds$ for each $q = 1, \ldots, N_\varphi$. Hence, the constitutive equation is governed with $\{\psi_q\}_{q=1}^{N_\varphi}$ by

$$\underline{\sigma}(t) = D\nabla\left(u(t) - \sum_{q=1}^{N_\varphi} \psi_q(t)\right). \tag{2.1.7}$$

By the definition of $\{\psi_q\}_{q=1}^{N_\varphi}$, we can derive the following ODE

$$\dot{\psi}_q(t) = \frac{\varphi_q}{\tau_q}u(t) - \frac{1}{\tau_q}\psi_q(t) \tag{2.1.8}$$

with the zero initial condition, $\forall q \in \{1, \ldots, N_\varphi\}$. From these above results, our model problem (2.1.1) and (2.1.3) can be rewritten as

$$\rho\ddot{u} - \nabla\cdot D\nabla\left(u - \sum_{q=1}^{N_\varphi} \psi_q\right) = f \quad \text{in} \quad (0,T]\times\Omega, \tag{2.1.9}$$

$$D\nabla\left(u-\sum_{q=1}^{N_\varphi}\psi_q\right)\cdot\underline{n}=g_N\quad\text{on}\quad[0,T]\times\Gamma_N,\tag{2.1.10}$$

respectively.

Now, we consider the variational formulation of (2.1.9) and so let us define the test space V such that

$$V=\left\{v\in H^1(\Omega)\mid v=0\ \text{ on }\Gamma_D\right\}.$$

Then multiplying $v\in V$ leads us to derive the following weak form,

$$(\rho\ddot{u}(t),v)_{L_2(\Omega)}+a(u(t),v)-\sum_{q=1}^{N_\varphi}a(\psi_q(t),v)=F_d(t;v),\tag{2.1.11}$$

where the symmetric bilinear form $a(\cdot,\cdot)$ and the linear form $F_d(\cdot)$ are defined by

$$\begin{aligned}a(w,v)&=(D\nabla w,\nabla v)_{L_2(\Omega)},\\F_d(t;v)&=(f(t),v)_{L_2(\Omega)}+(g_N(t),v)_{L_2(\Gamma_N)}.\end{aligned}$$

It is easy to check that (2.1.11) is the weak form of (2.1.1) by integration by parts.

$$\begin{aligned}-\int_\Omega\nabla\cdot D\nabla\left(u-\sum_{q=1}^{N_\varphi}\psi_q\right)v\,d\Omega&=\int_\Omega D\nabla\left(u-\sum_{q=1}^{N_\varphi}\psi_q\right)\cdot\nabla v\,d\Omega\\&\quad-\int_{\partial\Omega}\left(D\nabla\left(u-\sum_{q=1}^{N_\varphi}\psi_q\right)\cdot\underline{n}\right)v\,d\Gamma,\\&=\int_\Omega D\nabla\left(u-\sum_{q=1}^{N_\varphi}\psi_q\right)\cdot\nabla v\,d\Omega\\&\quad-\int_{\Gamma_D}\left(D\nabla\left(u-\sum_{q=1}^{N_\varphi}\psi_q\right)\cdot\underline{n}\right)v\,d\Gamma\\&\quad-\int_{\Gamma_N}\left(D\nabla\left(u-\sum_{q=1}^{N_\varphi}\psi_q\right)\cdot\underline{n}\right)v\,d\Gamma,\\&=\int_\Omega D\nabla\left(u-\sum_{q=1}^{N_\varphi}\psi_q\right)\cdot\nabla v\,d\Omega-\int_{\Gamma_N}g_Nv\,d\Gamma,\end{aligned}$$

since the boundary condition (2.1.10) is imposed, $\forall v\in V$. Thus, multiplying $v\in V$ on (2.1.9) and integrating it give (2.1.11). Furthermore, in a similar way, (2.1.8) implies

$$a(\tau_q\dot{\psi}_q(t)+\psi_q(t),v)=a(\varphi_qu(t),v)\tag{2.1.12}$$

for any $v \in V$ and $q = 1, \ldots, N_\varphi$. Therefore, we have the weak problem:

(P1) Find $u(t)$ and $\{\psi_q(t)\}_{q=1}^{N_\varphi}$ such that for all $v \in V$

$$(\rho \ddot{u}(t), v)_{L_2(\Omega)} + a(u(t), v) - \sum_{q=1}^{N_\varphi} a(\psi_q(t), v) = F_d(t; v),$$
$$\tau_q a(\dot{\psi}_q(t), v) + a(\psi_q(t), v) = \varphi_q a(u(t), v) \ \forall q \in \{1, \ldots, N_\varphi\},$$

with $u(0) = u_0$, $\dot{u}(0) = w_0$ and $\psi_q(0) = 0,\ \forall q \in \{1, \ldots, N_\varphi\}$.

Now, we shall consider *a priori* bound for **(P1)**. Due to Lemma 1.1, we have the norm equivalence between H^1 norm and the energy norm, defined by

$$\|v\|_V^2 = a(v, v),\ \forall v \in V.$$

In other words, $\forall v \in V$

$$\kappa \|v\|_{H^1(\Omega)}^2 \leq \|v\|_V^2 \leq D \|v\|_{H^1(\Omega)}^2 .$$

This result will be used to verify *a priori* bounds. To be specific, by Trace inequality and coercivity,

$$\|v\|_{L_2(\partial\Omega)} \leq C \|v\|_V . \tag{2.1.13}$$

Lemma 2.1. *Suppose the weak solution $u \in H^2(0, T; L_2(\Omega)) \cap H^1(0, T; V)$. Then for any $0 \leq t \leq T$*

$$\frac{\rho}{2} \|\dot{u}(t)\|_{L_2(\Omega)}^2 + \frac{1}{2} \|u(t)\|_V^2 = \frac{\rho}{2} \|w_0\|_{L_2(\Omega)}^2 + \frac{1}{2} \|u_0\|_V^2 + \int_0^t F_d(t'; \dot{u}(t'))dt'$$
$$+ \sum_{q=1}^{N_\varphi} \int_0^t a(\psi_q(t'), \dot{u}(t'))dt'.$$

Proof. Choosing $v = \dot{u}(t') \in V$ in (2.1.11) gives

$$\left(\rho \ddot{u}(t'), \dot{u}(t')\right)_{L_2(\Omega)} + a(u(t'), \dot{u}(t')) - \sum_{q=1}^{N_\varphi} a(\psi_q(t'), \dot{u}(t')) = F_d(t'; \dot{u}(t')).$$

Note that by Leibniz's integral rule, for any differentiable w we have

$$\begin{aligned}\frac{1}{2} \frac{d}{dt'} \left\|w(t')\right\|_{L_2(\Omega)}^2 &= \frac{1}{2} \frac{d}{dt'} \int_\Omega w^2(t')\ d\Omega,\\ &= \frac{1}{2} \int_\Omega \frac{\partial}{\partial t'}(w^2(t'))\ d\Omega,\\ &= \int_\Omega \dot{w}(t') w(t')\ d\Omega = \left(\dot{w}(t'), w(t')\right)_{L_2(\Omega)} .\end{aligned}$$

and similarly, $\frac{1}{2}\frac{d}{dt'}\left\|w(t')\right\|_V^2 = a(\dot{w}(t'), w(t'))$. Hence it yields

$$\frac{\rho}{2}\frac{d}{dt'}\left\|\dot{u}(t')\right\|_{L_2(\Omega)}^2 + \frac{1}{2}\frac{d}{dt'}\left\|u(t')\right\|_V^2 = F_d(t'; \dot{u}(t')) + \sum_{q=1}^{N_\varphi} a(\psi_q(t'), \dot{u}(t')).$$

Thus, from integration with respect to time from 0 to t where $t \in [0, T]$,

$$\begin{aligned}\frac{\rho}{2}\left\|\dot{u}(t)\right\|_{L_2(\Omega)}^2 + \frac{1}{2}\left\|u(t)\right\|_V^2 =& \frac{\rho}{2}\left\|w_0\right\|_{L_2(\Omega)}^2 + \frac{1}{2}\left\|u_0\right\|_V^2 + \int_0^t F_d(t'; \dot{u}(t'))dt' \\ &+ \sum_{q=1}^{N_\varphi}\int_0^t a(\psi_q(t'), \dot{u}(t'))dt'\end{aligned}$$

where we used the initial data as in (2.1.4) and (2.1.5). □

Lemma 2.2. *For any $q \in \{1, \ldots, N_\varphi\}$, assume that $\psi_q(t) \in H^1(0, T; V)$. Then for any $0 \leq t \leq T$,*

$$\int_0^t a(\psi_q(t'), \dot{u}(t'))dt' = a(u(t), \psi_q(t)) - \frac{1}{2\varphi_q}\left\|\psi_q(t)\right\|_V^2 - \frac{\tau_q}{\varphi_q}\int_0^t \left\|\dot{\psi}_q(t')\right\|_V^2 dt'.$$

Proof. Set $v = \dot{\psi}_q(t')$. Then (2.1.12) yields

$$\tau_q\left\|\dot{\psi}_q(t')\right\|_V^2 + a(\psi_q(t'), \dot{\psi}_q(t')) = \varphi_q a(u(t'), \dot{\psi}_q(t')),$$

and so

$$\tau_q\left\|\dot{\psi}_q(t')\right\|_V^2 + \frac{1}{2}\frac{d}{dt'}\left\|\psi_q(t')\right\|_V^2 = \varphi_q a(u(t'), \dot{\psi}_q(t')).$$

Integration by parts yields

$$\begin{aligned}\int_0^t \tau_q\left\|\dot{\psi}_q(t')\right\|_V^2 dt' + \frac{1}{2}\left(\left\|\psi_q(t)\right\|_V^2 - \left\|\psi_q(0)\right\|_V^2\right) =& \int_0^t \varphi_q a(u(t'), \dot{\psi}_q(t'))dt', \\ =& \varphi_q a(u(t), \psi_q(t)) - \varphi_q a(u(0), \psi_q(0)) \\ &- \int_0^t \varphi_q a(\dot{u}(t'), \psi_q(t'))dt'.\end{aligned}$$

Since $\psi_q(0) = 0$, we have

$$\int_0^t \tau_q\left\|\dot{\psi}_q(t')\right\|_V^2 dt' + \frac{1}{2}\left\|\psi_q(t)\right\|_V^2 = \varphi_q a(u(t), \psi_q(t)) - \int_0^t \varphi_q a(\dot{u}(t'), \psi_q(t'))dt',$$

and therefore,

$$\int_0^t a(\psi_q(t'), \dot{u}(t'))dt' = a(u(t), \psi_q(t)) - \frac{1}{2\varphi_q}\left\|\psi_q(t)\right\|_V^2 - \frac{\tau_q}{\varphi_q}\int_0^t \left\|\dot{\psi}_q(t')\right\|_V^2 dt.$$

□

Theorem 2.1. *If $u(t) \in H^2(0,T;L_2(\Omega)) \cap H^1(0,T;V)$, $\psi_q(t) \in H^1(0,T;V)$ $\forall q \in \{1,\dots,N_\varphi\}$, then we have the following stability bound: for any $t \in [0,T]$*

$$\begin{aligned}
&\frac{\rho}{2}\|\dot{u}(t)\|^2_{L_2(\Omega)} + \frac{\varphi_0}{8}\|u(t)\|^2_V + \sum_{q=1}^{N_\varphi}\frac{\varphi_0}{4\varphi_q^2 N_\varphi + 2\varphi_0\varphi_q}\|\psi_q(t)\|^2_V \\
&+ \sum_{q=1}^{N_\varphi}\frac{\tau_q}{\varphi_q}\int_0^t \left\|\dot{\psi}_q(t')\right\|^2_V dt' \\
\le & C\Bigg(\|w_0\|^2_{L_2(\Omega)} + \|u_0\|^2_V + \int_0^t \|f(t')\|^2_{L_2(\Omega)} dt' + \int_0^t \|\dot{g}_N(t')\|^2_{L_2(\Gamma_N)} dt' \\
&+ \|g_N(t)\|^2_{L_2(\Gamma_N)} + \|g_N(0)\|^2_{L_2(\Gamma_N)}\Bigg),
\end{aligned}$$

for some positive constant C which is independent of the weak solution but depends on the domain, its boundary and exponential of time.

Proof. By applying Lemma 2.2 into Lemma 2.1,

$$\begin{aligned}
&\frac{\rho}{2}\|\dot{u}(t)\|^2_{L_2(\Omega)} + \frac{1}{2}\|u(t)\|^2_V + \sum_{q=1}^{N_\varphi}\left(\frac{1}{2\varphi_q}\|\psi_q(t)\|^2_V + \frac{\tau_q}{\varphi_q}\int_0^T \left\|\dot{\psi}_q(t)\right\|^2_V dt\right) \\
=&\frac{\rho}{2}\|w_0\|^2_{L_2(\Omega)} + \frac{1}{2}\|u_0\|^2_V + \int_0^t F_d(\dot{u})dt' + \sum_{q=1}^{N_\varphi} a(u(t),\psi_q(t)).
\end{aligned} \tag{2.1.14}$$

First, observe $\int_0^t F_d(\dot{u})dt'$. From the definition,

$$\begin{aligned}
\int_0^t F_d(t;\dot{u}(t'))dt' =& \int_0^t \big(f(t'),\dot{u}(t')\big)_{L_2(\Omega)} + (g_N(t'),\dot{u}(t'))_{L_2(\Gamma_N)}dt' \\
=& \int_0^t \big(f(t'),\dot{u}(t')\big)_{L_2(\Omega)} dt' + (g_N(t'),u(t'))_{L_2(\Gamma_N)}\big|_{t'=0}^{t'=t} \\
&- \int_0^t (\dot{g}_N(t'),u(t'))_{L_2(\Gamma_N)}dt' \\
&\text{(by integration by parts)} \\
\le& \int_0^t \left\|f(t')\right\|_{L_2(\Omega)}\left\|\dot{u}(t')\right\|_{L_2(\Omega)} dt' + \|g_N(t)\|_{L_2(\Gamma_N)}\|u(t)\|_{L_2(\Gamma_N)} \\
&+ \|g_N(0)\|_{L_2(\Gamma_N)}\|u_0\|_{L_2(\Gamma_N)} + \int_0^t \left\|\dot{g}_N(t')\right\|_{L_2(\Gamma_N)}\left\|u(t')\right\|_{L_2(\Gamma_N)} dt' \\
&\text{(by Cauchy-Schwarz inequality)} \\
\le& \int_0^t \left\|f(t')\right\|_{L_2(\Omega)}\left\|\dot{u}(t')\right\|_{L_2(\Omega)} dt' + \|g_N(t)\|_{L_2(\Gamma_N)} C\|u(t)\|_V \\
&+ \|g_N(0)\|_{L_2(\Gamma_N)} C\|u_0\|_V + \int_0^t \left\|\dot{g}_N(t')\right\|_{L_2(\Gamma_N)} C\left\|u(t')\right\|_V dt'
\end{aligned}$$

$$
\begin{aligned}
&\text{(by (2.1.13))}\\
&\leq \int_0^t \left(\frac{1}{2} \left\| f(t') \right\|_{L_2(\Omega)}^2 + \frac{1}{2} \left\| \dot{u}(t') \right\|_{L_2(\Omega)}^2 \right) dt' + \frac{C}{2\epsilon} \left\| g_N(t) \right\|_{L_2(\Gamma_N)}^2 \\
&\quad + \frac{C\epsilon}{2} \left\| u(t) \right\|_V^2 + \frac{C}{2} \left\| g_N(0) \right\|_{L_2(\Gamma_N)}^2 + \frac{C}{2} \left\| u_0 \right\|_V^2 \\
&\quad + \frac{C}{2} \int_0^t \left\| \dot{g}_N(t') \right\|_{L_2(\Gamma_N)}^2 dt' + \frac{C}{2} \int_0^t \left\| u(t) \right\|_V^2 dt' \\
&\text{(by Young's inequality).}
\end{aligned}
$$

Take $\epsilon = \varphi_0/(4C) > 0$. Then

$$
\begin{aligned}
\int_0^t F_d(t'; \dot{u}(t'))dt' \leq &\frac{1}{2} \int_0^t \left\| f(t') \right\|_{L_2(\Omega)}^2 dt' + \frac{1}{2} \int_0^t \left\| \dot{u}(t') \right\|_{L_2(\Omega)}^2 dt' + \frac{4C^2}{2\varphi_0} \left\| g_N(t) \right\|_{L_2(\Gamma_N)}^2 \\
&+ \frac{\varphi_0}{8} \left\| u(t) \right\|_V^2 + \frac{C}{2} \left\| g_N(0) \right\|_{L_2(\Gamma_N)}^2 + \frac{C}{2} \left\| u_0 \right\|_V^2 \\
&+ \frac{C}{2} \int_0^t \left\| \dot{g}_N(t') \right\|_{L_2(\Gamma_N)}^2 dt' + \frac{C}{2} \int_0^t \left\| u(t') \right\|_V^2 dt'.
\end{aligned}
$$

On the other hand, in the same sense, Cauchy-Schwarz inequality and Young's inequality allow us to have

$$
\begin{aligned}
\sum_{q=1}^{N_\varphi} a(u(t), \psi_q(t)) &\leq \sum_{q=1}^{N_\varphi} \left\| u(t) \right\|_V \left\| \psi_q(t) \right\|_V \\
&\leq \sum_{q=1}^{N_\varphi} \frac{\epsilon_q}{2} \left\| u(t) \right\|_V^2 + \sum_{q=1}^{N_\varphi} \frac{1}{2\epsilon_q} \left\| \psi_q(t) \right\|_V^2 ,
\end{aligned}
$$

for positive constants $\{\epsilon_q\}$. Choose $\epsilon_q = \varphi_q + \frac{\varphi_0}{2N_\varphi} > 0$ for each q. Then

$$
\begin{aligned}
1 - \sum_{q=1}^{N_\varphi} \epsilon_q &= 1 - \left(\sum_{q=1}^{N_\varphi} \varphi_q + \frac{\varphi_0}{2} \right) \\
&= \varphi_0 - \frac{\varphi_0}{2} = \frac{\varphi_0}{2} > 0,
\end{aligned}
$$

since $\sum_{q=0}^{N_\varphi} \varphi_q = 1$. Also, for each q

$$
\begin{aligned}
\frac{1}{2\varphi_q} - \frac{1}{2\epsilon_q} &= \frac{1}{2\varphi_q} - \frac{1}{2\varphi_q + \varphi_0/N_\varphi} \\
&= \frac{\varphi_0/N_\varphi}{2\varphi_q(2\varphi_q + \varphi_0/N_\varphi)} \\
&= \frac{\varphi_0}{4\varphi_q^2 N_\varphi + 2\varphi_0\varphi_q} > 0.
\end{aligned}
$$

Consequently, (2.1.14) gives

$$
\begin{aligned}
&\frac{\rho}{2}\left\|\dot{u}(t)\right\|_{L_2(\Omega)}^2+\frac{1}{2}\left\|u(t)\right\|_V^2+\sum_{q=1}^{N_\varphi}\left(\frac{1}{2\varphi_q}\left\|\psi_q(t)\right\|_V^2+\frac{\tau_q}{\varphi_q}\int_0^t\left\|\dot{\psi}_q(t')\right\|_V^2 dt'\right)\\
\leq&\frac{\rho}{2}\left\|w_0\right\|_{L_2(\Omega)}^2+\frac{1}{2}\left\|u_0\right\|_V^2+\frac{1}{2}\int_0^t\left\|f(t')\right\|_{L_2(\Omega)}^2 dt'+\frac{1}{2}\int_0^t\left\|\dot{u}(t')\right\|_{L_2(\Omega)}^2 dt'\\
&+\frac{4C^2}{2\varphi_0}\left\|g_N(t)\right\|_{L_2(\Gamma_N)}^2+\frac{\varphi_0}{8}\left\|u(t)\right\|_V^2+\frac{C}{2}\left\|g_N(0)\right\|_{L_2(\Gamma_N)}^2+\frac{C}{2}\left\|u_0\right\|_V^2\\
&+\frac{C}{2}\int_0^t\left\|\dot{g}_N(t')\right\|_{L_2(\Gamma_N)}^2 dt'+\frac{C}{2}\int_0^t\left\|u(t')\right\|_V^2 dt'+\sum_{q=1}^{N_\varphi}\frac{\epsilon_q}{2}\left\|u(t)\right\|_V^2+\sum_{q=1}^{N_\varphi}\frac{1}{2\epsilon_q}\left\|\psi_q(t)\right\|_V^2,
\end{aligned}
$$

hence our choice of $\{\epsilon_q\}$ implies

$$
\begin{aligned}
&\frac{\rho}{2}\left\|\dot{u}(t)\right\|_{L_2(\Omega)}^2+\frac{\varphi_0}{8}\left\|u(t)\right\|_V^2+\sum_{q=1}^{N_\varphi}\frac{\varphi_0}{4\varphi_q^2N_\varphi+2\varphi_0\varphi_q}\left\|\psi_q(t)\right\|_V^2\\
&+\sum_{q=1}^{N_\varphi}\frac{\tau_q}{\varphi_q}\int_0^t\left\|\dot{\psi}_q(t')\right\|_V^2 dt',\\
\leq&\frac{\rho}{2}\left\|w_0\right\|_{L_2(\Omega)}^2+\frac{1}{2}\left\|u_0\right\|_V^2+\frac{1}{2}\int_0^t\left\|f(t')\right\|_{L_2(\Omega)}^2 dt'+\frac{1}{2}\int_0^t\left\|\dot{u}(t')\right\|_{L_2(\Omega)}^2 dt'\\
&+\frac{2C^2}{\varphi_0}\left\|g_N(t)\right\|_{L_2(\Gamma_N)}^2+\frac{C}{2}\left\|g_N(0)\right\|_{L_2(\Gamma_N)}^2+\frac{C}{2}\left\|u_0\right\|_V^2\\
&+\frac{C}{2}\int_0^t\left\|\dot{g}_N(t')\right\|_{L_2(\Gamma_N)}^2 dt'+\frac{C}{2}\int_0^t\left\|u(t')\right\|_V^2 dt'.
\end{aligned}
$$

Finally, Grönwall's inequality with respect to $\|\dot{u}(t')\|_{L_2(\Omega)}^2$ and $\|u(t')\|_V^2$ implies

$$
\begin{aligned}
&\frac{\rho}{2}\left\|\dot{u}(t)\right\|_{L_2(\Omega)}^2+\frac{\varphi_0}{8}\left\|u(t)\right\|_V^2+\sum_{q=1}^{N_\varphi}\frac{\varphi_0}{4\varphi_q^2N_\varphi+2\varphi_0\varphi_q}\left\|\psi_q(t)\right\|_V^2\\
&+\sum_{q=1}^{N_\varphi}\frac{\tau_q}{\varphi_q}\int_0^t\left\|\dot{\psi}_q(t')\right\|_V^2 dt'\\
\leq&C\Bigg(\left\|w_0\right\|_{L_2(\Omega)}^2+\left\|u_0\right\|_V^2+\int_0^t\left\|f(t')\right\|_{L_2(\Omega)}^2 dt'+\int_0^t\left\|\dot{g}_N(t')\right\|_{L_2(\Gamma_N)}^2 dt'\\
&+\left\|g_N(t)\right\|_{L_2(\Gamma_N)}^2+\left\|g_N(0)\right\|_{L_2(\Gamma_N)}^2\Bigg),
\end{aligned}
$$

for some positive C. Indeed, this C increases exponentially in time but is independent of solutions. □

Theorem 2.1 states the boundedness of solutions by initial conditions, boundary conditions and source terms. However, the bound constant C in the theorem exponentially grows in time by Grönwall's inequality. It is understood that for large final time T, the stability becomes meaningless in practice. However, instead of L_2 estimation in time, it is also able to obtain other stability bounds based on L_∞ norm in time. Let us define the norm for $v \in L_\infty(0,T;V)$

$$\|v\|_{L_\infty(0,T;V)} = \text{ess} \sup_{0\le t\le T} \|v\|_V .$$

Therefore, we will prove the stability bound without using Grönwall's inequality so that we have a non-exponentially growing bound constant C in time.

Theorem 2.2. *Suppose $u \in W^1_\infty(0,T;L_2(\Omega)) \cap L_\infty(0,T;V)$, $\psi_q \in H^1(0,T;V)$, $\forall q \in \{1,\dots,N_\varphi\}$. Then we have the stability bound for any $t \in [0,T]$, as*

$$\begin{aligned}
&\frac{\rho}{4}\|\dot{u}\|^2_{L_\infty(0,T;L_2(\Omega))} + \frac{\varphi_0}{8}\|u\|^2_{L_\infty(0,T;V)} + \sum_{q=1}^{N_\varphi}\frac{\varphi_0}{4\varphi_q^2 N_\varphi + 2\varphi_0\varphi_q}\|\psi_q(t)\|_V^2 \\
&\quad + \sum_{q=1}^{N_\varphi}\frac{\tau_q}{\varphi_q}\int_0^t \left\|\dot{\psi}_q(t')\right\|_V^2 dt' \\
&\le C\Bigg(\|w_0\|^2_{L_2(\Omega)} + \|u_0\|^2_V + \|f\|^2_{L_2(0,T;L_2(\Omega))} + \|g_N\|^2_{L_\infty(0,T;L_2(\Gamma_N))} \\
&\quad + \|\dot{g}_N\|^2_{L_2(0,T;L_2(\Gamma_N))}\Bigg),
\end{aligned}$$

where C is a positive constant independent of u and $\{\psi_q\}_{q=1}^{N_\varphi}$ but depending on the final time T. C is increasing in time but not exponentially.

Proof. Recall the proof of Theorem 2.1. In a similar way with (2.1.14), for $0 \le t \le T$

$$\begin{aligned}
&\frac{\rho}{2}\|\dot{u}(t)\|^2_{L_2(\Omega)} + \frac{1}{2}\|u(t)\|_V^2 + \sum_{q=1}^{N_\varphi}\left(\frac{1}{2\varphi_q}\|\psi_q(t)\|_V^2 + \frac{\tau_q}{\varphi_q}\int_0^t\left\|\dot{\psi}_q(t')\right\|_V^2 dt'\right) \\
&= \frac{\rho}{2}\|w_0\|^2_{L_2(\Omega)} + \frac{1}{2}\|u_0\|_V^2 + \int_0^t F_d(t';\dot{u}(t'))dt' + \sum_{q=1}^{N_\varphi} a(u(t),\psi_q(t)). \qquad (2.1.15)
\end{aligned}$$

On the other hand we can observe for any $t \in [0,T]$,

$$\begin{aligned}
\int_0^t F_d(t';\dot{u}(t'))dt' \le& \int_0^t \left\|f(t')\right\|_{L_2(\Omega)}\left\|\dot{u}(t')\right\|_{L_2(\Omega)} dt' + \frac{C}{2\epsilon}\|g_N(t)\|^2_{L_2(\Gamma_N)} \\
&+ \frac{C\epsilon}{2}\|u(t)\|_V^2 + \frac{C}{2}\|g_N(0)\|^2_{L_2(\Gamma_N)} + \frac{C}{2}\|u_0\|_V^2 \\
&+ \int_0^t \left\|\dot{g}_N(t')\right\|_{L_2(\Gamma_N)} C\left\|u(t')\right\|_V dt'
\end{aligned}$$

$$
\begin{aligned}
&\text{(as follows the proof of Theorem 2.1 with } \epsilon > 0 \\
&\ \text{and a positive constant } C \text{ from (2.1.13))} \\
&\le \int_0^T \left\| f(t') \right\|_{L_2(\Omega)} \left\| \dot{u}(t') \right\|_{L_2(\Omega)} dt' + \frac{C}{2\epsilon} \left\| g_N(t) \right\|^2_{L_2(\Gamma_N)} \\
&\quad + \frac{C\epsilon}{2} \left\| u(t) \right\|^2_V + \frac{C}{2} \left\| g_N(0) \right\|^2_{L_2(\Gamma_N)} + \frac{C}{2} \left\| u_0 \right\|^2_V \\
&\quad + \int_0^T \left\| \dot{g}_N(t') \right\|_{L_2(\Gamma_N)} C \left\| u(t') \right\|_V dt' \\
&\text{(since } 0 \le t \le T) \\
&\le \left\| \dot{u} \right\|_{L_\infty(0,T;L_2(\Omega))} \int_0^T \left\| f(t') \right\|_{L_2(\Omega)} dt' + \frac{C}{2\epsilon} \left\| g_N \right\|^2_{L_\infty(0,T;L_2(\Gamma_N))} \\
&\quad + \frac{C\epsilon}{2} \left\| u \right\|^2_{L_\infty(0,T;V)} + \frac{C}{2} \left\| g_N \right\|^2_{L_\infty(0,T;L_2(\Gamma_N))} + \frac{C}{2} \left\| u_0 \right\|^2_V \\
&\quad + C \left\| u \right\|_{L_\infty(0,T;V)} \int_0^T \left\| \dot{g}_N(t') \right\|_{L_2(\Gamma_N)} dt' \\
&\text{(by definition of the } L_\infty \text{ norm)} \\
&\le \frac{\epsilon_a}{2} \left\| \dot{u} \right\|^2_{L_\infty(0,T;L_2(\Omega))} + \frac{1}{2\epsilon_a} \left(\int_0^T \left\| f(t') \right\|_{L_2(\Omega)} dt' \right)^2 + \frac{C}{2} \left\| u_0 \right\|^2_V \\
&\quad + \frac{C}{2} \left\| g_N \right\|^2_{L_\infty(0,T;L_2(\Gamma_N))} + \frac{C\epsilon}{2} \left\| u \right\|^2_{L_\infty(0,T;V)} + \frac{C}{2\epsilon} \left\| g_N \right\|^2_{L_\infty(0,T;L_2(\Gamma_N))} \\
&\quad + \frac{C\epsilon_b}{2} \left\| u \right\|^2_{L_\infty(0,T;V)} + \frac{C}{2\epsilon_b} \left(\int_0^T \left\| \dot{g}_N(t') \right\|_{L_2(\Gamma_N)} dt' \right)^2 \\
&\text{(by Young's inequality for positive } \epsilon_a \text{ and } \epsilon_b) \\
&\le \frac{\epsilon_a}{2} \left\| \dot{u} \right\|^2_{L_\infty(0,T;L_2(\Omega))} + \frac{T}{2\epsilon_a} \left\| f \right\|^2_{L_2(0,T;L_2(\Omega))} + \frac{C}{2} \left\| u_0 \right\|^2_V \\
&\quad + \frac{C}{2} \left\| g_N \right\|^2_{L_\infty(0,T;L_2(\Gamma_N))} + \frac{C\epsilon}{2} \left\| u \right\|^2_{L_\infty(0,T;V)} + \frac{C}{2\epsilon} \left\| g_N \right\|^2_{L_\infty(0,T;L_2(\Gamma_N))} \\
&\quad + \frac{C\epsilon_b}{2} \left\| u \right\|^2_{L_\infty(0,T;V)} + \frac{CT}{2\epsilon_b} \left\| \dot{g}_N \right\|_{L_2(0,T;L_2(\Gamma_N))}, \qquad (2.1.16) \\
&\text{(by Cauchy-Schwarz inequality).}
\end{aligned}
$$

Note that we also have

$$
\sum_{q=1}^{N_\varphi} a(u(t), \psi_q(t)) \le \sum_{q=1}^{N_\varphi} \frac{\epsilon_q}{2} \left\| u(t) \right\|^2_V + \sum_{q=1}^{N_\varphi} \frac{1}{2\epsilon_q} \left\| \psi_q(t) \right\|^2_V, \qquad (2.1.17)
$$

with $\epsilon_q = \varphi_q + \frac{\varphi_0}{2N_\varphi} > 0$ for each q, $\forall t \in [0,T]$. Hence, the choice $\{\epsilon_q\}_{q=1}^{N_\varphi}$ and combining (2.1.16) and (2.1.17) in (2.1.15) lead us to have

$$
\frac{\rho}{2} \left\| \dot{u}(t) \right\|^2_{L_2(\Omega)} + \frac{\varphi_0}{4} \left\| u(t) \right\|^2_V + \sum_{q=1}^{N_\varphi} \frac{\varphi_0}{4\varphi_q^2 N_\varphi + 2\varphi_0\varphi_q} \left\| \psi_q(t) \right\|^2_V + \sum_{q=1}^{N_\varphi} \frac{\tau_q}{\varphi_q} \int_0^t \left\| \dot{\psi}_q(t') \right\|^2_V dt',
$$

$$
\begin{aligned}
\leq &\frac{\rho}{2}\left\|w_0\right\|_{L_2(\Omega)}^2+\frac{1}{2}\left\|u_0\right\|_V^2+\frac{\epsilon_a}{2}\|\dot{u}\|_{L_\infty(0,T;L_2(\Omega))}^2+\frac{T}{2\epsilon_a}\|f\|_{L_2(0,T;L_2(\Omega))}^2 \\
&+\frac{C\epsilon}{2}\|u\|_{L_\infty(0,T;V)}^2+\frac{C}{2}\left(1+\frac{1}{\epsilon}\right)\left\|g_N\right\|_{L_\infty(0,T;L_2(\Gamma_N))}^2+\frac{C}{2}\left\|u_0\right\|_V^2 \\
&+\frac{C\epsilon_b}{2}\|u\|_{L_\infty(0,T;V)}^2+\frac{CT}{2\epsilon_b}\left\|\dot{g}_N\right\|_{L_2(0,T;L_2(\Gamma_N))}^2 .
\end{aligned}
$$

If we consider essential supremum with respect to t, then

$$
\begin{aligned}
\operatorname*{ess\ sup}_{0\leq t\leq T} &\left\{\frac{\rho}{2}\|\dot{u}(t)\|_{L_2(\Omega)}^2+\frac{\varphi_0}{4}\|u(t)\|_V^2\right\} \\
&+\sum_{q=1}^{N_\varphi}\frac{\varphi_0}{4\varphi_q^2N_\varphi+2\varphi_0\varphi_q}\left\|\psi_q(t)\right\|_V^2+\sum_{q=1}^{N_\varphi}\frac{\tau_q}{\varphi_q}\int_0^t\left\|\dot{\psi}_q(t')\right\|_V^2dt' \\
\leq &\frac{\rho}{2}\left\|w_0\right\|_{L_2(\Omega)}^2+\frac{1}{2}\left\|u_0\right\|_V^2+\frac{\epsilon_a}{2}\|\dot{u}\|_{L_\infty(0,T;L_2(\Omega))}^2+\frac{T}{2\epsilon_a}\|f\|_{L_2(0,T;L_2(\Omega))}^2 \\
&+\frac{C\epsilon}{2}\|u\|_{L_\infty(0,T;V)}^2+\frac{C}{2}\left(1+\frac{1}{\epsilon}\right)\left\|g_N\right\|_{L_\infty(0,T;L_2(\Gamma_N))}^2+\frac{C}{2}\left\|u_0\right\|_V^2 \\
&+\frac{C\epsilon_b}{2}\|u\|_{L_\infty(0,T;V)}^2+\frac{CT}{2\epsilon_b}\left\|\dot{g}_N\right\|_{L_2(0,T;L_2(\Gamma_N))}^2 ,
\end{aligned}
$$

since the right hand side is independent of t. Note that for any non-negative $f(t)$ and $g(t)$,

$$
\operatorname{ess\,sup}\{f(t)\}\leq\operatorname{ess\,sup}\{f(t)+g(t)\},
$$

So if $\operatorname{ess\,sup}\{f(t)+g(t)\}$ is bounded, so is $\operatorname{ess\,sup}\{f(t)\}$ by that of upper bounds. Turning to the proof, by the property of essential supremum and the definition of L_∞ norm in time, it is seen that for any $t\in[0,T]$

$$
\begin{aligned}
&\frac{\rho}{2}\|\dot{u}\|_{L_\infty(0,T;L_2(\Omega))}^2+\frac{\varphi_0}{4}\|u\|_{L_\infty(0,T;V)}^2+\sum_{q=1}^{N_\varphi}\frac{\varphi_0}{4\varphi_q^2N_\varphi+2\varphi_0\varphi_q}\left\|\psi_q(t)\right\|_V^2 \\
&\quad+\sum_{q=1}^{N_\varphi}\frac{\tau_q}{\varphi_q}\int_0^t\left\|\dot{\psi}_q(t')\right\|_V^2dt' \\
&\leq 3\Bigg(\frac{\rho}{2}\left\|w_0\right\|_{L_2(\Omega)}^2+\frac{1}{2}\left\|u_0\right\|_V^2+\frac{\epsilon_a}{2}\|\dot{u}\|_{L_\infty(0,T;L_2(\Omega))}^2+\frac{T}{2\epsilon_a}\|f\|_{L_2(0,T;L_2(\Omega))}^2 \\
&\quad+\frac{C\epsilon}{2}\|u\|_{L_\infty(0,T;V)}^2+\frac{C}{2}\left(1+\frac{1}{\epsilon}\right)\left\|g_N\right\|_{L_\infty(0,T;L_2(\Gamma_N))}^2+\frac{C}{2}\left\|u_0\right\|_V^2 \\
&\quad+\frac{C\epsilon_b}{2}\|u\|_{L_\infty(0,T;V)}^2+\frac{CT}{2\epsilon_b}\left\|\dot{g}_N\right\|_{L_2(0,T;L_2(\Gamma_N))}^2\Bigg).
\end{aligned}
$$

In the end, if we set $\epsilon_a=\rho/6>0$ and $\epsilon=\epsilon_b=\varphi_0/(24C)>0$,

$$
\frac{\rho}{4}\|\dot{u}\|_{L_\infty(0,T;L_2(\Omega))}^2+\frac{\varphi_0}{8}\|u\|_{L_\infty(0,T;V)}^2+\sum_{q=1}^{N_\varphi}\frac{\varphi_0}{4\varphi_q^2N_\varphi+2\varphi_0\varphi_q}\left\|\psi_q(t)\right\|_V^2
$$

$$+\sum_{q=1}^{N_\varphi}\frac{\tau_q}{\varphi_q}\int_0^t \left\|\dot\psi_q(t')\right\|_V^2 dt'$$

$$\le\frac{3\rho}{2}\|w_0\|^2_{L_2(\Omega)}+\frac{3(1+C)}{2}\|u_0\|_V^2+\frac{6T}{\rho}\|f\|^2_{L_2(0,T;L_2(\Omega))}$$

$$+\frac{3C}{2}\left(1+\frac{48C}{\varphi_0}\right)\|g_N\|^2_{L_\infty(0,T;L_2(\Gamma_N))}+\frac{36C^2T}{\varphi_0}\|\dot g_N\|^2_{L_2(0,T;L_2(\Gamma_N))}.$$

Therefore, there exists positive C such that

$$\frac{\rho}{4}\|\dot u\|^2_{L_\infty(0,T;L_2(\Omega))}+\frac{\varphi_0}{8}\|u\|^2_{L_\infty(0,T;V)}+\sum_{q=1}^{N_\varphi}\frac{\varphi_0}{4\varphi_q^2N_\varphi+2\varphi_0\varphi_q}\|\psi_q(t)\|_V^2$$

$$+\sum_{q=1}^{N_\varphi}\frac{\tau_q}{\varphi_q}\int_0^t \left\|\dot\psi_q(t')\right\|_V^2 dt',$$

$$\le C\Bigg(\|w_0\|^2_{L_2(\Omega)}+\|u_0\|_V^2+\|f\|^2_{L_2(0,T;L_2(\Omega))}+\|g_N\|^2_{L_\infty(0,T;L_2(\Gamma_N))}$$

$$+\|\dot g_N\|^2_{L_2(0,T;L_2(\Gamma_N))}\Bigg).$$

□

In Theorem 2.1 and 2.2, we can observe the stability bounds for the weak formulation **(P1)**. However, we can derive an alternative form of internal variables so we are going to define the velocity form and take into account its stability bounds.

2.1.2 Velocity Form

Recall $\{\psi_q\}_{q=1}^{N_\varphi}$. Note that

$$\begin{aligned}\psi_q(t)&=\frac{\varphi_q}{\tau_q}\int_0^t e^{-(t-t')/\tau_q}u(t')\,dt'\\&=\frac{\varphi_q}{\tau_q}\left(\tau_q u(t)-\tau_q e^{-t/\tau_q}u_0-\tau_q\int_0^t e^{-(t-t')/\tau_q}\dot u(t')\,dt'\right)\\&=\varphi_q\left(u(t)-e^{-t/\tau_q}u_0-\int_0^t e^{-(t-t')/\tau_q}\dot u(t')\,dt'\right)\end{aligned}$$

by integration by parts then we define

$$\zeta_q(t)=\int_0^t\varphi_q e^{-(t-t')/\tau_q}\dot u(t')\,dt'$$

for each $q=1,\ldots,N_\varphi$ so we can have

$$\dot\zeta_q(t)=\varphi_q\dot u(t)-\frac{1}{\tau_q}\int_0^t\varphi_q e^{-(t-t')/\tau_q}\dot u(t')\,dt'$$

$$= \varphi_q \dot{u}(t) - \frac{1}{\tau_q}\zeta_q(t) \tag{2.1.18}$$

with $\zeta_q(0) = 0$.

Since $\psi_q(t) = \varphi_q u(t) - \varphi_q e^{-t/\tau_q} u_0 - \zeta_q(t)$ and

$$\sum_{q=0}^{N_\varphi} \varphi_q = \varphi_0 + \sum_{q=1}^{N_\varphi} \varphi_q = 1,$$

(2.1.9) can be replaced by

$$\begin{aligned}
\rho\ddot{u}(t) - \nabla \cdot D\nabla \left(u(t) - \sum_{q=1}^{N_\varphi} \psi_q(t) \right) =& \rho\ddot{u}(t) - \nabla \cdot D\nabla u(t) \\
&+ \nabla \cdot D\nabla \left(\sum_{q=1}^{N_\varphi} (\varphi_q u(t) - \varphi_q e^{-t/\tau_q} u_0 - \zeta_q(t)) \right) \\
=& \rho\ddot{u}(t) - \nabla \cdot D\nabla \left(1 - \sum_{q=1}^{N_\varphi} \varphi_q \right) u(t) \\
&- \nabla \cdot D\nabla \left(\sum_{q=1}^{N_\varphi} (\varphi_q e^{-t/\tau_q} u_0 + \zeta_q(t)) \right) \\
=& \rho\ddot{u}(t) - \nabla \cdot D\nabla \varphi_0 u(t) \\
&- \nabla \cdot D\nabla \left(\sum_{q=1}^{N_\varphi} (\varphi_q e^{-t/\tau_q} u_0 + \zeta_q(t)) \right) \\
=& f(t)
\end{aligned}$$

and (2.1.7) yields

$$D\nabla \left(\varphi_0 u(t) + \sum_{q=1}^{N_\varphi} \zeta_q(t) + \sum_{q=1}^{N_\varphi} \varphi_q e^{-t/\tau_q} u_0 \right) \cdot \underline{n} = g_N(t) \text{ on } \Gamma_N. \tag{2.1.19}$$

Thus we have the alternative weak formulation

$$(\rho\ddot{u}(t), v)_{L_2(\Omega)} + \varphi_0 a(u(t), v) + \sum_{q=1}^{N_\varphi} a(\zeta_q(t), v) = F_v(t; v) \tag{2.1.20}$$

for all $v \in V$ where

$$\begin{aligned}
a(w, v) =& (D\nabla w, \nabla v)_{L_2(\Omega)}, \\
F_v(t; v) =& (f(t), v)_{L_2(\Omega)} - \sum_{q=1}^{N_\varphi} \varphi_q e^{-t/\tau_q} a(u_0, v) + (g_N(t), v)_{L_2(\Gamma_N)},
\end{aligned}$$

but also we have for each $q = 1, \ldots, N_\varphi$

$$a(\tau_q \dot{\zeta}_q(t) + \zeta_q(t), v) = a(\tau_q \varphi_q \dot{u}(t), v). \tag{2.1.21}$$

Consequently, we obtain the weak problem of velocity form:

(P2) Find u and $\{\zeta_q\}_{q=1}^{N_\varphi}$ such that for all $v \in V$

$$(\rho \ddot{u}(t), v)_{L_2(\Omega)} + \varphi_0 a(u(t), v) + \sum_{q=1}^{N_\varphi} a(\zeta_q(t), v) = F_v(t; v),$$
$$\tau_q a(\dot{\zeta}_q(t), v) + a(\zeta_q(t), v) = \tau_q \varphi_q a(\dot{u}(t), v),\ \forall q = 1, \ldots, N_\varphi,$$

with $u(0) = u_0$, $\dot{u}(0) = w_0$ and $\zeta_q(0) = 0,\ \forall q = 1, \ldots, N_\varphi$.

In a similar way with Theorem 2.1, it is able to observe the stability bounds for **(P2)**. In other words, a weak solution for **(P2)** is bounded by given data such as initial conditions and boundary conditions.

Lemma 2.3. *Suppose the weak solution* $u \in H^2(0, T; L_2(\Omega)) \cap H^1(0, T; V)$. *Then it holds*

$$\begin{aligned}
&\frac{\rho}{2} \|\dot{u}(t)\|^2_{L_2(\Omega)} + \frac{\varphi_0}{2} \|u(t)\|^2_V + \sum_{q=1}^{N_\varphi} \int_0^t a(\zeta_q(t'), \dot{u}(t'))dt' \\
=&\frac{\rho}{2} \|w_0\|^2_{L_2(\Omega)} + \frac{\varphi_0}{2} \|u_0\|^2_V + \int_0^t F_v(t'; \dot{u}(t'))dt'
\end{aligned}$$

for any $t \in [0, T]$.

Proof. Put $v = \dot{u}(t')$ into (2.1.20). Then we have

$$\frac{\rho}{2} \frac{d}{dt'} \|\dot{u}(t')\|^2_{L_2(\Omega)} + \frac{\varphi_0}{2} \frac{d}{dt'} \|u(t')\|^2_V + \sum_{q=1}^{N_\varphi} a(\zeta_q(t'), \dot{u}(t')) = F_v(t'; \dot{u}(t')).$$

Integrating the equation from 0 to t with respect to time, and using the initial condition,

$$\begin{aligned}
&\frac{\rho}{2} \|\dot{u}(t)\|^2_{L_2(\Omega)} + \frac{\varphi_0}{2} \|u(t)\|^2_V + \sum_{q=1}^{N_\varphi} \int_0^t a(\zeta_q(t'), \dot{u}(t'))dt' \\
=&\frac{\rho}{2} \|w_0\|^2_{L_2(\Omega)} + \frac{\varphi_0}{2} \|u_0\|^2_V + \int_0^t F_v(t'; \dot{u}(t'))dt'.
\end{aligned}$$

□

Lemma 2.4. *For any* $q \in \{1, \ldots, N_\varphi\}$, *assume that* $\zeta_q \in H^1(0, T; V)$. *Then we have*

$$\sum_{q=1}^{N_\varphi} \int_0^t a(\zeta_q(t'), \dot{u}(t'))dt' = \sum_{q=1}^{N_\varphi} \frac{1}{2\varphi_q} \|\zeta_q(t)\|^2_V + \sum_{q=1}^{N_\varphi} \int_0^t \frac{1}{\tau_q \varphi_q} \|\zeta_q(t')\|^2_V dt'$$

for $t \in [0, T]$.

Proof. For each q, set $v = \zeta_q(t')$ and substitute it into (2.1.21), then it yields

$$\frac{\tau_q}{2}\frac{d}{dt'}\|\zeta_q(t')\|_V^2 + \|\zeta_q(t')\|_V^2 = \tau_q\varphi_q a(\dot{u}(t'), \zeta_q(t')).$$

Taking into account the integration of this equation, by zero initial condition,

$$\frac{\tau_q}{2}\|\zeta_q(t)\|_V^2 + \int_0^t \|\zeta_q(t')\|_V^2 dt' = \int_0^t \tau_q\varphi_q a(\dot{u}_h(t'), \zeta_q(t'))dt',$$
$$\Rightarrow \sum_{q=1}^{N_\varphi}\int_0^t a(\zeta_q(t'), \dot{u}_h(t'))dt' = \sum_{q=1}^{N_\varphi}\left(\frac{1}{2\varphi_q}\|\zeta_q(t)\|_V^2 + \int_0^t \frac{1}{\tau_q\varphi_q}\|\zeta_q(t')\|_V^2 dt'\right).$$

□

Theorem 2.3. *Assume $u \in H^2(0;T;L_2(\Omega)) \cap H^1(0,T;V)$ and $\zeta_q \in H^1(0,T;V)$, $\forall q \in \{1,\ldots,N_\varphi\}$. Then a weak solution to* **(P2)** *has the stability bound such that for some positive C*

$$\frac{\rho}{2}\|\dot{u}(t)\|_{L_2(\Omega)}^2 + \frac{\varphi_0}{4}\|u(t)\|_V^2 + \sum_{q=1}^{N_\varphi}\frac{1}{2\varphi_q}\|\zeta_q(t)\|_V^2 + \sum_{q=1}^{N_\varphi}\int_0^t \frac{1}{\tau_q\varphi_q}\|\zeta_q(t')\|_V^2 dt'$$
$$\leq C\bigg(\|w_0\|_{L_2(\Omega)}^2 + \|u_0\|_V^2 + \int_0^t \|f(t')\|_{L_2(\Omega)}^2 dt' + \int_0^t \|\dot{g}_N(t')\|_{L_2(\Gamma_N)}^2 dt'$$
$$+ \|g_N(0)\|_{L_2(\Gamma_N)}^2 + \|g_N(t)\|_{L_2(\Gamma_N)}^2\bigg)$$

for any $t \in [0,T]$.

Proof. From Lemma 2.3 and 2.4, we have the equality such that

$$\frac{\rho}{2}\|\dot{u}(t)\|_{L_2(\Omega)}^2 + \frac{\varphi_0}{2}\|u(t)\|_V^2 + \sum_{q=1}^{N_\varphi}\frac{1}{2\varphi_q}\|\zeta_q(t)\|_V^2 + \sum_{q=1}^{N_\varphi}\int_0^t \frac{1}{\tau_q\varphi_q}\|\zeta_q(t')\|_V^2 dt'$$
$$= \frac{\rho}{2}\|w_0\|_{L_2(\Omega)}^2 + \frac{\varphi_0}{2}\|u_0\|_V^2 + \int_0^t F_v(t'; \dot{u}(t'))dt'. \quad (2.1.22)$$

Recall the definition of F_v. Then,

$$\int_0^t F_v(t'; \dot{u}(t'))dt' = \int_0^t \Big[\big(f(t'), \dot{u}(t')\big)_{L_2(\Omega)} + (g_N(t'), \dot{u}(t'))_{L_2(\Gamma_N)}$$
$$- \sum_{q=1}^{N_\varphi}\varphi_q e^{-t'/\tau_q} a(u_0, \dot{u}(t'))\Big]dt'$$
$$= \int_0^t \big(f(t'), \dot{u}(t')\big)_{L_2(\Omega)} dt' + (g_N(t'), u(t'))_{L_2(\Gamma_N)}\big|_{t'=0}^{t'=t}$$

$$
\begin{aligned}
&- \int_0^t (\dot{g}_N(t'), u(t'))_{L_2(\Gamma_N)} dt' \\
&- \sum_{q=1}^{N_\varphi} \varphi_q e^{-t'/\tau_q} a(u_0, u)|_{t'=0}^{t'=t} - \int_0^t \sum_{q=1}^{N_\varphi} \frac{\varphi_q}{\tau_q} e^{-t'/\tau_q} a(u_0, u(t')) dt' \\
&\text{(by integration by parts),} \\
\leq &\int_0^t \left\| f(t') \right\|_{L_2(\Omega)} \left\| \dot{u}(t') \right\|_{L_2(\Omega)} dt' + \int_0^t \left\| \dot{g}_N(t') \right\|_{L_2(\Gamma_N)} \left\| u(t') \right\|_{L_2(\Gamma_N)} dt' \\
&+ \int_0^t \sum_{q=1}^{N_\varphi} \frac{\varphi_q}{\tau_q} e^{-t'/\tau_q} \left\| u_0 \right\|_V \left\| u(t') \right\|_V dt' + \sum_{q=1}^{N_\varphi} \varphi_q e^{-t/\tau_q} \left\| u_0 \right\|_V \left\| u(t) \right\|_V \\
&+ \sum_{q=1}^{N_\varphi} \varphi_q \left\| u_0 \right\|_V^2 + \left\| g_N(t) \right\|_{L_2(\Gamma_N)} \left\| u(t) \right\|_{L_2(\Gamma_N)} \\
&+ \left\| g_N(0) \right\|_{L_2(\Gamma_N)} \left\| u_0 \right\|_{L_2(\Gamma_N)} \\
&\text{(by Cauchy-Schwarz inequality),} \\
\leq &\int_0^t \left\| f(t') \right\|_{L_2(\Omega)} \left\| \dot{u}(t') \right\|_{L_2(\Omega)} dt' + \int_0^t \left\| \dot{g}_N(t') \right\|_{L_2(\Gamma_N)} \left\| u(t') \right\|_{L_2(\Gamma_N)} dt' \\
&+ \int_0^t \sum_{q=1}^{N_\varphi} \frac{\varphi_q}{\tau_q} \left\| u_0 \right\|_V \left\| u(t') \right\|_V dt' + \left\| u_0 \right\|_V \left\| u(t) \right\|_V + \left\| u_0 \right\|_V^2 \\
&+ \left\| g_N(t) \right\|_{L_2(\Gamma_N)} \left\| u(t) \right\|_{L_2(\Gamma_N)} + \left\| g_N(0) \right\|_{L_2(\Gamma_N)} \left\| u_0 \right\|_{L_2(\Gamma_N)} \quad (2.1.23) \\
&\left(\text{since } 0 < \sum_{q=1}^{N_\varphi} \varphi_q < 1 \text{ and } 0 < e^{-t/\tau_q} \leq 1,\ \forall t \geq 0, \forall q \in \{1, \ldots, N_\varphi\}\right).
\end{aligned}
$$

Moreover, by (2.1.13) and Young's inequality, we can obtain

$$
\begin{aligned}
\int_0^t F_v(t'; \dot{u}(t')) dt' \leq &\frac{1}{2} \int_0^t \left\| f(t') \right\|_{L_2(\Omega)}^2 dt' + \frac{1}{2} \int_0^t \left\| \dot{u}(t') \right\|_{L_2(\Omega)}^2 dt' + \sum_{q=1}^{N_\varphi} \frac{\varphi_q}{2\tau_q} \int_0^t \left\| u_0 \right\|_V^2 dt' \\
&+ \sum_{q=1}^{N_\varphi} \frac{\varphi_q}{2\tau_q} \int_0^t \left\| u(t') \right\|_V^2 dt' + \frac{C}{2} \int_0^t \left\| \dot{g}_N(t') \right\|_{L_2(\Gamma_N)}^2 dt' \\
&+ \frac{C}{2} \int_0^t \left\| u(t') \right\|_V^2 dt' + \frac{1}{2\epsilon_a} \left\| u_0 \right\|_V^2 + \frac{\epsilon_a}{2} \left\| u(t) \right\|_V^2 + \left\| u_0 \right\|_V^2 \\
&+ \frac{C}{2\epsilon} \left\| g_N(t) \right\|_{L_2(\Gamma_N)}^2 + \frac{C\epsilon}{2} \left\| u(t) \right\|_V^2 + \frac{C}{2} \left\| g_N(0) \right\|_{L_2(\Gamma_N)}^2 + \frac{C}{2} \left\| u_0 \right\|_V^2,
\end{aligned}
$$

for any positive ϵ, ϵ_a and for some positive C. If we choose $\epsilon = \varphi_0/(4C) > 0$ and $\epsilon_a = \varphi_0/4 > 0$, then

$$
\frac{\epsilon_a}{2} + \frac{C\epsilon}{2} = \frac{\varphi_0}{8} + \frac{\varphi_0}{8} = \frac{\varphi_0}{4}.
$$

Hence (2.1.22) implies

$$
\begin{aligned}
&\frac{\rho}{2}\|\dot{u}(t)\|^2_{L_2(\Omega)} + \frac{\varphi_0}{4}\|u(t)\|^2_V + \sum_{q=1}^{N_\varphi}\frac{1}{2\varphi_q}\|\zeta_q(t)\|^2_V + \sum_{q=1}^{N_\varphi}\int_0^t \frac{1}{\tau_q\varphi_q}\|\zeta_q(t')\|^2_V dt' \\
\leq &\frac{\rho}{2}\|w_0\|^2_{L_2(\Omega)} + \frac{\varphi_0}{2}\|u_0\|^2_V + \frac{1}{2}\int_0^t \left\|f(t')\right\|^2_{L_2(\Omega)} dt' + \frac{1}{2}\int_0^t \left\|\dot{u}(t')\right\|^2_{L_2(\Omega)} dt' \\
&+ \sum_{q=1}^{N_\varphi}\frac{\varphi_q}{2\tau_q}\int_0^t \|u_0\|^2_V\, dt' + \sum_{q=1}^{N_\varphi}\frac{\varphi_q}{2\tau_q}\int_0^t \left\|u(t')\right\|^2_V dt' + \frac{C}{2}\int_0^t \left\|\dot{g}_N(t')\right\|^2_{L_2(\Gamma_N)} dt' \\
&+ \frac{C}{2}\int_0^t \left\|u(t')\right\|^2_V dt' + \left(1 + \frac{C}{2} + \frac{1}{2\epsilon_a}\right)\|u_0\|^2_V + \frac{C}{2\epsilon}\|g_N(t)\|^2_{L_2(\Gamma_N)} \\
&+ \left(\frac{C}{2} + \frac{C}{2\epsilon}\right)\|g_N(0)\|^2_{L_2(\Gamma_N)}.
\end{aligned}
$$

In the end, it is shown that

$$
\begin{aligned}
&\frac{\rho}{2}\|\dot{u}(t)\|^2_{L_2(\Omega)} + \frac{\varphi_0}{4}\|u(t)\|^2_V + \sum_{q=1}^{N_\varphi}\frac{1}{2\varphi_q}\|\zeta_q(t)\|^2_V + \sum_{q=1}^{N_\varphi}\int_0^t \frac{1}{\tau_q\varphi_q}\|\zeta_q(t')\|^2_V dt' \\
\leq &C\Bigg(\|w_0\|^2_{L_2(\Omega)} + \|u_0\|^2_V + \int_0^t \left\|f(t')\right\|^2_{L_2(\Omega)} dt' + \int_0^t \|u_0\|^2_V\, dt' + \int_0^t \left\|\dot{g}_N(t')\right\|^2_{L_2(\Gamma_N)} dt' \\
&+ \|g_N(0)\|^2_{L_2(\Gamma_N)} + \|g_N(t)\|^2_{L_2(\Gamma_N)}\Bigg)
\end{aligned}
$$

with applying Grönwall's inequality with respect to $\|\dot{u}\|^2_{L_2(\Omega)}$ and $\|u\|^2_V$ terms. Since u_0 is independent of time variable t and $t \leq T$,

$$
\begin{aligned}
&\frac{\rho}{2}\|\dot{u}(t)\|^2_{L_2(\Omega)} + \frac{\varphi_0}{4}\|u(t)\|^2_V + \sum_{q=1}^{N_\varphi}\frac{1}{2\varphi_q}\|\zeta_q(t)\|^2_V + \sum_{q=1}^{N_\varphi}\int_0^t \frac{1}{\tau_q\varphi_q}\|\zeta_q(t')\|^2_V dt' \\
\leq &C\Bigg(\|w_0\|^2_{L_2(\Omega)} + \|u_0\|^2_V + \int_0^t \left\|f(t')\right\|^2_{L_2(\Omega)} dt' + \int_0^t \left\|\dot{g}_N(t')\right\|^2_{L_2(\Gamma_N)} dt' \\
&+ \|g_N(0)\|^2_{L_2(\Gamma_N)} + \|g_N(t)\|^2_{L_2(\Gamma_N)}\Bigg).
\end{aligned}
$$

□

In Theorem 2.3, we use Grönwall's inequality for the stability bound hence the positive constant C increases exponentially in time. In a similar way with Theorem 2.2, we consider stability bounds in L_∞ norm in time for **(P2)** so that we improve the constant C such that even increases but not exponentially in time.

Theorem 2.4. *Suppose $u \in W^1_\infty(0,T;L_2(\Omega)) \cap L_\infty(0,T;V)$ and $\zeta_q \in H^1(0,T;V)$, $\forall q \in \{1,\ldots,N_\varphi\}$. Then we have the stability bound as for any $t \in [0,T]$*

$$\frac{\rho}{4}\|\dot{u}\|^2_{L_\infty(0,T;L_2(\Omega))} + \frac{\varphi_0}{4}\|u\|^2_{L_\infty(0,T;V)} + \sum_{q=1}^{N_\varphi}\frac{1}{2\varphi_q}\|\zeta_q(t)\|^2_V + \sum_{q=1}^{N_\varphi}\int_0^t \frac{1}{\tau_q\varphi_q}\|\zeta_q(t')\|^2_V dt'$$
$$\leq C\Big(\|w_0\|^2_{L_2(\Omega)} + \|u_0\|^2_V + \|f\|^2_{L_2(0,T;L_2(\Omega))} + \|\dot{g}_N\|^2_{L_2(0,T;L_2(\Gamma_N))} + \|g_N\|^2_{L_\infty(0,T;L_2(\Gamma_N))}\Big).$$

Proof. As following the proof in Theorem 2.3, recall (2.1.23). Then we have for any $0 \leq t \leq T$

$$\begin{aligned}\int_0^t F_v(t';\dot{u}(t'))dt' \leq &\int_0^t \left\|f(t')\right\|_{L_2(\Omega)}\left\|\dot{u}(t')\right\|_{L_2(\Omega)} dt' + C\int_0^t \left\|\dot{g}_N(t')\right\|_{L_2(\Gamma_N)}\left\|u(t')\right\|_V dt' \\ &+ \int_0^t \sum_{q=1}^{N_\varphi}\frac{\varphi_q}{\tau_q}\|u_0\|_V\left\|u(t')\right\|_V dt' + \|u_0\|_V\|u(t)\|_V + \|u_0\|^2_V \\ &+ C\|g_N(t)\|_{L_2(\Gamma_N)}\|u(t)\|_V + C\|g_N(0)\|_{L_2(\Gamma_N)}\|u_0\|_V\end{aligned}$$

with positive constant C by (2.1.13). Taking into account L_∞ norm in time and a property of integration, it yields

$$\begin{aligned}\int_0^t F_v(t';\dot{u}(t'))dt' \leq \|\dot{u}\|_{L_\infty(0,T;L_2(\Omega))}&\int_0^T \left\|f(t')\right\|_{L_2(\Omega)} dt' \\ &+ C\|u\|_{L_\infty(0,T;V)}\int_0^T \left\|\dot{g}_N(t')\right\|_{L_2(\Gamma_N)} dt' \\ &+ \sum_{q=1}^{N_\varphi}\frac{\varphi_q}{\tau_q}\|u\|_{L_\infty(0,T;V)}\int_0^T \|u_0\|_V dt' + \|u_0\|_V\|u\|_{L_\infty(0,T;V)} + \|u_0\|^2_V \\ &+ C\|g_N\|_{L_\infty(0,T;L_2(\Gamma_N))}\|u\|_{L_\infty(0,T;V)} + C\|g_N\|_{L_\infty(0,T;L_2(\Gamma_N))}\|u_0\|_V.\end{aligned}$$

Then Young's inequality allows us to obtain

$$\begin{aligned}\int_0^t F_v(t';\dot{u}(t'))dt' \leq &\frac{\epsilon_a}{2}\|\dot{u}\|^2_{L_\infty(0,T;L_2(\Omega))} + \frac{1}{2\epsilon_a}\left(\int_0^T \left\|f(t')\right\|_{L_2(\Omega)} dt'\right)^2 \\ &+ \frac{C\epsilon_b}{2}\|u\|^2_{L_\infty(0,T;V)} + \frac{C}{2\epsilon_b}\left(\int_0^T \left\|\dot{g}_N(t')\right\|_{L_2(\Gamma_N)} dt'\right)^2 \\ &+ \sum_{q=1}^{N_\varphi} T\frac{\varphi_q}{\tau_q}\left(\frac{1}{2\epsilon_q}\|u_0\|^2_V + \frac{\epsilon_q}{2}\|u\|^2_{L_\infty(0,T;V)}\right) \\ &+ \frac{1}{2\epsilon_c}\|u_0\|^2_V + \frac{\epsilon_c}{2}\|u\|^2_{L_\infty(0,T;V)} + \frac{C\epsilon_d}{2}\|u\|^2_{L_\infty(0,T;V)} \\ &+ \frac{C}{2}\left(1 + \frac{1}{\epsilon_d}\right)\|g_N\|^2_{L_\infty(0,T;L_2(\Gamma_N))} + \left(1 + \frac{C}{2}\right)\|u_0\|^2_V,\end{aligned}$$

for positive $\epsilon_a,\ \epsilon_b,\ \epsilon_c,\ \epsilon_d$ and $\{\epsilon_q\}$. Recall (2.1.22), we have

$$\begin{aligned}&\frac{\rho}{2}\left\|\dot{u}(t)\right\|^2_{L_2(\Omega)}+\frac{\varphi_0}{2}\left\|u(t)\right\|^2_V+\sum_{q=1}^{N_\varphi}\frac{1}{2\varphi_q}\|\zeta_q(t)\|^2_V+\sum_{q=1}^{N_\varphi}\int_0^t\frac{1}{\tau_q\varphi_q}\|\zeta_q(t')\|^2_V dt'\\ =&\frac{\rho}{2}\left\|w_0\right\|^2_{L_2(\Omega)}+\frac{\varphi_0}{2}\left\|u_0\right\|^2_V+\int_0^t F_v(t';\dot{u}(t'))dt',\end{aligned}$$

hence

$$\begin{aligned}&\frac{\rho}{2}\left\|\dot{u}(t)\right\|^2_{L_2(\Omega)}+\frac{\varphi_0}{2}\left\|u(t)\right\|^2_V+\sum_{q=1}^{N_\varphi}\frac{1}{2\varphi_q}\|\zeta_q(t)\|^2_V+\sum_{q=1}^{N_\varphi}\int_0^t\frac{1}{\tau_q\varphi_q}\|\zeta_q(t')\|^2_V dt'\\ \leq&\frac{\rho}{2}\left\|w_0\right\|^2_{L_2(\Omega)}+\frac{\varphi_0}{2}\left\|u_0\right\|^2_V+\frac{\epsilon_a}{2}\left\|\dot{u}\right\|^2_{L_\infty(0,T;L_2(\Omega))}+\frac{1}{2\epsilon_a}\left(\int_0^T\left\|f(t')\right\|_{L_2(\Omega)}dt'\right)^2\\ &+\left(1+\frac{C}{2}+\frac{1}{2\epsilon_c}+\sum_{q=1}^{N_\varphi}\frac{T}{2\epsilon_q}\frac{\varphi_q}{\tau_q}\right)\left\|u_0\right\|^2_V+\frac{C}{2\epsilon_b}\left(\int_0^T\left\|\dot{g}_N(t')\right\|_{L_2(\Gamma_N)}dt'\right)^2\\ &+\frac{C}{2}\left(1+\frac{1}{\epsilon_d}\right)\left\|g_N\right\|^2_{L_\infty(0,T;L_2(\Gamma_N))}\\ &+\left(\frac{C\epsilon_b}{2}+\frac{\epsilon_c}{2}+\frac{C\epsilon_d}{2}+\sum_{q=1}^{N_\varphi}\frac{T\epsilon_q}{2}\frac{\varphi_q}{\tau_q}\right)\left\|u\right\|^2_{L_\infty(0,T;V)},\end{aligned}$$

for any $t\in[0,T]$. Suppose

$$\begin{aligned}\epsilon_a&=\frac{\rho}{6}>0,\\ \epsilon_b&=\frac{\varphi_0}{24C}>0,\\ \epsilon_c&=\frac{\varphi_0}{24}>0,\\ \epsilon_d&=\frac{\varphi_0}{24C}>0,\\ \epsilon_q&=\frac{\varphi_0}{24TN_\varphi}\frac{\tau_q}{\varphi_q}>0,\ \forall q\in\{1,\ldots,N_\varphi\},\end{aligned}$$

then it is able to observe that

$$\frac{C\epsilon_b}{2}+\frac{\epsilon_c}{2}+\frac{C\epsilon_d}{2}+\sum_{q=1}^{N_\varphi}\frac{T\epsilon_q}{2}\frac{\varphi_q}{\tau_q}=\frac{\varphi_0}{48}+\frac{\varphi_0}{48}+\frac{\varphi_0}{48}+\frac{\varphi_0}{48}=\frac{\varphi_0}{12}.$$

Thus we have

$$\frac{\rho}{2}\left\|\dot{u}(t)\right\|^2_{L_2(\Omega)}+\frac{\varphi_0}{2}\left\|u(t)\right\|^2_V+\sum_{q=1}^{N_\varphi}\frac{1}{2\varphi_q}\|\zeta_q(t)\|^2_V+\sum_{q=1}^{N_\varphi}\int_0^t\frac{1}{\tau_q\varphi_q}\|\zeta_q(t')\|^2_V dt'$$

$$
\begin{aligned}
\leq & \frac{\rho}{2}\left\|w_0\right\|_{L_2(\Omega)}^2+\left(1+\frac{C}{2}+\frac{\varphi_0}{2}+\frac{1}{2\epsilon_c}+\sum_{q=1}^{N_\varphi}\frac{T}{2\epsilon_q}\frac{\varphi_q}{\tau_q}\right)\left\|u_0\right\|_V^2 \\
& +\frac{1}{2\epsilon_a}\left(\int_0^T\left\|f(t')\right\|_{L_2(\Omega)}dt'\right)^2+\frac{C}{2\epsilon_b}\left(\int_0^T\left\|\dot{g}_N(t')\right\|_{L_2(\Gamma_N)}dt'\right)^2 \\
& +\frac{C}{2}\left(1+\frac{1}{\epsilon_d}\right)\left\|g_N\right\|_{L_\infty(0,T;L_2(\Gamma_N))}^2+\frac{\rho}{12}\left\|\dot{u}\right\|_{L_\infty(0,T;L_2(\Omega))}^2+\frac{\varphi_0}{12}\left\|u\right\|_{L_\infty(0,T;V)}^2.
\end{aligned}
$$

Now, let us consider the essential supremums then we have

$$
\begin{aligned}
& \frac{\rho}{2}\left\|\dot{u}\right\|_{L_\infty(0,T;L_2(\Omega))}^2+\frac{\varphi_0}{2}\left\|u\right\|_{L_\infty(0,T;V)}^2+\sum_{q=1}^{N_\varphi}\frac{1}{2\varphi_q}\left\|\zeta_q(t)\right\|_V^2+\sum_{q=1}^{N_\varphi}\int_0^t\frac{1}{\tau_q\varphi_q}\|\zeta_q(t')\|_V^2dt' \\
\leq & 3\Bigg(\frac{\rho}{2}\left\|w_0\right\|_{L_2(\Omega)}^2+\left(1+\frac{C}{2}+\frac{\varphi_0}{2}+\frac{1}{2\epsilon_c}+\sum_{q=1}^{N_\varphi}\frac{T}{2\epsilon_q}\frac{\varphi_q}{\tau_q}\right)\left\|u_0\right\|_V^2 \\
& +\frac{1}{2\epsilon_a}\left(\int_0^T\left\|f(t')\right\|_{L_2(\Omega)}dt'\right)^2+\frac{C}{2\epsilon_b}\left(\int_0^T\left\|\dot{g}_N(t')\right\|_{L_2(\Gamma_N)}dt'\right)^2 \\
& +\frac{C}{2}\left(1+\frac{1}{\epsilon_d}\right)\left\|g_N\right\|_{L_\infty(0,T;L_2(\Gamma_N))}^2+\frac{\rho}{12}\left\|\dot{u}\right\|_{L_\infty(0,T;L_2(\Omega))}^2+\frac{\varphi_0}{12}\left\|u\right\|_{L_\infty(0,T;V)}^2\Bigg).
\end{aligned}
$$

Thus, by subtracting $\frac{\rho}{4}\left\|\dot{u}\right\|_{L_\infty(0,T;L_2(\Omega))}^2$ and $\frac{\varphi_0}{4}\left\|u\right\|_{L_\infty(0,T;V)}^2$, there exists a positive constant C such that

$$
\begin{aligned}
& \frac{\rho}{4}\left\|\dot{u}\right\|_{L_\infty(0,T;L_2(\Omega))}^2+\frac{\varphi_0}{4}\left\|u\right\|_{L_\infty(0,T;V)}^2+\sum_{q=1}^{N_\varphi}\frac{1}{2\varphi_q}\left\|\zeta_q(t)\right\|_V^2+\sum_{q=1}^{N_\varphi}\int_0^t\frac{1}{\tau_q\varphi_q}\|\zeta_q(t')\|_V^2dt' \\
\leq & C\Bigg(\left\|w_0\right\|_{L_2(\Omega)}^2+\left\|u_0\right\|_V^2+\left(\int_0^T\left\|f(t')\right\|_{L_2(\Omega)}dt'\right)^2+\left(\int_0^T\left\|\dot{g}_N(t')\right\|_{L_2(\Gamma_N)}dt'\right)^2 \\
& +\left\|g_N\right\|_{L_\infty(0,T;L_2(\Gamma_N))}^2\Bigg) \\
\leq & C\Bigg(\left\|w_0\right\|_{L_2(\Omega)}^2+\left\|u_0\right\|_V^2+T\left\|f\right\|_{L_2(0,T;L_2(\Omega))}^2+T\left\|\dot{g}_N\right\|_{L_2(0,T;L_2(\Gamma_N))}^2 \\
& +\left\|g_N\right\|_{L_\infty(0,T;L_2(\Gamma_N))}^2\Bigg) \\
& \text{(by Cauchy-Schwarz inequality)} \\
\leq & C\Big(\left\|w_0\right\|_{L_2(\Omega)}^2+\left\|u_0\right\|_V^2+\left\|f\right\|_{L_2(0,T;L_2(\Omega))}^2+\left\|\dot{g}_N\right\|_{L_2(0,T;L_2(\Gamma_N))}^2+\left\|g_N\right\|_{L_\infty(0,T;L_2(\Gamma_N))}^2\Big).
\end{aligned}
$$

□

Theorem 2.3 and 2.4 lead us to have stability bounds for **(P2)**. In particular, Theorem 2.4 is proved without Grönwall's inequality hence our weak solutions are bounded by data with non-exponential increased in time.

For the both formulation, we are going to use CGFEM for spatial discretisation and so we recall Chapter 1.4. In the next section, we introduce the semidiscrete formulations and observe the stability bounds and error bounds.

2.2 Semidiscrete Formulation for CGFEM

Let us define V^h such that consists of continuous local basis functions with respect to Lagrange finite elements[11]. Hence we can define $V^h \subset V$ with its global basis functions $\{\Phi_i\}_{i=1}^{N_{V^h}}$ by

$$V^h = \text{span}\{\Phi_i \mid 1 \leq i \leq N_{V^h}\} \cap V$$

where Φ_i is a continuous piecewise polynomial of degree $k \in \mathbb{N}$, $\forall i \in \{1, \ldots, N_{V^h}\}$. In this section, we approximate the solution $u(t)$ to (2.1.1)-(2.1.5) by $u_h(t)$ which belongs to the finite dimensional space V^h for all $t \geq 0$. Also, we should consider the internal variables.

2.2.1 Displacement Form

Using the global basis functions, for any function $v \in V^h$, v can be expressed as

$$v(\boldsymbol{x}) = \sum_{i=1}^{N_{V^h}} v_i \Phi_i(\boldsymbol{x}),$$

for $v_i \in \mathbb{R}$, $\forall i \in \{1, \ldots, N_{V^h}\}$. In this sense, the approximate solution and internal variables for the displacement form **(P1)** are given as

$$u_h(\boldsymbol{x}, t) = \sum_{i=1}^{N_{V^h}} \mathfrak{u}_i(t) \Phi_i(\boldsymbol{x}),$$

$$\psi_{hq}(\boldsymbol{x}, t) = \sum_{i=1}^{N_{V^h}} \psi_{hq,i}(t) \Phi_i(\boldsymbol{x}),$$

which satisfy

$$\begin{aligned}
(\rho \ddot{u}_h(t), v)_{L_2(\Omega)} + a(u_h, v) \quad - \sum_{q=1}^{N_\varphi} a(\psi_{hq}(t), v) \quad &= F_d(t; v), && (2.2.1)\\
\tau_q a\left(\dot{\psi}_{hq}(t), v\right) + a(\psi_q(t), v) \quad &= \varphi_q a(u_h(t), v) \quad \forall q = 1, \ldots, N_\varphi, && (2.2.2)\\
a(u_h(0), v) \quad &= a(u_0, v)\ , && (2.2.3)\\
(\dot{u}_h(0), v)_{L_2(\Omega)} \quad &= (w_0, v)_{L_2(\Omega)}\ , && (2.2.4)
\end{aligned}$$

for any $v \in V^h$ and with $\psi_{hq}(0) = 0$, $\forall q \in \{1, \ldots, N_\varphi\}$.

In order to obtain the solutions, we should determine the degrees of freedom $\underline{\mathfrak{u}}$ and $\underline{\psi}_q$, for each q where

$$\underline{\mathfrak{u}}(t) = (\mathfrak{u}_i(t))_{i=1}^{N_{V^h}},\ \underline{\psi}_{hq}(t) = (\psi_{hq,i}(t))_{i=1}^{N_{V^h}}.$$

Put $v = \Phi_i$ into (2.2.1) for $i = 1, \ldots, N_{V^h}$. It implies the second order ODE system such that

$$\rho M \ddot{\underline{\mathrm{u}}}(t) + A\underline{\mathrm{u}}(t) - \sum_{q=1}^{N_\varphi} A\underline{\psi}_{hq}(t) = \underline{F}(t), \tag{2.2.5}$$

where the mass matrix M and the stiffness matrix A are defined by for $i, j = 1, \ldots, N_{V^h}$

$$M_{ij} = (\Phi_j, \Phi_i)_{L_2(\Omega)}, \ A_{ij} = a(\Phi_j, \Phi_i),$$

and $F_i(t) = F_d(t; \Phi_i)$ for $1 \leq i \leq N_{V^h}$.

Remark Since $V^h \subset V$, $a(\cdot, \cdot)$ is coercive on V^h by Lemma 1.1.

Theorem 2.5. *The mass matrix M and the stiffness matrix A are symmetric positive definite. Thus, they are invertible.*

Proof. Note that L_2 inner product and the bilinear form $a(\cdot, \cdot)$ are symmetric hence M and A are symmetric.

Let $\underline{v} \in \mathbb{R}^{N_{V^h}}$. Then

$$\underline{v}^\intercal M \underline{v} = \sum_{i,j=1}^{N_{V^h}} v_j M_{ij} v_i = \sum_{i,j=1}^{N_{V^h}} v_j (\Phi_j, \Phi_i)_{L_2(\Omega)} v_i = \sum_{i,j=1}^{N_{V^h}} (v_j\Phi_j, v_i\Phi_i)_{L_2(\Omega)} = \|v\|^2_{L_2(\Omega)} \geq 0,$$

where $v = \sum_{i=1}^{N_{V^h}} v_i \Phi_i \in V^h$. By the norm axiom, $\underline{v}^\intercal M \underline{v} = 0$ if and only if $\underline{v} = \underline{0}$. Thus M is symmetric positive definite and hence M is invertible.

On the other hand, $a(\cdot, \cdot)$ is coercive, so

$$\underline{v}^\intercal A \underline{v} = \sum_{i,j=1}^{N_{V^h}} v_j A_{ij} v_i = \sum_{i,j=1}^{N_{V^h}} v_j a(\Phi_j, \Phi_i) v_i = \sum_{i,j=1}^{N_{V^h}} a(v_j\Phi_j, v_i\Phi_i) = a(v, v) \geq \kappa \|v\|^2_{H^1(\Omega)} \geq 0$$

for some positive constant κ. It implies that also $\underline{v}^\intercal A \underline{v} = 0$ if and only if $\underline{v} = 0$, therefore A is symmetric positive definite and so invertible. $\square$

Turning to the semidiscrete formula, (2.2.2) and invertible A yield

$$\tau_q \dot{\underline{\psi}}_{hq}(t) + \underline{\psi}_{hq}(t) = \varphi_q \underline{\mathrm{u}}(t) \tag{2.2.6}$$

for each q and (2.2.3) and (2.2.4) provide the initial condition by solving

$$A\underline{\mathrm{u}}(0) = \underline{U}_0, \ M\dot{\underline{\mathrm{u}}}(0) = \underline{W}_0$$

where $(U_0)_i = a(u_0, \Phi_i)$ and $(W_0)_i = (w_0, \Phi_i)_{L_2(\Omega)}$ for $i = 1, \ldots, N_{V^h}$. Without dealing with many details in terms of solving second order ODE system, since M and A are invertible, and initial conditions are given, this system can be solved uniquely(e.g. see [57] in detail for the theory of ODEs).

Theorem 2.6. (Stability bound for the semidiscrete solution of **(P1)**)
Let u_h and $\{\psi_{hq}\}_{q=1}^{N_\varphi}$ be the semidiscrete solution of **(P1)**. *For any $t \in [0,T]$, it holds*

$$\begin{aligned}
&\frac{\rho}{2}\|\dot{u}_h(t)\|^2_{L_2(\Omega)} + \frac{\varphi_0}{8}\|u_h(t)\|^2_V + \sum_{q=1}^{N_\varphi}\frac{\varphi_0}{4\varphi_q^2 N_\varphi + 2\varphi_0\varphi_q}\|\psi_{hq}(t)\|^2_V \\
&+ \sum_{q=1}^{N_\varphi}\frac{\tau_q}{\varphi_q}\int_0^t \left\|\dot{\psi}_{hq}(t')\right\|^2_V dt' \\
\le& C\bigg(\|w_0\|^2_{L_2(\Omega)} + \|u_0\|^2_V + \int_0^t \|f(t')\|^2_{L_2(\Omega)} dt' + \int_0^t \|\dot{g}_N(t')\|^2_{L_2(\Gamma_N)} dt' \\
&+ \|g_N(t)\|^2_{L_2(\Gamma_N)} + \|g_N(0)\|^2_{L_2(\Gamma_N)} \bigg),
\end{aligned}$$

for some positive constant C which is independent of the weak solutions but depends on the domain, its boundary and the time.

Proof. Since $V^h \subset V$, the proof of Theorem 2.6 follows that of Theorem 2.1. In the proof of Theorem 2.1, there is the initial condition such that $u(0) = u_0$ and $\dot{u}(0) = w_0$ and so we use it. However, $u_h(0) \neq u_0$ and $\dot{u}_h(0) \neq w_0$ hence we cannot replace $u_h(0)$ and $\dot{u}_h(0)$ by u_0 and w_0, respectively. Note that (2.2.3) and (2.2.4) imply that

$$\begin{aligned}
\|u_h(0)\|^2_V &= a(u_h(0), u_h(0)) = a(u_0, u_h(0)) \le \|u_0\|_V \|u_h(0)\|_V \\
\|\dot{u}_h(0)\|^2_{L_2(\Omega)} &= (\dot{u}_h(0), \dot{u}_h(0))_{L_2(\Omega)} = (w_0, \dot{u}_h(0))_{L_2(\Omega)} \le \|w_0\|_{L_2(\Omega)} \|\dot{u}_h(0)\|_{L_2(\Omega)},
\end{aligned}$$

by Cauchy-Schwarz inequality with taking $v = u_h(0)$ and $\dot{u}_h(0)$, respectively. Hence we have $\|u_h(0)\|_V \le \|u_0\|_V$ and $\|\dot{u}_h(0)\|_{L_2(\Omega)} \le \|w_0\|_{L_2(\Omega)}$. Therefore,

$$\begin{aligned}
&\frac{\rho}{2}\|\dot{u}_h(t)\|^2_{L_2(\Omega)} + \frac{\varphi_0}{8}\|u_h(t)\|^2_V + \sum_{q=1}^{N_\varphi}\frac{\varphi_0}{4\varphi_q^2 N_\varphi + 2\varphi_0\varphi_q}\|\psi_{hq}(t)\|^2_V \\
&+ \sum_{q=1}^{N_\varphi}\frac{\tau_q}{\varphi_q}\int_0^t \left\|\dot{\psi}_{hq}(t')\right\|^2_V dt' \\
\le& C\bigg(\|\dot{u}_h(0)\|^2_{L_2(\Omega)} + \|u_h(0)\|^2_V + \int_0^t \|f(t')\|^2_{L_2(\Omega)} dt' + \int_0^t \|\dot{g}_N(t')\|^2_{L_2(\Gamma_N)} dt' \\
&+ \|g_N(t)\|^2_{L_2(\Gamma_N)} + \|g_N(0)\|^2_{L_2(\Gamma_N)} \bigg), \\
\le& C\bigg(\|w_0\|^2_{L_2(\Omega)} + \|u_0\|^2_V + \int_0^t \|f(t')\|^2_{L_2(\Omega)} dt' + \int_0^t \|\dot{g}_N(t')\|^2_{L_2(\Gamma_N)} dt' \\
&+ \|g_N(t)\|^2_{L_2(\Gamma_N)} + \|g_N(0)\|^2_{L_2(\Gamma_N)} \bigg).
\end{aligned}$$

□

Here, the constant C is given with using Grönwall's inequality so that it increases exponentially in time. We can improve this as shown in Theorem 2.2 with based on L_∞ norm in time.

Theorem 2.7. *Suppose u_h and $\{\psi_{hq}\}_{q=1}^{N_\varphi}$ are the semidiscrete solution to* **(P1)**. *In addition, we assume that $u_h \in W_\infty^1(0,T;L_2(\Omega)) \cap L_\infty(0,T;V^h)$, and $\psi_{hq} \in H^1(0,T;V^h)$ $\forall q \in \{1,\dots,N_\varphi\}$. Then we have the stability bound as*

$$\begin{aligned}
&\frac{\rho}{4}\|\dot{u}_h\|^2_{L_\infty(0,T;L_2(\Omega))} + \frac{\varphi_0}{8}\|u_h\|^2_{L_\infty(0,T;V)} + \sum_{q=1}^{N_\varphi}\frac{\varphi_0}{4\varphi_q^2 N_\varphi + 2\varphi_0\varphi_q}\|\psi_{hq}(t)\|_V^2 \\
&\quad + \sum_{q=1}^{N_\varphi}\frac{\tau_q}{\varphi_q}\int_0^t \left\|\dot{\psi}_{hq}(t')\right\|_V^2 dt', \\
&\le C\bigg(\|w_0\|^2_{L_2(\Omega)} + \|u_0\|_V^2 + \|f\|^2_{L_2(0,T;L_2(\Omega))} + \|g_N\|^2_{L_\infty(0,T;L_2(\Gamma_N))} \\
&\quad + \|\dot{g}_N\|^2_{L_2(0,T;L_2(\Gamma_N))}\bigg),
\end{aligned}$$

for any $t \in [0,T]$.

Proof. As follows the proof of Theorem 2.2, our claim is shown with the facts,

$$\|\dot{u}_h(0)\|_{L_2(\Omega)} \le \|w_0\|_{L_2(\Omega)} \text{ and } \|u_h(0)\|_V \le \|u_0\|_V .$$

In other words, we have for any t

$$\begin{aligned}
&\frac{\rho}{4}\|\dot{u}_h\|^2_{L_\infty(0,T;L_2(\Omega))} + \frac{\varphi_0}{8}\|u_h\|^2_{L_\infty(0,T;V)} + \sum_{q=1}^{N_\varphi}\frac{\varphi_0}{4\varphi_q^2 N_\varphi + 2\varphi_0\varphi_q}\|\psi_{hq}(t)\|_V^2 \\
&\quad + \sum_{q=1}^{N_\varphi}\frac{\tau_q}{\varphi_q}\int_0^t \left\|\dot{\psi}_{hq}(t')\right\|_V^2 dt', \\
&\le C\bigg(\|\dot{u}_h(0)\|^2_{L_2(\Omega)} + \|u_h(0)\|_V^2 + \|f\|^2_{L_2(0,T;L_2(\Omega))} + \|g_N\|^2_{L_\infty(0,T;L_2(\Gamma_N))} \\
&\quad + \|\dot{g}_N\|^2_{L_2(0,T;L_2(\Gamma_N))}\bigg) \\
&\le C\bigg(\|w_0\|^2_{L_2(\Omega)} + \|u_0\|_V^2 + \|f\|^2_{L_2(0,T;L_2(\Omega))} + \|g_N\|^2_{L_\infty(0,T;L_2(\Gamma_N))} \\
&\quad + \|\dot{g}_N\|^2_{L_2(0,T;L_2(\Gamma_N))}\bigg).
\end{aligned}$$

□

We proved the stability bounds for the semidiscrete formulation of the displacement form with/without Groïwall inequality. It is observed that our semidiscrete solution is bounded by the given data so that it is sufficient to show the existence and uniqueness of the solution. In a similar way, we can derive error estimates. First of all, recall the elliptic projection R in Chapter 1.4. Let us define notations as following

$$\begin{aligned}
\theta &= u - Ru, \\
\vartheta_q &= \psi_q - R\psi_q,\ \forall q \in \{1, \ldots, N_\varphi\}, \\
\chi &= u_h - Ru, \\
\varsigma_q &= \psi_{hq} - R\psi_q,\ \forall q\{1, \ldots, N_\varphi\},
\end{aligned}$$

where R is the elliptic projection operator such that satisfies for $w \in V$

$$R : V \mapsto V^h \text{ by } a(Rw, v) = a(w, v),\ \forall v \in V^h.$$

By the elliptic orthogonality, $a(\theta, v) = 0$ and $a(\vartheta_q, v) = 0,$ for each q, and $\forall v \in V^h$.

Lemma 2.5. *For any $w(t) \in V$ and $\dot{w}(t) \in V$,*

$$\frac{\partial}{\partial t}(Rw(t)) = R\dot{w}(t).$$

Proof. Let $w(t),\ \dot{w}(t) \in V$. By definition of elliptic projection, for any $v \in V^h$

$$a\left(w, v\right) = a\left(Rw, v\right) \text{ and } a\left(\dot{w}, v\right) = a\left(R\dot{w}, v\right).$$

Consider time differentiation,

$$\begin{aligned}
\frac{\partial}{\partial t} a\left(w, v\right) =& \frac{\partial}{\partial t} \int_\Omega D\nabla w \cdot \nabla v \ d\Omega \\
=& \int_\Omega \frac{\partial}{\partial t}(D\nabla w \cdot \nabla v) \ d\Omega \\
& \text{(by Leibniz's integral rule)} \\
=& \int_\Omega \frac{\partial}{\partial t}(D\nabla w) \cdot \nabla v \ d\Omega \\
& \text{(since } v \text{ is time independent)} \\
=& \int_\Omega D\nabla \dot{w} \cdot \nabla v \ d\Omega \\
& \text{(by Leibniz's integral rule)} \\
=& a\left(\dot{w}, v\right) \\
=& a\left(R\dot{w}, v\right).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\frac{\partial}{\partial t} a\left(w, v\right) =& \frac{\partial}{\partial t} a\left(Rw, v\right) \\
=& \int_\Omega \frac{\partial}{\partial t}(D\nabla Rw \nabla v) \ d\Omega,
\end{aligned}$$

$$
\begin{aligned}
&\text{(by Leibniz's integral rule)}\\
=&\int_\Omega \frac{\partial}{\partial t}(D\nabla Rw)\nabla v \; d\Omega,\\
&\text{(since } v \text{ is time independent)}\\
=&\int_\Omega D\nabla\left(\frac{\partial}{\partial t}Rw\right)\nabla v \; d\Omega,\\
&\text{(by Leibniz's integral rule)}\\
=&a\left(\frac{\partial}{\partial t}Rw, v\right).
\end{aligned}
$$

Hence for any $v \in V^h$,

$$
a\left(\frac{\partial}{\partial t}Rw, v\right) = a\left(R\dot{w}, v\right) \Rightarrow a\left(\frac{\partial}{\partial t}Rw - R\dot{w}, v\right) = 0.
$$

Moreover, $\frac{\partial}{\partial t}Rw,\ R\dot{w} \in V^h$ so $\frac{\partial}{\partial t}Rw - R\dot{w} \in V^h$. It yields

$$
0 = a\left(\frac{\partial}{\partial t}Rw - R\dot{w}, \frac{\partial}{\partial t}Rw - R\dot{w}\right) \geq \kappa \left\|\frac{\partial}{\partial t}Rw - R\dot{w}\right\|^2_{H^1(\Omega)} \geq 0
$$

by coercivity. It implies $\frac{\partial}{\partial t}Rw - R\dot{w} = 0$, therefore we can conclude

$$
\frac{\partial}{\partial t}Rw = R\dot{w}.
$$

□

Lemma 2.6. *Suppose $u \in H^2(0,T;L_2(\Omega)) \cap W^1_\infty(0,T;H^s(\Omega))$. We can observe that for the semidiscrete formulation of* **(P1)** *with $\chi(t) = u_h(t) - Ru(t)$,*

$$
\|\dot{\chi}\|_{L_\infty(0,T;L_2(\Omega))} + \|\chi\|_{L_\infty(0,T;V)} \leq Ch^{\min(k+1,s)-1}
$$

and if elliptic regularity provided,

$$
\|\dot{\chi}\|_{L_\infty(0,T;L_2(\Omega))} + \|\chi\|_{L_\infty(0,T;V)} \leq Ch^{\min(k+1,s)}
$$

where C is some positive constant.

Proof. By subtracting (2.1.11) from (2.2.1), we have for any $v \in V^h$

$$
(\rho(\ddot{u}_h(t) - \ddot{u}(t)), v)_{L_2(\Omega)} + a(u_h(t) - u(t), v) - \sum_{q=1}^{N_\varphi} a(\psi_{hq}(t) - \psi_q(t), v) = 0.
$$

Since

$$
u_h(t) - u(t) - = (u_h(t) - Ru(t)) - (u(t) - Ru(t)) = \chi(t) - \theta(t)
$$

and

$$\psi_{hq}(t) - \psi_q(t) = (\psi_{hq}(t) - R\psi_q(t)) - (\psi_q(t) - R\psi_q(t)) = \varsigma_q(t) - \vartheta_q(t),\ \forall q = 1, \dots, N_\varphi,$$

the equality yields

$$\begin{aligned}
&\rho\left(\ddot{\chi}(t), v\right)_{L_2(\Omega)} + a(\chi(t), v) - \sum_{q=1}^{N_\varphi} a(\varsigma_q(t), v) \\
=&\rho\left(\ddot{\theta}(t), v\right)_{L_2(\Omega)} + a(\theta(t), v) - \sum_{q=1}^{N_\varphi} a(\vartheta_q(t), v)
\end{aligned} \tag{2.2.7}$$

for any $v \in V^h$. In a similar way, from (2.2.2), we can also have

$$a(\tau_q \dot{\varsigma}_q + \varsigma_q, v) - \varphi_q a(\chi, v) = a(\tau_q \dot{\vartheta}_q + \vartheta_q, v) - \varphi_q a(\theta, v) \tag{2.2.8}$$

for each q and for any $v \in V^h$. Put $v = \dot{\chi}(t)$ into (2.2.7) to get

$$\begin{aligned}
&\rho\left(\ddot{\chi}(t), \dot{\chi}(t)\right)_{L_2(\Omega)} + a(\chi(t), \dot{\chi}(t)) - \sum_{q=1}^{N_\varphi} a(\varsigma_q(t), \dot{\chi}(t)) \\
=& \rho\left(\ddot{\theta}(t), \dot{\chi}(t)\right)_{L_2(\Omega)} + a(\theta(t), \dot{\chi}(t)) - \sum_{q=1}^{N_\varphi} a(\vartheta_q(t), \dot{\chi}(t)).
\end{aligned}$$

Integrate with respect to time to obtain for $0 \le t \le T$

$$\begin{aligned}
&\frac{\rho}{2}\left(\|\dot{\chi}(t)\|^2_{L_2(\Omega)} - \|\dot{\chi}(0)\|^2_{L_2(\Omega)}\right) + \frac{1}{2}\left(\|\chi(t)\|^2_V - \|\chi(0)\|^2_V\right) - \sum_{q=1}^{N_\varphi} \int_0^t a(\varsigma_q(t'), \dot{\chi}(t'))dt' \\
=&\frac{\rho}{2}\left(\|\dot{\chi}(t)\|^2_{L_2(\Omega)} - \|\dot{\chi}(0)\|^2_{L_2(\Omega)}\right) + \frac{1}{2}\left(\|\chi(t)\|^2_V - \|\chi(0)\|^2_V\right) - \sum_{q=1}^{N_\varphi} a(\varsigma_q(t), \chi(t)) \\
&+ \sum_{q=1}^{N_\varphi} a(\varsigma_q(0), \chi(0)) + \sum_{q=1}^{N_\varphi} \int_0^t a(\dot{\varsigma}_q(t'), \chi(t'))dt' \\
=& \int_0^t \left(\rho\left(\ddot{\theta}(t'), \dot{\chi}(t')\right)_{L_2(\Omega)} + a(\theta(t'), \dot{\chi}(t')) - \sum_{q=1}^{N_\varphi} a(\vartheta_q(t'), \dot{\chi}(t'))\right)dt'.
\end{aligned} \tag{2.2.9}$$

Next, if we choose $v = \dot{\varsigma}_q(t)$ in (2.2.8), we obtain

$$a(\chi(t), \dot{\varsigma}_q(t)) = \frac{1}{\varphi_q} a(\tau_q \dot{\varsigma}_q(t) + \varsigma_q(t), \dot{\varsigma}_q(t)) - \frac{1}{\varphi_q} a(\tau_q \dot{\vartheta}_q(t) + \vartheta_q(t), \dot{\varsigma}_q(t)) + a(\theta(t), \dot{\varsigma}_q(t))$$

and integration over time gives

$$\int_0^t a(\chi(t'), \dot{\varsigma}_q(t'))dt' = \int_0^t \frac{\tau_q}{\varphi_q} \left\|\dot{\varsigma}_q(t')\right\|^2_V dt' + \frac{1}{2\varphi_q}\left(\|\varsigma_q(t)\|^2_V - \|\varsigma_q(0)\|^2_V\right)$$

$$+\int_0^t \Big(-\frac{1}{\varphi_q} a(\tau_q \dot{\vartheta}_q(t') + \vartheta_q(t'), \dot{\varsigma}_q(t')) + a(\theta(t'), \dot{\varsigma}_q(t')) \Big) dt' \tag{2.2.10}$$

for each q. Hence, substitution of (2.2.10) into (2.2.9) and Galerkin orthogonality imply that

$$\frac{\rho}{2}\|\dot{\chi}(t)\|^2_{L_2(\Omega)} + \frac{1}{2}\|\chi(t)\|^2_V + \sum_{q=1}^{N_\varphi} \frac{1}{2\varphi_q}\|\varsigma_q(t)\|^2_V + \sum_{q=1}^{N_\varphi}\int_0^t \frac{\tau_q}{\varphi_q}\left\|\dot{\varsigma}_q(t')\right\|^2_V dt'$$
$$= \frac{\rho}{2}\|\dot{\chi}(0)\|^2_{L_2(\Omega)} + \frac{1}{2}\|\chi(0)\|^2_V + \sum_{q=1}^{N_\varphi} a(\varsigma_q(t), \chi(t)) - \sum_{q=1}^{N_\varphi} a(\varsigma_q(0), \chi(0)) + \sum_{q=1}^{N_\varphi}\frac{1}{2\varphi_q}\|\varsigma_q(0)\|^2_V$$
$$+ \int_0^t \rho\left(\ddot{\theta}(t'), \dot{\chi}(t')\right)_{L_2(\Omega)} dt'.$$

More precisely, Galerkin orthogonality gives the following facts,

$$a(\theta(t'), \dot{\chi}(t')) = 0,\ a(\vartheta_q(t'), \dot{\chi}(t')) = 0,\ a(\tau_q\dot{\vartheta}_q(t') + \vartheta_q(t'), \dot{\varsigma}_q(t')) = 0,\ a(\theta(t'), \dot{\varsigma}_q(t')) = 0,$$

for any s, $\forall q \in \{1, \dots, N_\varphi\}$. Moreover, since $\varsigma_q(0) = \psi_{hq}(0) - R\psi_q(0) = 0 - 0 = 0$ for each q by initial conditions, we have

$$\frac{\rho}{2}\|\dot{\chi}(t)\|^2_{L_2(\Omega)} + \frac{1}{2}\|\chi(t)\|^2_V + \sum_{q=1}^{N_\varphi} \frac{1}{2\varphi_q}\|\varsigma_q(t)\|^2_V + \sum_{q=1}^{N_\varphi}\int_0^t \frac{\tau_q}{\varphi_q}\left\|\dot{\varsigma}_q(t')\right\|^2_V dt'$$
$$= \frac{\rho}{2}\|\dot{\chi}(0)\|^2_{L_2(\Omega)} + \frac{1}{2}\|\chi(0)\|^2_V + \sum_{q=1}^{N_\varphi} a(\varsigma_q(t), \chi(t)) + \int_0^t \rho\left(\ddot{\theta}(t'), \dot{\chi}(t')\right)_{L_2(\Omega)} dt'.$$

Note that (2.2.3) and elliptic projection lead

$$a(u_0, v) = a(u_h(0), v) \text{ and } a(Ru_0, v) = a(u_0, v)\ \forall v \in V^h$$

so that

$$a(u_h(0), v) = a(Ru_0, v) \text{ and } a(u_h(0) - Ru_0, v) = 0\ \forall v \in V^h.$$

Thus,

$$\|\chi(0)\|^2_V = a(u_h(0) - Ru_0, u_h(0) - Ru_0) = 0,$$

since $u_h(0) - Ru_0 \in V^h$. On the other hand,

$$\begin{aligned}\|\dot{\chi}(0)\|^2_{L_2(\Omega)} &= (\dot{u}_h(0) - Rw_0, \dot{u}_h(0) - Rw_0)_{L_2(\Omega)},\\ &= (\dot{u}_h(0) - Rw_0, w_0 - Rw_0)_{L_2(\Omega)},\\ &\le \|\dot{u}_h(0) - Rw_0\|_{L_2(\Omega)}\|w_0 - Rw_0\|_{L_2(\Omega)},\end{aligned}$$

$$= \|\dot{\chi}(0)\|_{L_2(\Omega)} \left\|\dot{\theta}(0)\right\|_{L_2(\Omega)},$$

since (2.2.4) satisfied hence $\|\dot{\chi}(0)\|_{L_2(\Omega)} \leq \left\|\dot{\theta}(0)\right\|_{L_2(\Omega)}$. As a result we can obtain

$$\frac{\rho}{2}\|\dot{\chi}(t)\|^2_{L_2(\Omega)} + \frac{1}{2}\|\chi(t)\|^2_V + \sum_{q=1}^{N_\varphi}\frac{1}{2\varphi_q}\|\varsigma_q(t)\|^2_V + \sum_{q=1}^{N_\varphi}\int_0^t \frac{\tau_q}{\varphi_q}\|\dot{\varsigma}_q(t')\|^2_V\,dt'$$
$$\leq \frac{\rho}{2}\left\|\dot{\theta}(0)\right\|^2_{L_2(\Omega)} + \sum_{q=1}^{N_\varphi} a(\varsigma_q(t),\chi(t)) + \int_0^t \rho\left(\ddot{\theta},\dot{\chi}\right)_{L_2(\Omega)} dt'.$$

With applying Cauchy-Schwarz inequality and Young's inequality, we have

$$\frac{\rho}{2}\|\dot{\chi}(t)\|^2_{L_2(\Omega)} + \frac{1}{2}\|\chi(t)\|^2_V + \sum_{q=1}^{N_\varphi}\frac{1}{2\varphi_q}\|\varsigma_q(t)\|^2_V + \sum_{q=1}^{N_\varphi}\int_0^t \frac{\tau_q}{\varphi_q}\|\dot{\varsigma}_q(t')\|^2_V\,dt'$$
$$\leq \frac{\rho}{2}\left\|\dot{\theta}(0)\right\|^2_{L_2(\Omega)} + \sum_{q=1}^{N_\varphi}\frac{\epsilon_q}{2}\|\chi(t)\|^2_V + \sum_{q=1}^{N_\varphi}\frac{1}{2\epsilon_q}\|\varsigma_q(t)\|^2_V + \int_0^t \rho\left\|\ddot{\theta}(t')\right\|_{L_2(\Omega)}\|\dot{\chi}(t')\|_{L_2(\Omega)}\,dt',$$

for positive constants $\{\epsilon_q\}$ for $q = 1,\ldots,N_\varphi$. In a similar way with the previous proofs, choosing $\epsilon_q = \varphi_q + \frac{\varphi_0}{2N_\varphi}$ provides

$$\frac{1}{2} - \sum_{q=1}^{N_\varphi}\frac{\epsilon_q}{2} = \frac{\varphi_0}{4} \text{ and } \frac{1}{2\varphi_q} - \frac{1}{2\epsilon_q} = \frac{\varphi_0}{4\varphi_q^2 + N_\varphi + \varphi_0} > 0 \text{ for } q = 1,\ldots,N_\varphi$$

so that

$$\frac{\rho}{2}\|\dot{\chi}(t)\|^2_{L_2(\Omega)} + \frac{\varphi_0}{4}\|\chi(t)\|^2_V + \sum_{q=1}^{N_\varphi}\frac{\varphi_0}{4\varphi_q^2 + N_\varphi + \varphi_0}\|\varsigma_q(t)\|^2_V + \sum_{q=1}^{N_\varphi}\int_0^t \frac{\tau_q}{\varphi_q}\|\dot{\varsigma}_q(t')\|^2_V\,dt'$$
$$\leq \frac{\rho}{2}\left\|\dot{\theta}(0)\right\|^2_{L_2(\Omega)} + \int_0^t \rho\left\|\ddot{\theta}(t')\right\|_{L_2(\Omega)}\|\dot{\chi}(t')\|_{L_2(\Omega)}\,dt',$$

Taking into account the L_∞ norm in time with Young's inequality on right hand side,

$$\frac{\rho}{2}\|\dot{\chi}(t)\|^2_{L_2(\Omega)} + \frac{\varphi_0}{4}\|\chi(t)\|^2_V + \sum_{q=1}^{N_\varphi}\frac{\varphi_0}{4\varphi_q^2 + N_\varphi + \varphi_0}\|\varsigma_q(t)\|^2_V + \sum_{q=1}^{N_\varphi}\int_0^t \frac{\tau_q}{\varphi_q}\|\dot{\varsigma}_q(t')\|^2_V\,dt'$$
$$\leq \frac{\rho}{2}\left\|\dot{\theta}(0)\right\|^2_{L_2(\Omega)} + \int_0^t \rho\left\|\ddot{\theta}(t')\right\|_{L_2(\Omega)}\,dt'\,\|\dot{\chi}\|_{L_\infty(0,T;L_2(\Omega))}$$
$$\leq \frac{\rho}{2}\left\|\dot{\theta}(0)\right\|^2_{L_2(\Omega)} + \rho\left(\int_0^T dt'\right)^{1/2}\left(\int_0^T \left\|\ddot{\theta}(t')\right\|^2_{L_2(\Omega)}\,dt'\right)^{1/2}\|\dot{\chi}\|_{L_\infty(0,T;L_2(\Omega))}$$
$$= \frac{\rho}{2}\left\|\dot{\theta}(0)\right\|^2_{L_2(\Omega)} + \rho\sqrt{T}\left\|\ddot{\theta}\right\|_{L_2(0,T;L_2(\Omega))}\|\dot{\chi}\|_{L_\infty(0,T;L_2(\Omega))}$$

$$\leq \frac{\rho}{2} \left\| \dot{\theta}(0) \right\|_{L_2(\Omega)}^2 + \frac{\rho T}{2\epsilon} \left\| \ddot{\theta} \right\|_{L_2(0,T;L_2(\Omega))}^2 + \frac{\rho\epsilon}{2} \left\| \dot{\chi} \right\|_{L_\infty(0,T;L_2(\Omega))}^2 ,$$

with a positive constant ϵ. If we set $\epsilon = \frac{1}{6}$ and consider the L_∞ norm in time with respect to $\|\dot{\chi}(t)\|_{L_2(\Omega)}$ and $\|\chi(t)\|_V$ on the left hand side, it is given that

$$\begin{aligned}
&\frac{\rho}{4} \|\dot{\chi}\|_{L_\infty(0,T;L_2(\Omega))}^2 + \frac{\varphi_0}{4} \|\chi\|_{L_\infty(0,T;V)}^2 + \sum_{q=1}^{N_\varphi} \frac{\varphi_0}{4\varphi_q^2 + N_\varphi + \varphi_0} \|\varsigma_q(t)\|_V^2 \\
&+ \sum_{q=1}^{N_\varphi} \int_0^t \frac{\tau_q}{\varphi_q} \|\dot{\varsigma}_q(t')\|_V^2 \, dt' \\
&\leq \frac{3\rho}{2} \left\| \dot{\theta}(0) \right\|_{L_2(\Omega)}^2 + 9\rho T \left\| \ddot{\theta} \right\|_{L_2(0,T;L_2(\Omega))}^2 .
\end{aligned}$$

Consequently, (1.4.8) leads us to have

$$\left\| \dot{\theta}(0) \right\|_{L_2(\Omega)}, \left\| \ddot{\theta} \right\|_{L_2(0,T;L_2(\Omega))} \leq Ch^{\min(k+1,s)-1}$$

so we can conclude

$$\begin{aligned}
&\frac{\rho}{4} \|\dot{\chi}\|_{L_\infty(0,T;L_2(\Omega))}^2 + \frac{\varphi_0}{4} \|\chi\|_{L_\infty(0,T;V)}^2 + \sum_{q=1}^{N_\varphi} \frac{\varphi_0}{4\varphi_q^2 + N_\varphi + \varphi_0} \|\varsigma_q(t)\|_V^2 \\
&+ \sum_{q=1}^{N_\varphi} \int_0^t \frac{\tau_q}{\varphi_q} \|\dot{\varsigma}_q(t')\|_V^2 \, dt' \\
&\leq Ch^{2(\min(k+1,s)-1)},
\end{aligned}$$

for some positive constant C. Furthermore, if elliptic regularity is satisfied,

$$\begin{aligned}
&\frac{\rho}{4} \|\dot{\chi}\|_{L_\infty(0,T;L_2(\Omega))}^2 + \frac{\varphi_0}{4} \|\chi\|_{L_\infty(0,T;V)}^2 + \sum_{q=1}^{N_\varphi} \frac{\varphi_0}{4\varphi_q^2 + N_\varphi + \varphi_0} \|\varsigma_q(t)\|_V^2 \\
&+ \sum_{q=1}^{N_\varphi} \int_0^t \frac{\tau_q}{\varphi_q} \|\dot{\varsigma}_q(t')\|_V^2 \, dt' \\
&\leq Ch^{2(\min(k+1,s))},
\end{aligned}$$

since

$$\left\| \dot{\theta}(0) \right\|_{L_2(\Omega)}, \left\| \ddot{\theta} \right\|_{L_2(0,T;L_2(\Omega))} = O(h^{\min(k+1,s)}).$$

Therefore, we can show that

$$\|\dot{\chi}\|_{L_\infty(0,T;L_2(\Omega))} + \|\chi\|_{L_\infty(0,T;V)} \leq Ch^{\min(k+1,s)-1}$$

and if elliptic regularity provided,

$$\|\dot{\chi}\|_{L_\infty(0,T;L_2(\Omega))} + \|\chi\|_{L_\infty(0,T;V)} \leq Ch^{\min(k+1,s)}$$

for some positive constant C. $\square$

Theorem 2.8. *Suppose* $u \in H^2(0,T;L_2(\Omega)) \cap W^1_\infty(0,T;H^s(\Omega))$. *Then*

$$\|u - u_h\|_{L_\infty(0,T;V)} \leq Ch^{\min(k+1,s)-1} \text{ and } \|\dot{u} - \dot{u}_h\|_{L_\infty(0,T;L_2(\Omega))} \leq Ch^{\min(k+1,s)-1}.$$

Furthermore, if elliptic regularity is given, we have

$$\|\dot{u} - \dot{u}_h\|_{L_\infty(0,T;L_2(\Omega))} \leq Ch^{\min(k+1,s)}.$$

Proof. By triangular inequality, we can derive

$$\begin{aligned}\|u - u_h\|_{L_\infty(0,T;V)} &= \|u - Ru - (u_h - Ru)\|_{L_\infty(0,T;V)} = \|\theta - \chi\|_{L_\infty(0,T;V)} \\ &\leq \|\theta\|_{L_\infty(0,T;V)} + \|\chi\|_{L_\infty(0,T;V)} \\ &\leq Ch^{\min(k+1,s)-1}\end{aligned}$$

for some positive C as following Lemma 2.6 and (1.4.8). In this manner, we can also obtain

$$\begin{aligned}\|\dot{u} - \dot{u}_h\|_{L_\infty(0,T;L_2(\Omega))} &= \|\dot{u} - R\dot{u} - (\dot{u}_h - R\dot{u})\|_{L_\infty(0,T;L_2(\Omega))} = \left\|\dot{\theta} - \dot{\chi}\right\|_{L_\infty(0,T;L_2(\Omega))} \\ &\leq \left\|\dot{\theta}\right\|_{L_\infty(0,T;L_2(\Omega))} + \|\dot{\chi}\|_{L_\infty(0,T;L_2(\Omega))} \\ &\leq Ch^{\min(k+1,s)-1},\end{aligned}$$

and if elliptic regularity is satisfied, (1.4.9) leads

$$\|\dot{u} - \dot{u}_h\|_{L_\infty(0,T;L_2(\Omega))} \leq Ch^{\min(k+1,s)}$$

for some positive C. $\square$

For the finite dimensional space V^h, we can derive the semidiscrete solution for the displacement form by solving the second order ODE system. By theory of ODEs and the stability bounds for **(P1)**, we solve the ODE system uniquely. Also, we can observe the error between the exact solution and the semidiscrete solution in L_2 estimates and H^1 estimates in space with using elliptic projection but without Grönwall's inequality. In this manner, we are going to deal with the velocity form.

2.2.2 Velocity Form

In a similar way with the semidiscrete formulation for **(P1)**, the semidiscrete formulation for **(P2)** is as follows: Find $u_h(t) \in V^h$ and $\zeta_{hq}(t) \in V^h$ for $hq = h1, h2, \ldots, hN_\varphi$, and for $0 \le t \le T$ such that for all $v \in V^h$,

$$(\rho\ddot{u}_h(t), v)_{L_2(\Omega)} + \varphi_0 a(u_h(t), v) \quad + \sum_{q=1}^{N_\varphi} a(\zeta_{hq}(t), v) \quad = F_v(t; v), \qquad (2.2.11)$$

$$\tau_q a\left(\dot{\zeta}_{hq}(t), v\right) + a(\zeta_{hq}(t), v) \quad = \tau_q\varphi_q a\left(\dot{u}_h(t), v\right), \qquad (2.2.12)$$

$$a(u_h(0), v) \quad = a(u_0, v), \qquad (2.2.13)$$

$$(\dot{u}_h(0), v)_{L_2(\Omega)} \quad = (w_0, v)_{L_2(\Omega)}, \qquad (2.2.14)$$

with $\zeta_{hq}(0) = 0$. Hence our approximate solution $u_h(\boldsymbol{x}, t)$ and $\{\zeta_{hq}\}_{q=1}^{N_\varphi}$ can be written as

$$u_h(\boldsymbol{x}, t) = \sum_{i=1}^{N_{Vh}} \mathfrak{u}_i(t)\Phi_i(\boldsymbol{x}), \qquad \zeta_{hq}(\boldsymbol{x}, t) = \sum_{i=1}^{N_{Vh}} \zeta_{hq,i}(t)\Phi_i(\boldsymbol{x}),$$

for each q and we have $\underline{\mathfrak{u}}(t) = (\mathfrak{u}_i(t))_{i=1}^{N_{Vh}}$ and $\underline{\zeta}_{hq}(t) = (\zeta_{hq,i}(t))_{i=1}^{N_{Vh}}$ for each q. From these results, (2.2.11)-(2.2.14) yield the following second order ODE system

$$\begin{aligned}
\rho M\ddot{\underline{\mathfrak{u}}}(t) + \varphi_0 A\underline{\mathfrak{u}}(t) + \sum_{q=1}^{N_\varphi} A\underline{\zeta}_{hq}(t) &= \tilde{\underline{F}}(t), \\
\tau_q\dot{\underline{\zeta}}_{hq}(t) + \underline{\zeta}_{hq}(t) &= \tau_q\varphi_q\dot{\underline{\mathfrak{u}}}(t), \text{ for each } q, \\
A\underline{\mathfrak{u}}(0) &= \underline{U}_0, \\
M\dot{\underline{\mathfrak{u}}}(0) &= \underline{W}_0, \\
\underline{\zeta}_{hq}(0) &= \underline{0}, \text{ for each } q,
\end{aligned}$$

where $(\tilde{\underline{F}}(t))_i = F_v(t; \Phi_i)$, and with $\underline{\zeta}_{hq}(0) = \underline{0}$, $\forall t$. Note that we know the mass matrix M and stiffness matrix A are invertible and the theory of second order ordinary differential equations allows us to have the existence and uniqueness of the solutions [57].

Theorem 2.9. (Stability bound for the semidiscrete solution of **(P2)**)
Let u_h and $\{\zeta_{hq}\}_{q=1}^{N_\varphi}$ be the semidiscrete solution of **(P2)**. *For any $t \in [0, T]$,*

$$\begin{aligned}
&\frac{\rho}{2}\|\dot{u}_h(t)\|_{L_2(\Omega)}^2 + \frac{\varphi_0}{4}\|u_h(t)\|_V^2 + \sum_{q=1}^{N_\varphi}\frac{1}{2\varphi_q}\|\zeta_{hq}(t)\|_V^2 + \sum_{q=1}^{N_\varphi}\int_0^t \frac{1}{\tau_q\varphi_q}\left\|\zeta_{hq}(t')\right\|_V^2 dt' \\
\le& C\bigg(\|w_0\|_{L_2(\Omega)}^2 + \|u_0\|_V^2 + \int_0^t \left\|f(t')\right\|_{L_2(\Omega)}^2 dt' + \int_0^t \left\|\dot{g}_N(t')\right\|_{L_2(\Gamma_N)}^2 dt' \\
&+ \|g_N(0)\|_{L_2(\Gamma_N)}^2 + \|g_N(t)\|_{L_2(\Gamma_N)}^2\bigg)
\end{aligned}$$

for some positive constant C which is independent of the weak solutions but depends on the domain, its boundary and the final time T.

Proof. The proof follows Theorem 2.3 since $V^h \subset V$ but $u_h(0)$ and $\dot{u}_h(0)$ should be dealt with more carefully here since $u_h(0) \neq u_0$ and $\dot{u}_h(0) \neq w_0$. Taking $v = \dot{u}_h(t)$ into (2.2.11). Then we have

$$\frac{\rho}{2}\|\dot{u}_h(t)\|^2_{L_2(\Omega)} + \frac{\varphi_0}{2}\|u_h(t)\|^2_V + \sum_{q=1}^{N_\varphi}\int_0^t a(\zeta_{hq}(t'), \dot{u}_h(t'))dt'$$
$$= \frac{\rho}{2}\|\dot{u}_h(0)\|^2_{L_2(\Omega)} + \frac{\varphi_0}{2}\|u_h(0)\|^2_V + \int_0^t F_v(t'; \dot{u}_h(t'))dt' \qquad (2.2.15)$$

with integration. As following the proof of Theorem 2.3 in exactly same way but u and ζ_q are replaced by u_h and ζ_{hq} for $q = 1, \ldots, N_\varphi$, (2.2.15) yields

$$\frac{\rho}{2}\|\dot{u}_h(t)\|^2_{L_2(\Omega)} + \frac{\varphi_0}{4}\|u_h(t)\|^2_V + \sum_{q=1}^{N_\varphi}\frac{1}{2\varphi_q}\|\zeta_{hq}(t)\|^2_V + \sum_{q=1}^{N_\varphi}\int_0^t \frac{1}{\tau_q\varphi_q}\left\|\zeta_{hq}(t')\right\|^2_V dt'$$
$$\leq C\Bigg(\|\dot{u}_h(0)\|^2_{L_2(\Omega)} + \|u_h(0)\|^2_V + \|u_0\|^2_V + \int_0^t \left\|f(t')\right\|^2_{L_2(\Omega)} dt' + \int_0^t \left\|\dot{g}_N(t')\right\|^2_{L_2(\Gamma_N)} dt'$$
$$+ \|g_N(0)\|^2_{L_2(\Gamma_N)} + \|g_N(t)\|^2_{L_2(\Gamma_N)}\Bigg),$$

for some positive C. Recall the facts that

$$\|\dot{u}_h(0)\|_{L_2(\Omega)} \leq \|w_0\|_{L_2(\Omega)} \text{ and } \|u_h(0)\|_V \leq \|u_0\|_V.$$

Therefore,

$$\frac{\rho}{2}\|\dot{u}_h(t)\|^2_{L_2(\Omega)} + \frac{\varphi_0}{4}\|u_h(t)\|^2_V + \sum_{q=1}^{N_\varphi}\frac{1}{2\varphi_q}\|\zeta_{hq}(t)\|^2_V + \sum_{q=1}^{N_\varphi}\int_0^t \frac{1}{\tau_q\varphi_q}\left\|\zeta_{hq}(t')\right\|^2_V dt'$$
$$\leq C\Bigg(\|w_0\|^2_{L_2(\Omega)} + \|u_0\|^2_V + \int_0^t \left\|f(t')\right\|^2_{L_2(\Omega)} dt' + \int_0^t \left\|\dot{g}_N(t')\right\|^2_{L_2(\Gamma_N)} dt'$$
$$+ \|g_N(0)\|^2_{L_2(\Gamma_N)} + \|g_N(t)\|^2_{L_2(\Gamma_N)}\Bigg).$$

□

In Theorem 2.9, we used Grönwall's inequality so that the constant C increases exponentially in time. However, we can also obtain the stability bound without Grönwall's inequality.

Theorem 2.10. *Suppose* $u_h \in W^1_\infty(0,T;L_2(\Omega)) \cap L_\infty(0,T;V)$ *and* $\zeta_{hq} \in H^1(0,T;V)$, $\forall q \in \{1,\ldots,N_\varphi\}$, *which are the semidiscrete solution to* **(P2)**. *Then we have the stability bound as*

$$\frac{\rho}{4}\|\dot{u}_h\|^2_{L_\infty(0,T;L_2(\Omega))} + \frac{\varphi_0}{4}\|u_h\|^2_{L_\infty(0,T;V)} + \sum_{q=1}^{N_\varphi}\frac{1}{2\varphi_q}\|\zeta_{hq}(t)\|^2_V + \sum_{q=1}^{N_\varphi}\int_0^t \frac{1}{\tau_q\varphi_q}\|\zeta_{hq}(t')\|^2_V dt'$$

$$\leq C\Big(\|w_0\|^2_{L_2(\Omega)} + \|u_0\|^2_V + \|f\|^2_{L_2(0,T;L_2(\Omega))} + \|\dot{g}_N\|^2_{L_2(0,T;L_2(\Gamma_N))} + \|g_N\|^2_{L_\infty(0,T;L_2(\Gamma_N))} \Big),$$

for any $t \in [0, T]$.

Proof. In a similar way with the proof of Theorem 2.4, since

$$\|\dot{u}_h(0)\|_{L_2(\Omega)} \leq \|w_0\|_{L_2(\Omega)} \text{ and } \|u_h(0)\|_V \leq \|u_0\|_V ,$$

we have

$$\frac{\rho}{4} \|\dot{u}_h\|^2_{L_\infty(0,T;L_2(\Omega))} + \frac{\varphi_0}{4} \|u_h\|^2_{L_\infty(0,T;V)} + \sum_{q=1}^{N_\varphi} \frac{1}{2\varphi_q} \|\zeta_{hq}(t)\|^2_V + \sum_{q=1}^{N_\varphi} \int_0^t \frac{1}{\tau_q\varphi_q} \|\zeta_{hq}(t')\|^2_V dt'$$
$$\leq C\Big(\|\dot{u}_h(0)\|^2_{L_2(\Omega)} + \|u_h(0)\|^2_V + \|u_0\|^2_V + \|f\|^2_{L_2(0,T;L_2(\Omega))} + \|\dot{g}_N\|^2_{L_2(0,T;L_2(\Gamma_N))}$$
$$+ \|g_N\|^2_{L_\infty(0,T;L_2(\Gamma_N))} \Big)$$
$$\leq C\Big(\|w_0\|^2_{L_2(\Omega)} + \|u_0\|^2_V + \|f\|^2_{L_2(0,T;L_2(\Omega))} + \|\dot{g}_N\|^2_{L_2(0,T;L_2(\Gamma_N))} + \|g_N\|^2_{L_\infty(0,T;L_2(\Gamma_N))} \Big).$$

□

As shown in Theorem 2.4, we proved the stability bounds for semidiscrete solution to **(P2)** without Grönwall's inequality hence also the constant C in Theorem 2.10 does not exponentially grow in time.

In order to consider the error bounds for the semidiscrete formulation of **(P2)**, we shall define

$$\begin{aligned} \theta &= u - Ru, \\ \nu_q &= \zeta_q - R\zeta_q, \ \forall q \in \{1, \dots, N_\varphi\}, \\ \chi &= u_h - Ru, \\ \Upsilon_q &= \zeta_{hq} - R\zeta_q, \ \forall q \in \{1, \dots, N_\varphi\}, \\ e_h &= u - u_h = \theta - \chi, \end{aligned}$$

where R is the elliptic projection operator.

Lemma 2.7. *Suppose* $u \in H^2(0, T; L_2(\Omega)) \cap W^1_\infty(0, T; H^s(\Omega))$. *Then we have*

$$\|\dot{\chi}\|_{L_\infty(0,T;L_2(\Omega))} + \|\chi\|_{L_\infty(0,T;V)} \leq Ch^{\min(k+1,s)-1}$$

and if elliptic regularity provided,

$$\|\dot{\chi}\|_{L_\infty(0,T;L_2(\Omega))} + \|\chi\|_{L_\infty(0,T;V)} \leq Ch^{\min(k+1,s)}$$

where C *is some positive constant.*

Proof. Subtracting (2.1.20) from (2.2.11) gives us for any $v \in V^h$

$$\begin{aligned}
&\rho\left(\ddot{\chi}(t), v\right)_{L_2(\Omega)} + \varphi_0 a\left(\chi(t), v\right) + \sum_{q=1}^{N_\varphi} a\left(\Upsilon_q(t), v\right) \\
&= \rho\left(\ddot{\theta}(t), v\right)_{L_2(\Omega)} + \varphi_0 a\left(\theta(t), v\right) + \sum_{q=1}^{N_\varphi} a\left(\nu_q(t), v\right),
\end{aligned} \tag{2.2.16}$$

so if we take $v = \dot{\chi}(t)$ in (2.2.16) then

$$\begin{aligned}
&\rho\left(\ddot{\chi}(t), \dot{\chi}(t)\right)_{L_2(\Omega)} + \varphi_0 a\left(\chi(t), \dot{\chi}(t)\right) + \sum_{q=1}^{N_\varphi} a\left(\Upsilon_q(t), \dot{\chi}(t)\right) \\
=&\rho\left(\ddot{\theta}(t), \dot{\chi}(t)\right)_{L_2(\Omega)} + \varphi_0 a\left(\theta(t), \dot{\chi}(t)\right) + \sum_{q=1}^{N_\varphi(t)} a\left(\nu_q(t), \dot{\chi}(t)\right), \\
=&\rho\left(\ddot{\theta}(t), \dot{\chi}(t)\right)_{L_2(\Omega)},
\end{aligned} \tag{2.2.17}$$

because of Galerkin orthogonality,

$$a\left(\theta(t), v\right) = 0, \qquad a\left(\nu_q(t), v\right) = 0,\ \forall q \in \{1, \ldots, N_\varphi\},$$

for any $v \in V^h$.

On the other hand, from the subtraction between (2.1.21) and (2.2.12), we have

$$\begin{aligned}
&\tau_q a\left(\dot{\Upsilon}_q(t), v\right) + a\left(\Upsilon_q(t), v\right) - \tau_q \varphi_q a\left(\dot{\chi}(t), v\right) \\
&= \tau_q a\left(\dot{\nu}_q(t), v\right) + a\left(\nu_q(t), v\right) - \tau_q \varphi_q a\left(\dot{\theta}(t), v\right)
\end{aligned}$$

for any $v \in V^h$, $\forall q \in \{1, \ldots, N_\varphi\}$. Set $v = \Upsilon_q(t)$ then

$$\begin{aligned}
&\tau_q a\left(\dot{\Upsilon}_q(t), \Upsilon_q(t)\right) + a\left(\Upsilon_q(t), \Upsilon_q(t)\right) - \tau_q \varphi_q a\left(\dot{\chi}(t), \Upsilon_q(t)\right), \\
=&\tau_q a\left(\dot{\nu}_q(t), \Upsilon_q(t)\right) + a\left(\nu_q(t), \Upsilon_q(t)\right) - \tau_q \varphi_q a\left(\dot{\theta}(t), \Upsilon_q(t)\right), \\
=&0,
\end{aligned}$$

since

$$a\left(\dot{\nu}_q(t), v\right) = 0,\ a\left(\nu_q(t), v\right) = 0,\ a\left(\dot{\theta}(t), v\right) = 0,$$

for any $v \in V^h$ by Galerkin orthogonality. Hence we obtain

$$a\left(\Upsilon_q(t), \dot{\chi}(t)\right) = \frac{1}{\varphi_q} a\left(\dot{\Upsilon}_q(t), \Upsilon_q(t)\right) + \frac{1}{\tau_q \varphi_q} \left\|\Upsilon_q(t)\right\|_V^2. \tag{2.2.18}$$

Turning to the main proof, with integration of (2.2.17) over time, using (2.2.18) gives

$$
\begin{aligned}
&\rho\int_0^t \left(\ddot{\chi}(t'),\dot{\chi}(t')\right)_{L_2(\Omega)} dt' + \varphi_0\int_0^t a\left(\chi(t'),\dot{\chi}(t')\right) dt' + \sum_{q=1}^{N_\varphi}\int_0^t \frac{1}{\varphi_q} a\left(\dot{\Upsilon}_q(t'),\Upsilon_q(t')\right) dt' \\
&\quad+\sum_{q=1}^{N_\varphi}\int_0^t \frac{1}{\tau_q\varphi_q}\left\|\Upsilon_q(t')\right\|_V^2 dt', \\
&=\frac{\rho}{2}\left(\|\dot{\chi}(t)\|_{L_2(\Omega)}^2 - \|\dot{\chi}(0)\|_{L_2(\Omega)}^2\right) + \frac{\varphi_0}{2}\left(\|\chi(t)\|_V^2 - \|\chi(0)\|_V^2\right) \\
&\quad+\sum_{q=1}^{N_\varphi}\frac{1}{2\varphi_q}\left(\|\Upsilon_q(t)\|_V^2 - \|\Upsilon_q(0)\|_V^2\right) + \sum_{q=1}^{N_\varphi}\int_0^t \frac{1}{\tau_q\varphi_q}\left\|\Upsilon_q(t')\right\|_V^2 dt', \\
&=\rho\int_0^t \left(\ddot{\theta}(t'),\dot{\chi}(t')\right)_{L_2(\Omega)} dt'.
\end{aligned}
$$

Note that (2.2.13) and elliptic projection lead

$$a(u_h(0),v) = a(Ru_0,v) \text{ and } a(u_h(0)-Ru_0,v) = 0,\ \forall v \in V^h.$$

so that

$$\|\chi(0)\|_V^2 = a(u_h(0)-Ru_0, u_h(0)-Ru_0) = 0.$$

Also, the initial condition $\zeta_q(0) = 0 = \zeta_{hq}(0)$ implies

$$\|\Upsilon_q(0)\|_V^2 = 0,\ \forall q \in \{1,\ldots,N_\varphi\}.$$

In addition,

$$
\begin{aligned}
\|\dot{\chi}(0)\|_{L_2(\Omega)}^2 &= \left(\dot{u}_h(0)-Rw_0, \dot{u}_h(0)-Rw_0\right)_{L_2(\Omega)}, \\
&= \left(\dot{u}_h(0)-Rw_0, w_0-Rw_0\right)_{L_2(\Omega)}, \\
&\leq \|\dot{u}_h(0)-Rw_0\|_{L_2(\Omega)} \|w_0-Rw_0\|_{L_2(\Omega)}, \\
&= \|\dot{\chi}(0)\|_{L_2(\Omega)} \left\|\dot{\theta}(0)\right\|_{L_2(\Omega)},
\end{aligned}
$$

since (2.2.14) satisfied, then $\|\dot{\chi}(0)\|_{L_2(\Omega)} \leq \left\|\dot{\theta}(0)\right\|_{L_2(\Omega)}$. From these above results, with applying Cauchy-Schwarz inequality and Young's inequality, we can obtain

$$
\begin{aligned}
&\frac{\rho}{2}\|\dot{\chi}(t)\|_{L_2(\Omega)}^2 + \frac{\varphi_0}{2}\|\chi(t)\|_V^2 + \sum_{q=1}^{N_\varphi}\frac{1}{2\varphi_q}\|\Upsilon_q(t)\|_V^2 + \sum_{q=1}^{N_\varphi}\int_0^t \frac{1}{\tau_q\varphi_q}\left\|\Upsilon_q(t')\right\|_V^2 dt', \\
&\leq\frac{\rho}{2}\left\|\dot{\theta}(0)\right\|_{L_2(\Omega)}^2 + \frac{\rho}{2\epsilon}\int_0^t \left\|\ddot{\theta}(t')\right\|_{L_2(\Omega)}^2 dt' + \frac{\rho\epsilon}{2}\int_0^t \|\dot{\chi}(t')\|_{L_2(\Omega)}^2 dt', \\
&\leq\frac{\rho}{2}\left\|\dot{\theta}(0)\right\|_{L_2(\Omega)}^2 + \frac{\rho}{2\epsilon}\left\|\ddot{\theta}\right\|_{L_2(0,T;L_2(\Omega))}^2 + \frac{\rho\epsilon}{2}\|\dot{\chi}\|_{L_\infty(0,T;L_2(\Omega))}^2\int_0^T dt'
\end{aligned}
$$

(by L_∞ norm in time and for $t \leq T$)

$$\leq \frac{\rho}{2}\left\|\dot{\theta}(0)\right\|^2_{L_2(\Omega)} + \frac{\rho}{2\epsilon}\left\|\ddot{\theta}\right\|^2_{L_2(0,T;L_2(\Omega))} + \frac{\rho\epsilon}{2}T\left\|\dot{\chi}\right\|^2_{L_\infty(0,T;L_2(\Omega))},$$

for some positive ϵ. Thus, if we take $\epsilon = \frac{1}{6T}$ and consider L_∞ norm in time with respect to $\|\dot{\chi}(t)\|_{L_2(\Omega)}$ and $\|\chi(t)\|_V$ on the left hand side, we can obtain

$$\frac{\rho}{4}\|\dot{\chi}\|^2_{L_\infty(0,T;L_2(\Omega))} + \frac{\varphi_0}{2}\|\chi\|^2_{L_\infty(0,T;V)} + \sum_{q=1}^{N_\varphi}\frac{1}{2\varphi_q}\|\Upsilon_q(t)\|^2_V + \sum_{q=1}^{N_\varphi}\int_0^t \frac{1}{\tau_q\varphi_q}\left\|\Upsilon_q(t')\right\|^2_V dt',$$
$$\leq \frac{3\rho}{2}\left\|\dot{\theta}(0)\right\|^2_{L_2(\Omega)} + 9\rho T\left\|\ddot{\theta}\right\|^2_{L_\infty(0,T;L_2(\Omega))}$$
$$\leq Ch^{2(\min(k+1,s)-1)}.$$

Therefore it is also true that

$$\|\dot{\chi}\|_{L_\infty(0,T;L_2(\Omega))} + \|\chi\|_{L_\infty(0,T;V)} \leq Ch^{\min(k+1,s)-1}.$$

If we assume elliptic regularity, we have

$$\left\|\dot{\theta}(0)\right\|_{L_2(\Omega)} \leq Ch^{\min(k+1,s)}, \qquad \left\|\ddot{\theta}\right\|_{L_2(0,T;L_2(\Omega))} \leq Ch^{\min(k+1,s)}$$

for some positive C and so

$$\|\dot{\chi}\|_{L_\infty(0,T;L_2(\Omega))} + \|\chi\|_{L_\infty(0,T;V)} \leq Ch^{\min(k+1,s)}.$$

□

From Lemma 2.7, we can show error estimates by using the properties of elliptic projection (1.4.8) and (1.4.9).

Theorem 2.11. *Suppose $u \in H^2(0,T;L_2(\Omega)) \cap W^1_\infty(0,T;H^s(\Omega))$ for $s \in \mathbb{N}$. Then*

$$\|u - u_h\|_{L_\infty(0,T;V)} \leq Ch^{\min(k+1,s)-1} \text{ and } \|\dot{u} - \dot{u}_h\|_{L_\infty(0,T;L_2(\Omega))} \leq Ch^{\min(k+1,s)-1}.$$

Moreover,

$$\|\dot{u} - \dot{u}_h\|_{L_\infty(0,T;L_2(\Omega))} \leq Ch^{\min(k+1,s)}$$

if elliptic regularity is satisfied.

Proof. Note that Lemma 2.7 gives us

$$\|\chi\|_{L_\infty(0,T;V)} + \|\dot{\chi}\|_{L_\infty(0,T;L_2(\Omega))} \leq Ch^{\min(k+1,s)-1}$$

for some positive C.

By the definition and triangular inequality, we can derive

$$\|u - u_h\|_{L_\infty(0,T;V)} = \|u - Ru - (u_h - Ru)\|_{L_\infty(0,T;V)} = \|\theta - \chi\|_{L_\infty(0,T;V)}$$

$$
\begin{aligned}
&\leq \|\theta\|_{L_\infty(0,T;V)} + \|\chi\|_{L_\infty(0,T;V)} \\
&\leq C h^{\min(k+1,s)-1}
\end{aligned}
$$

for some positive C by (1.4.8) and Lemma 2.7. In this same way, we can also obtain

$$
\begin{aligned}
\|\dot{u} - \dot{u}_h\|_{L_\infty(0,T;L_2(\Omega))} &= \|\dot{u} - R\dot{u} - (\dot{u}_h - R\dot{u})\|_{L_\infty(0,T;L_2(\Omega))} = \left\|\dot{\theta} - \dot{\chi}\right\|_{L_\infty(0,T;L_2(\Omega))} \\
&\leq \left\|\dot{\theta}\right\|_{L_\infty(0,T;L_2(\Omega))} + \|\dot{\chi}\|_{L_\infty(0,T;L_2(\Omega))} \\
&\leq C h^{\min(k+1,s)-1}
\end{aligned}
$$

and if elliptic regularity is satisfied

$$
\|\dot{u} - \dot{u}_h\|_{L_\infty(0,T;L_2(\Omega))} \leq C h^{\min(k+1,s)}
$$

for some positive C from (1.4.9) and Lemma 2.7. □

In order to solve the model problem numerically, it is necessary to introduce discretisation in time. For the next step, we are dealing with Crank-Nicolson finite difference method for time discretisation and so define fully discrete formulations for the displacement form and the velocity form, respectively. In a similar way with semidiscrete formulations, we will consider the stability and the error bounds.

2.3 Fully Discrete Formulation for CGFEM

We can obtain fully discrete forms when we apply finite difference methods in time to the semidiscrete forms. A variety of finite difference schemes allow us to have various numerical simulations with different convergence rates and stability conditions with respect to time steps. Moreover, our numerical solution U_h can be expressed as

$$
U_h(\boldsymbol{x}, t_n) = U_h^n = \sum_{i=1}^{N_{Vh}} \mathfrak{u}_i^n \Phi_i(\boldsymbol{x}),
$$

for $t_n = n\Delta t$, where $\Delta t > 0$ such that $T = N\Delta t$, $N \in \mathbb{N}$, for $n = 0, \ldots, N$. With this in mind, the fully discrete formulation is determined by Crank-Nicolson method. Suppose W_h^n denotes the approximation to first derivative in time at $t = t_n$ with the relation

$$
\frac{W_h^{n+1} + W_h^n}{2} = \frac{U_h^{n+1} - U_h^n}{\Delta t}. \tag{2.3.1}
$$

Moreover, we will recall and use time average notation. For example,

$$
\bar{f}^n = \frac{f(t_{n+1}) + f(t_n)}{2} \text{ and } \bar{g}_N^n = \frac{g_N(t_{n+1}) + g_N(t_n)}{2},
$$

and for any v

$$
\bar{F}_d^n(v) = \frac{F_d(t_{n+1}; v) + F_d(t_n; v)}{2} \text{ and } \bar{F}_v^n(v) = \frac{F_v(t_{n+1}; v) + F_v(t_n; v)}{2}.
$$

2.3.1 Displacement Form

(P1) Find $u(t)$ and $\{\psi_q(t)\}_{q=1}^{N_\varphi}$ such that for all $v \in V$

$$(\rho\ddot{u}(t), v)_{L_2(\Omega)} + a(u(t), v) - \sum_{q=1}^{N_\varphi} a(\psi_q(t), v) = F_d(t; v),$$

$$\tau_q a(\dot{\psi}_q(t), v) + a(\psi_q(t), v) = \varphi_q a(u(t), v) \ \forall q \in \{1, \dots, N_\varphi\},$$

with $u(0) = u_0$, $\dot{u}(0) = w_0$ and $\psi_q(0) = 0$, $\forall q \in \{1, \dots, N_\varphi\}$.

With applying Crank-Nicolson method, the fully discrete formulation for **(P1)** can be defined as follows:
Find U_h^n, W_h^n and $\Psi_{hq}^n \in V^h$ for $n = 0, \dots, N$, $\forall q \in \{1, \dots, N_\varphi\}$ such that for $n = 0, \dots, N-1$

$$\left(\rho\frac{W_h^{n+1} - W_h^n}{\Delta t}, v\right)_{L_2(\Omega)} + a\left(\frac{U_h^{n+1} + U_h^n}{2}, v\right) - \sum_{q=1}^{N_\varphi} a\left(\frac{\Psi_{hq}^{n+1} + \Psi_{hq}^n}{2}, v\right) = \bar{F}_d^n(v), \tag{2.3.2}$$

$$\tau_q a\left(\frac{\Psi_{hq}^{n+1} - \Psi_{hq}^n}{\Delta t}, v\right) + a\left(\frac{\Psi_{hq}^{n+1} + \Psi_{hq}^n}{2}, v\right) = \varphi_q a\left(\frac{U_h^{n+1} + U_h^n}{2}, v\right), \ \forall q \in \{1, \dots, N_\varphi\}, \tag{2.3.3}$$

$$a(U_h^0, v) = a(u_0, v)\ , \tag{2.3.4}$$

$$\left(W_h^0, v\right)_{L_2(\Omega)} = (w_0, v)_{L_2(\Omega)}\ . \tag{2.3.5}$$

In a similar way with the semidiscrete formulation, we can derive

$$\underline{\mathfrak{u}}^0 = A^{-1}\underline{U}_0,$$

and if we set

$$W_h^n = \sum_{i=1}^{N_{Vh}} \mathfrak{w}_i^n \Phi_i(\boldsymbol{x}), \qquad \Psi_{hq}^n = \sum_{i=1}^{N_{Vh}} \Psi_{hq,i}^n \Phi_i(\boldsymbol{x}),$$

we have

$$\underline{\mathfrak{w}}^0 = M^{-1}\underline{W}_0.$$

Since A is invertible and $\underline{\Psi}_{hq}^0 = \underline{0}$, $\forall q \in \{1, \dots, N_\varphi\}$, for $n = 0$ (2.3.3) provides

$$\left(\frac{\tau_q}{\Delta t} + \frac{1}{2}\right)\underline{\Psi}_{hq}^1 = \varphi_q\left(\frac{\underline{\mathfrak{u}}^1 + \underline{\mathfrak{u}}^0}{2}\right). \tag{2.3.6}$$

And (2.3.2) implies

$$\frac{\rho}{\Delta t}M(\underline{\mathfrak{w}}^1-\underline{\mathfrak{w}}^0)+\frac{1}{2}A(\underline{\mathfrak{u}}^1+\underline{\mathfrak{u}}^0)-\sum_{q=1}^{N_\varphi}\frac{1}{2}A\underline{\Psi}_{hq}^1=\frac{1}{2}(\underline{F}^1+\underline{F}^0), \tag{2.3.7}$$

where $(\underline{F}^n)_i = F_d(t_n;\Phi_i)$. From the relation (2.3.1), we have

$$\underline{\mathfrak{w}}^1=\frac{2}{\Delta t}(\underline{\mathfrak{u}}^1-\underline{\mathfrak{u}}^0)-\underline{\mathfrak{w}}^0, \tag{2.3.8}$$

and so (2.3.6) yields

$$\begin{aligned}&\frac{\rho}{\Delta t}M\left(\frac{2}{\Delta t}(\underline{\mathfrak{u}}^1-\underline{\mathfrak{u}}^0)-2\underline{\mathfrak{w}}^0\right)+\frac{1}{2}A\left(\underline{\mathfrak{u}}^1+\underline{\mathfrak{u}}^0\right)\\&-\sum_{q=1}^{N_\varphi}\left(\frac{\tau_q}{\Delta t}+\frac{1}{2}\right)^{-1}\frac{\varphi_q}{2}A\left(\underline{\mathfrak{u}}^1+\underline{\mathfrak{u}}^0\right)=\frac{1}{2}(\underline{F}^1+\underline{F}^0),\end{aligned}$$

and so

$$\begin{aligned}&\left(\frac{2\rho}{\Delta t^2}M+\frac{1}{2}\left(1-\sum_{q=1}^{N_\varphi}\frac{\varphi_q}{2}\left(\frac{\tau_q}{\Delta t}+\frac{1}{2}\right)^{-1}\right)A\right)\underline{\mathfrak{u}}^1\\&=\frac{2\rho}{\Delta t}M\underline{\mathfrak{w}}^0+\left(\frac{2\rho}{\Delta t^2}M-\frac{1}{2}\left(1-\sum_{q=1}^{N_\varphi}\frac{\varphi_q}{2}\left(\frac{\tau_q}{\Delta t}+\frac{1}{2}\right)^{-1}\right)A\right)\underline{\mathfrak{u}}^0+\frac{1}{2}(\underline{F}^1+\underline{F}^0).\end{aligned}$$

Let us define the matrix $\mathcal{A}$ by

$$\mathcal{A}:=\frac{1}{2}\left(1-\sum_{q=1}^{N_\varphi}\frac{\varphi_q}{2}\left(\frac{\tau_q}{\Delta t}+\frac{1}{2}\right)^{-1}\right)A.$$

If the matrix $\frac{2\rho}{\Delta t^2}M+\mathcal{A}$ is invertible, since $\underline{\mathfrak{w}}^0$, $\underline{\mathfrak{u}}^0$ and $\underline{F}^n$ are known, we can obtain $\underline{\mathfrak{u}}^1$. Eventually, we can also derive $\underline{\mathfrak{w}}^1$ and $\underline{\Psi}_{hq}^1$, $\forall q\in\{1,\dots,N_\varphi\}$ by (2.3.8) and (2.3.6). In this manner, we can solve the following system for $n=1,\dots,N-1$

$$\begin{aligned}\underline{\mathfrak{u}}^{n+1}=&\left(\frac{2\rho}{\Delta t^2}M+\mathcal{A}\right)^{-1}\left[\frac{2\rho}{\Delta t}M\underline{\mathfrak{w}}^n+\left(\frac{2\rho}{\Delta t^2}M-\mathcal{A}\right)\underline{\mathfrak{u}}^n\right.\\&\left.+\sum_{q=1}^{N_\varphi}\frac{2\tau_q}{2\tau_q+\Delta t}A\underline{\Psi}_{hq}^n+\frac{1}{2}(\underline{F}^{n+1}+\underline{F}^n)\right] \qquad (2.3.9)\\\underline{\mathfrak{w}}^{n+1}=&\frac{2}{\Delta t}(\underline{\mathfrak{u}}^{n+1}-\underline{\mathfrak{u}}^n)-\underline{\mathfrak{w}}^n, \qquad (2.3.10)\\\underline{\Psi}_{hq}^{n+1}=&\frac{2\Delta t}{2\tau_q+\Delta t}\left(\frac{2\tau_q-\Delta t}{2\Delta t}\underline{\Psi}_{hq}^n+\frac{\varphi_q}{2}(\underline{\mathfrak{u}}^{n+1}+\underline{\mathfrak{u}}^n)\right),\ \forall q\in\{1,\dots,N_\varphi\}. \qquad (2.3.11)\end{aligned}$$

By (2.3.9)-(2.3.11), our approximation solution can be computed. But we shall show the matrix $\frac{2\rho}{\Delta t^2}M + \mathcal{A}$ is invertible to solve (2.3.9) uniquely. In order to do that, we should consider the stability bounds. The resulting linear system from (2.3.2)-(2.3.5) has the existence and uniqueness of the solution by the stability bounds. It would be dealt later in detail.

From now on, we will consider the stability for (2.3.2)-(2.3.5). In order to observe the stability theorem, following lemmas should be introduced.

Lemma 2.8. *For any $m \in \mathbb{N}$ such that $1 \le m \le N$,*

$$\rho \|W_h^m\|_{L_2(\Omega)}^2 + \|U_h^m\|_V^2 = \left\|W_h^0\right\|_{L_2(\Omega)}^2 + \left\|U_h^0\right\|_V^2 + \Delta t \sum_{n=1}^{m-1} \bar{F}_d^n \left(W_h^{n+1} + W_h^n\right)$$
$$+ \sum_{n=0}^{m-1} \sum_{q=1}^{N_\varphi} a(\Psi_{hq}^{n+1} + \Psi_{hq}^n, U_h^{n+1} - U_h^n).$$

Proof. Let $v = W_h^{n+1} + W_h^n$ for $0 \le n \le m-1$. Then (2.3.2) yields

$$\frac{\rho}{\Delta t}\left(\|W_h^{n+1}\|_{L_2(\Omega)}^2 - \|W_h^n\|_{L_2(\Omega)}^2\right) + \frac{1}{2}a(U_h^{n+1} + U_h^n, W_h^{n+1} + W_h^n)$$
$$= \bar{F}_d^n \left(W_h^{n+1} + W_h^n\right) + \frac{1}{2}\sum_{q=1}^{N_\varphi} a(\Psi_{hq}^{n+1} + \Psi_{hq}^n, W_h^{n+1} + W_h^n).$$

By (2.3.1), we have

$$W_h^{n+1} + W_h^n = \frac{2}{\Delta t}(U_h^{n+1} - U_h^n),$$

hence with multiplying Δt on both sides,

$$\rho\left(\|W_h^{n+1}\|_{L_2(\Omega)}^2 - \|W_h^n\|_{L_2(\Omega)}^2\right) + \left\|U_h^{n+1}\right\|_V^2 - \|U_h^n\|_V^2$$
$$= \Delta t \bar{F}_d^n \left(W_h^{n+1} + W_h^n\right) + \sum_{q=1}^{N_\varphi} a(\Psi_{hq}^{n+1} + \Psi_{hq}^n, U_h^{n+1} - U_h^n).$$

Taking into account the summation from $n = 0$ to $n = m-1$,

$$\rho\left(\|W_h^m\|_{L_2(\Omega)}^2 - \left\|W_h^0\right\|_{L_2(\Omega)}^2\right) + \|U_h^m\|_V^2 - \left\|U_h^0\right\|_V^2$$
$$= \Delta t \sum_{n=1}^{m-1} \bar{F}_d^n \left(W_h^{n+1} + W_h^n\right) + \sum_{n=0}^{m-1} \sum_{q=1}^{N_\varphi} a(\Psi_{hq}^{n+1} + \Psi_{hq}^n, U_h^{n+1} - U_h^n).$$

Thus,

$$\rho \|W_h^m\|_{L_2(\Omega)}^2 + \|U_h^m\|_V^2 = \rho \left\|W_h^0\right\|_{L_2(\Omega)}^2 + \left\|U_h^0\right\|_V^2 + \Delta t \sum_{n=1}^{m-1} \bar{F}_d^n \left(W_h^{n+1} + W_h^n\right)$$

$$+\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi} a(\Psi_{hq}^{n+1}+\Psi_{hq}^{n}, U_h^{n+1}-U_h^n).$$

□

Lemma 2.9. *For any $q \in \{1, \ldots, N_\varphi\}$, it is satisfied that for any $m \in \mathbb{N}$ such that $1 \leq m \leq N$,*

$$\sum_{n=0}^{m-1} a(U_h^{n+1}-U_h^n, \Psi_{hq}^{n+1}+\Psi_{hq}^{n}) = 2a(U_h^m, \Psi_{hq}^m) - \frac{2\tau_q}{\Delta t \varphi_q}\sum_{n=0}^{m-1}\left\|\Psi_{hq}^{n+1}-\Psi_{hq}^{n}\right\|_V^2 - \frac{1}{\varphi_q}\left\|\Psi_{hq}^m\right\|_V^2 .$$

Proof. Put $v = \Psi_{hq}^{n+1} - \Psi_{hq}^n$ into (2.3.3). Then

$$\frac{\tau_q}{\Delta t}\left\|\Psi_{hq}^{n+1}-\Psi_{hq}^{n}\right\|_V^2 + \frac{1}{2}\left(\left\|\Psi_{hq}^{n+1}\right\|_V^2 - \left\|\Psi_{hq}^{n}\right\|_V^2\right) = \frac{\varphi_q}{2} a(U_h^{n+1}+U_h^n, \Psi_{hq}^{n+1}-\Psi_{hq}^n). \tag{2.3.12}$$

Since

$$\begin{aligned} a(U_h^{n+1}, \Psi_{hq}^{n+1}-\Psi_{hq}^n) &= a(U_h^{n+1}, \Psi_{hq}^{n+1}) - a(U_h^n, \Psi_{hq}^n) + a(U_h^n, \Psi_{hq}^n) - a(U_h^{n+1}, \Psi_{hq}^n),\\ &= a(U_h^{n+1}, \Psi_{hq}^{n+1}) - a(U_h^n, \Psi_{hq}^n) - a(U_h^{n+1}-U_h^n, \Psi_{hq}^n), \end{aligned}$$

and

$$\begin{aligned} a(U_h^{n}, \Psi_{hq}^{n+1}-\Psi_{hq}^n) &= a(U_h^{n+1}, \Psi_{hq}^{n+1}) - a(U_h^{n+1}, \Psi_{hq}^{n+1}) - a(U_h^n, \Psi_{hq}^n) + a(U_h^{n}, \Psi_{hq}^{n+1}),\\ &= a(U_h^{n+1}, \Psi_{hq}^{n+1}) - a(U_h^n, \Psi_{hq}^n) - a(U_h^{n+1}-U_h^n, \Psi_{hq}^{n+1}), \end{aligned}$$

(2.3.12) implies

$$\begin{aligned} &\frac{\tau_q}{\Delta t}\left\|\Psi_{hq}^{n+1}-\Psi_{hq}^{n}\right\|_V^2 + \frac{1}{2}\left(\left\|\Psi_{hq}^{n+1}\right\|_V^2 - \left\|\Psi_{hq}^{n}\right\|_V^2\right)\\ =&\frac{\varphi_q}{2} a(U_h^{n+1}+U_h^n, \Psi_{hq}^{n+1}-\Psi_{hq}^n)\\ =&\frac{\varphi_q}{2}\left(a(U_h^{n+1}, \Psi_{hq}^{n+1}-\Psi_{hq}^n) + a(U_h^{n}, \Psi_{hq}^{n+1}-\Psi_{hq}^n)\right)\\ =&\varphi_q\left(a(U_h^{n+1}, \Psi_{hq}^{n+1}) - a(U_h^n, \Psi_{hq}^n)\right) - \frac{\varphi_q}{2} a(U_h^{n+1}-U_h^n, \Psi_{hq}^{n+1}+\Psi_{hq}^n). \end{aligned}$$

It yields

$$\begin{aligned} a(U_h^{n+1}-U_h^n, \Psi_{hq}^{n+1}+\Psi_{hq}^{n}) =& 2\left(a(U_h^{n+1}, \Psi_{hq}^{n+1}) - a(U_h^n, \Psi_{hq}^n)\right) - \frac{2\tau_q}{\Delta t\varphi_q}\left\|\Psi_{hq}^{n+1}-\Psi_{hq}^{n}\right\|_V^2\\ &- \frac{1}{\varphi_q}\left(\left\|\Psi_{hq}^{n+1}\right\|_V^2 - \left\|\Psi_{hq}^{n}\right\|_V^2\right). \end{aligned}$$

Thus, with taking summation from $n = 0$ to $n = m - 1$, we have

$$\begin{aligned}
&\sum_{n=0}^{m-1} a(U_h^{n+1} - U_h^n, \Psi_{hq}^{n+1} + \Psi_{hq}^n) \\
=&2a(U_h^m, \Psi_{hq}^m) - 2a(U_h^0, \Psi_{hq}^0) - \frac{2\tau_q}{\Delta t \varphi_q} \sum_{n=0}^{m-1} \left\| \Psi_{hq}^{n+1} - \Psi_{hq}^n \right\|_V^2 - \frac{1}{\varphi_q} \left(\left\| \Psi_{hq}^m \right\|_V^2 - \left\| \Psi_{hq}^0 \right\|_V^2 \right) \\
=&2a(U_h^m, \Psi_{hq}^m) - \frac{2\tau_q}{\Delta t \varphi_q} \sum_{n=0}^{m-1} \left\| \Psi_{hq}^{n+1} - \Psi_{hq}^n \right\|_V^2 - \frac{1}{\varphi_q} \left\| \Psi_{hq}^m \right\|_V^2
\end{aligned}$$

since $\Psi_{hq}^0 = 0$, $\forall q \in \{1, \dots, N_\varphi\}$. □

By Lemma 2.8 and 2.9, *a priori* bound for **(P1)** can be observed.
Note that (2.3.4) and (2.3.5) implies

$$\left\| U_h^0 \right\|_V^2 = a(U_h^0, U_h^0) = a(u_0, U_h^0) \le \left\| U_h^0 \right\|_V \left\| u_0 \right\|_V,$$

and

$$\left\| W_h^0 \right\|_{L_2(\Omega)}^2 = \left(W_h^0, W_h^0 \right)_{L_2(\Omega)} = \left(w_0, W_h^0 \right)_{L_2(\Omega)} \le \left\| W_h^0 \right\|_{L_2(\Omega)} \left\| w_0 \right\|_{L_2(\Omega)},$$

so that we have

$$\left\| U_h^0 \right\|_V \le \left\| u_0 \right\|_V, \tag{2.3.13}$$

$$\left\| W_h^0 \right\|_{L_2(\Omega)} \le \left\| w_0 \right\|_{L_2(\Omega)}. \tag{2.3.14}$$

Theorem 2.12. *For any $m \in \mathbb{N}$ such that $1 \le m \le N$, there exists a positive constant C such that*

$$\begin{aligned}
&\frac{\rho}{2} \left\| W_h^m \right\|_{L_2(\Omega)}^2 + \frac{\varphi_0}{4} \left\| U_h^m \right\|_V^2 + \sum_{q=1}^{N_\varphi} \frac{\varphi_0}{2\varphi_q^2 N_\varphi + \varphi_q \varphi_0} \left\| \Psi_{hq}^m \right\|_V^2 \\
&+ \sum_{q=1}^{N_\varphi} \sum_{n=0}^{m-1} \frac{2\tau_q}{\Delta t \varphi_q} \left\| \Psi_{hq}^{n+1} - \Psi_{hq}^n \right\|_V^2 \\
\le& C \Bigg(\left\| w_0 \right\|_{L_2(\Omega)}^2 + \left\| u_0 \right\|_V^2 + \left\| \bar{g}_N^{m-1} \right\|_{L_2(\Gamma_N)}^2 + \left\| \bar{g}_N^0 \right\|_{L_2(\Gamma_N)}^2 \\
&+ \Delta t \sum_{n=0}^{m-1} \left\| \bar{f}^n \right\|_{L_2(\Omega)}^2 + \int_0^{t_m} \left\| \dot{g}_N(t') \right\|_{L_2(\Gamma_N)}^2 dt' \Bigg).
\end{aligned}$$

Proof. Recall Lemma 2.8 and 2.9, then we have

$$\rho \left\| W_h^m \right\|_{L_2(\Omega)}^2 + \left\| U_h^m \right\|_V^2 + \sum_{q=1}^{N_\varphi} \frac{1}{\varphi_q} \left\| \Psi_{hq}^m \right\|_V^2 + \sum_{q=1}^{N_\varphi} \sum_{n=0}^{m-1} \frac{2\tau_q}{\Delta t \varphi_q} \left\| \Psi_{hq}^{n+1} - \Psi_{hq}^n \right\|_V^2$$

$$=\rho \left\|W_h^0\right\|_{L_2(\Omega)}^2 + \left\|U_h^0\right\|_V^2 + \Delta t \sum_{n=0}^{m-1} \bar{F}_d^n \left(W_h^{n+1} + W_h^n\right) + \sum_{q=1}^{N_\varphi} 2a(U_h^m, \Psi_{hq}^m).$$

First of all, we will consider $\sum_{n=0}^{m-1} \bar{F}_d^n \left(W_h^{n+1} + W_h^n\right)$. By the definition,

$$\begin{aligned}\Delta t \sum_{n=0}^{m-1} \bar{F}_d^n \left(W_h^{n+1} + W_h^n\right) =& \Delta t \sum_{n=0}^{m-1} \left(\bar{f}^n, W_h^{n+1} + W_h^n\right)_{L_2(\Omega)} \\ &+ \Delta t \sum_{n=0}^{m-1} \left(\bar{g}_N^n, W_h^{n+1} + W_h^n\right)_{L_2(\Gamma_N)}.\end{aligned}$$

Since $\Delta t(W_h^{n+1} + W_h^n) = 2(U_h^{n+1} - U_h^n)$ from (2.3.1),

$$\begin{aligned}\Delta t \sum_{n=0}^{m-1} \bar{F}_d^n \left(W_h^{n+1} + W_h^n\right) =& \Delta t \sum_{n=0}^{m-1} \left(\bar{f}^n, W_h^{n+1} + W_h^n\right)_{L_2(\Omega)} \\ &+ 2 \sum_{n=0}^{m-1} \left(\bar{g}_N^n, U_h^{n+1} - U_h^n\right)_{L_2(\Gamma_N)}.\end{aligned}$$

Note that we can apply summation by parts to our equations, which is an discrete analogue of integration by parts. This can yield

$$\begin{aligned}\sum_{n=0}^{m-1} \left(\bar{g}_N^n, U_h^{n+1} - U_h^n\right)_{L_2(\Gamma_N)} =& \left(\bar{g}_N^{m-1}, U_h^m\right)_{L_2(\Gamma_N)} - \left(\bar{g}_N^0, U_h^0\right)_{L_2(\Gamma_N)} \\ &- \sum_{n=1}^{m-1} \left(\bar{g}_N^n - \bar{g}_N^{n-1}, U_h^n\right)_{L_2(\Gamma_N)}.\end{aligned}$$

Here, since g_N is differentiable in time, we can obtain

$$\bar{g}_N^n - \bar{g}_N^{n-1} = \frac{g_N(t_{n+1}) - g_N(t_{n-1})}{2} = \frac{1}{2} \int_{t_{n-1}}^{t_{n+1}} \dot{g}_N(t')dt',$$

and then we have by Leibniz's integral rule,

$$\sum_{n=1}^{m-1} \left(\bar{g}_N^n - \bar{g}_N^{n-1}, U_h^n\right)_{L_2(\Gamma_N)} = \frac{1}{2} \sum_{n=1}^{m-1} \int_{t_{n-1}}^{t_{n+1}} \left(\dot{g}_N(t'), U_h^n\right)_{L_2(\Gamma_N)} dt'.$$

Hence we have

$$\Delta t \sum_{n=0}^{m-1} \bar{F}_d^n \left(W_h^{n+1} + W_h^n\right)$$

$$
\begin{aligned}
=&\Delta t \sum_{n=0}^{m-1} \left(\bar{f}^n, W_h^{n+1} + W_h^n\right)_{L_2(\Omega)} + 2\left(\bar{g}_N^{m-1}, U_h^m\right)_{L_2(\Gamma_N)} \\
&- 2\left(\bar{g}_N^0, U_h^0\right)_{L_2(\Gamma_N)} - \sum_{n=1}^{m-1} \int_{t_{n-1}}^{t_{n+1}} \left(\dot{g}_N(t'), U_h^n\right)_{L_2(\Gamma_N)} dt'.
\end{aligned}
$$

By applying Cauchy-Schwarz inequality and (2.1.13),

$$
\begin{aligned}
&\Delta t \sum_{n=0}^{m-1} \bar{F}_d^n \left(W_h^{n+1} + W_h^n\right) \\
\le &\Delta t \sum_{n=0}^{m-1} \left\|\bar{f}^n\right\|_{L_2(\Omega)} \left\|W_h^{n+1} + W_h^n\right\|_{L_2(\Omega)} + 2C \left\|\bar{g}_N^{m-1}\right\|_{L_2(\Gamma_N)} \|U_h^m\|_V \\
&+ 2C \left\|\bar{g}_N^0\right\|_{L_2(\Gamma_N)} \left\|U_h^0\right\|_V + C \sum_{n=1}^{m-1} \int_{t_{n-1}}^{t_{n+1}} \left\|\dot{g}_N(t')\right\|_{L_2(\Gamma_N)} \|U_h^n\|_V \, dt', \qquad (2.3.15)
\end{aligned}
$$

where C is a positive constant from (2.1.13). Note that U_h^n is independent of time so Cauchy-Schwarz inequality gives

$$
\begin{aligned}
\int_{t_{n-1}}^{t_{n+1}} \left\|\dot{g}_N(t')\right\|_{L_2(\Gamma_N)} \|U_h^n\|_V \, dt' &\le \left(\int_{t_{n-1}}^{t_{n+1}} \left\|\dot{g}_N(t')\right\|^2_{L_2(\Gamma_N)} \int_{t_{n-1}}^{t_{n+1}} \|U_h^n\|_V^2 \, dt' \right)^{1/2} \\
&= \left(\int_{t_{n-1}}^{t_{n+1}} \left\|\dot{g}_N(t')\right\|^2_{L_2(\Gamma_N)} dt' \right)^{1/2} \left(2\Delta t \, \|U_h^n\|_V^2\right)^{1/2}.
\end{aligned}
$$

Then Young's inequality and the triangular inequality allow us to obtain

$$
\begin{aligned}
&\Delta t \sum_{n=0}^{m-1} \bar{F}_d^n \left(W_h^{n+1} + W_h^n\right) \\
\le &\Delta t \sum_{n=0}^{m-1} \left\|\bar{f}^n\right\|_{L_2(\Omega)} \left\|W_h^{n+1}\right\|_{L_2(\Omega)} + \Delta t \sum_{n=0}^{m-1} \left\|\bar{f}^n\right\|_{L_2(\Omega)} \|W_h^n\|_{L_2(\Omega)} \\
&+ 2C \left\|\bar{g}_N^{m-1}\right\|_{L_2(\Gamma_N)} \|U_h^m\|_V + 2C \left\|\bar{g}_N^0\right\|_{L_2(\Gamma_N)} \left\|U_h^0\right\|_V \\
&+ C \sum_{n=1}^{m-1} \left(\int_{t_{n-1}}^{t_{n+1}} \left\|\dot{g}_N(t')\right\|^2_{L_2(\Gamma_N)} dt' \right)^{1/2} \left(2\Delta t \, \|U_h^n\|_V^2\right)^{1/2} \\
\le &\Delta t \sum_{n=0}^{m-2} \left\|\bar{f}^n\right\|_{L_2(\Omega)} \left\|W_h^{n+1}\right\|_{L_2(\Omega)} + \Delta t \left\|\bar{f}^{m-1}\right\|_{L_2(\Omega)} \|W_h^m\|_{L_2(\Omega)} \\
&+ \Delta t \sum_{n=0}^{m-1} \left\|\bar{f}^n\right\|_{L_2(\Omega)} \|W_h^n\|_{L_2(\Omega)} + 2C \left\|\bar{g}_N^{m-1}\right\|_{L_2(\Gamma_N)} \|U_h^m\|_V \\
&+ 2C \left\|\bar{g}_N^0\right\|_{L_2(\Gamma_N)} \left\|U_h^0\right\|_V
\end{aligned}
$$

$$
\begin{aligned}
&+ C \sum_{n=1}^{m-1} \left(\int_{t_{n-1}}^{t_{n+1}} \left\| \dot{g}_N(t') \right\|_{L_2(\Gamma_N)}^2 dt' \right)^{1/2} \left(2\Delta t \left\| U_h^n \right\|_V^2 \right)^{1/2} \\
\le& \frac{\Delta t}{2} \sum_{n=0}^{m-2} \left\| \bar{f}^n \right\|_{L_2(\Omega)}^2 + \frac{\Delta t}{2} \sum_{n=0}^{m-2} \left\| W_h^{n+1} \right\|_{L_2(\Omega)}^2 + \frac{\Delta t}{2\epsilon_a} \left\| \bar{f}^{m-1} \right\|_{L_2(\Omega)}^2 \\
&+ \frac{\Delta t \epsilon_a}{2} \left\| W_h^m \right\|_{L_2(\Omega)}^2 + \frac{\Delta t}{2} \sum_{n=0}^{m-1} \left\| \bar{f}^n \right\|_{L_2(\Omega)}^2 + \frac{\Delta t}{2} \sum_{n=0}^{m-1} \left\| W_h^n \right\|_{L_2(\Omega)}^2 \\
&+ \frac{C}{\epsilon_b} \left\| \bar{g}_N^{m-1} \right\|_{L_2(\Gamma_N)}^2 + C\epsilon_b \left\| U_h^m \right\|_V^2 + C \left\| \bar{g}_N^0 \right\|_{L_2(\Gamma_N)}^2 + C \left\| U_h^0 \right\|_V^2 \\
&+ C \int_0^{t_m} \left\| \dot{g}_N(t') \right\|_{L_2(\Gamma_N)}^2 dt' + C\Delta t \sum_{n=1}^{m-1} \left\| U_h^n \right\|_V^2 \\
\le& \Delta t \sum_{n=0}^{m-1} \left\| \bar{f}^n \right\|_{L_2(\Omega)}^2 + \Delta t \sum_{n=0}^{m-1} \left\| W_h^n \right\|_{L_2(\Omega)}^2 + \frac{\Delta t}{2\epsilon_a} \left\| \bar{f}^{m-1} \right\|_{L_2(\Omega)}^2 \\
&+ \frac{\Delta t \epsilon_a}{2} \left\| W_h^m \right\|_{L_2(\Omega)}^2 + \frac{C}{\epsilon_b} \left\| \bar{g}_N^{m-1} \right\|_{L_2(\Gamma_N)}^2 + C\epsilon_b \left\| U_h^m \right\|_V^2 \\
&+ C \left\| \bar{g}_N^0 \right\|_{L_2(\Gamma_N)}^2 + C \left\| U_h^0 \right\|_V^2 + C \int_0^{t_m} \left\| \dot{g}_N(t') \right\|_{L_2(\Gamma_N)}^2 dt' + C\Delta t \sum_{n=1}^{m-1} \left\| U_h^n \right\|_V^2 \\
\le& \left(\Delta t + \frac{\Delta t}{2\epsilon_a} \right) \sum_{n=0}^{m-1} \left\| \bar{f}^n \right\|_{L_2(\Omega)}^2 + \Delta t \sum_{n=0}^{m-1} \left\| W_h^n \right\|_{L_2(\Omega)}^2 + \frac{\Delta t \epsilon_a}{2} \left\| W_h^m \right\|_{L_2(\Omega)}^2 \\
&+ \frac{C}{\epsilon_b} \left\| \bar{g}_N^{m-1} \right\|_{L_2(\Gamma_N)}^2 + C\epsilon_b \left\| U_h^m \right\|_V^2 + C \left\| \bar{g}_N^0 \right\|_{L_2(\Gamma_N)}^2 + C \left\| U_h^0 \right\|_V^2 \\
&+ C \int_0^{t_m} \left\| \dot{g}_N(t') \right\|_{L_2(\Gamma_N)}^2 dt' + C\Delta t \sum_{n=1}^{m-1} \left\| U_h^n \right\|_V^2 ,
\end{aligned}
$$

with positive ϵ_a and ϵ_b.

Secondly, let us observe $\sum_{q=1}^{N_\varphi} 2a(U_h^m, \Psi_{hq}^m)$. In a similar way with the above result,

$$
\sum_{q=1}^{N_\varphi} 2a(U_h^m, \Psi_{hq}^m) \le \sum_{q=1}^{N_\varphi} \epsilon_q \left\| U_h^m \right\|_V^2 + \sum_{q=1}^{N_\varphi} \frac{1}{\epsilon_q} \left\| \Psi_{hq}^m \right\|_V^2 ,
$$

by Cauchy-Schwarz inequality and Young's inequality with positive ϵ_q for each q.

Finally, as tidying up the results, if we take $\epsilon_a = \rho/\Delta t > 0$, $\epsilon_b = \varphi_0/(4C) > 0$ and $\epsilon_q = \varphi_q + \varphi_0/(2N_\varphi) > 0$ for each q, since

$$
1 - \sum_{q=1}^{N_\varphi} \epsilon_q - C\epsilon_b = \frac{\varphi_0}{4}, \qquad \frac{1}{\varphi_q} - \frac{1}{\epsilon_q} = \frac{\varphi_0}{2\varphi_q^2 N_\varphi + \varphi_q \varphi_0} \text{ for each } q,
$$

we can obtain

$$
\begin{aligned}
&\frac{\rho}{2}\left\|W_h^m\right\|_{L_2(\Omega)}^2+\frac{\varphi_0}{4}\left\|U_h^m\right\|_V^2+\sum_{q=1}^{N_\varphi}\frac{\varphi_0}{2\varphi_q^2N_\varphi+\varphi_q\varphi_0}\left\|\Psi_{hq}^m\right\|_V^2\\
&\quad+\sum_{q=1}^{N_\varphi}\sum_{n=0}^{m-1}\frac{2\tau_q}{\Delta t\varphi_q}\left\|\Psi_{hq}^{n+1}-\Psi_{hq}^n\right\|_V^2\\
&\leq\rho\left\|W_h^0\right\|_{L_2(\Omega)}^2+(1+C)\left\|U_h^0\right\|_V^2+\frac{4C^2}{\varphi_0}\left\|\bar{g}_N^{m-1}\right\|_{L_2(\Gamma_N)}^2+C\left\|\bar{g}_N^0\right\|_{L_2(\Gamma_N)}^2\\
&\quad+\left(\Delta t+\frac{\Delta t^2}{2\rho}\right)\sum_{n=0}^{m-1}\left\|\bar{f}^n\right\|_{L_2(\Omega)}^2+C\int_0^{t_m}\left\|\dot{g}_N(t')\right\|_{L_2(\Gamma_N)}^2dt'\\
&\quad+\Delta t\sum_{n=0}^{m-1}\left\|W_h^n\right\|_{L_2(\Omega)}^2+C\Delta t\sum_{n=0}^{m-1}\left\|U_h^n\right\|_V^2.
\end{aligned}
$$

Consequently, with applying discrete Grönwall's inequality, we can derive

$$
\begin{aligned}
&\frac{\rho}{2}\left\|W_h^m\right\|_{L_2(\Omega)}^2+\frac{\varphi_0}{4}\left\|U_h^m\right\|_V^2+\sum_{q=1}^{N_\varphi}\frac{\varphi_0}{2\varphi_q^2N_\varphi+\varphi_q\varphi_0}\left\|\Psi_{hq}^m\right\|_V^2\\
&\quad+\sum_{q=1}^{N_\varphi}\sum_{n=0}^{m-1}\frac{2\tau_q}{\Delta t\varphi_q}\left\|\Psi_{hq}^{n+1}-\Psi_{hq}^n\right\|_V^2\\
&\leq C\Bigg(\left\|W_h^0\right\|_{L_2(\Omega)}^2+\left\|U_h^0\right\|_V^2+\left\|\bar{g}_N^{m-1}\right\|_{L_2(\Gamma_N)}^2+\left\|\bar{g}_N^0\right\|_{L_2(\Gamma_N)}^2\\
&\quad+\Delta t\sum_{n=0}^{m-1}\left\|\bar{f}^n\right\|_{L_2(\Omega)}^2+\int_0^{t_m}\left\|\dot{g}_N(t')\right\|_{L_2(\Gamma_N)}^2dt'\Bigg),\\
&\leq C\Bigg(\left\|w_0\right\|_{L_2(\Omega)}^2+\left\|u_0\right\|_V^2+\left\|\bar{g}_N^{m-1}\right\|_{L_2(\Gamma_N)}^2+\left\|\bar{g}_N^0\right\|_{L_2(\Gamma_N)}^2\\
&\quad+\Delta t\sum_{n=0}^{m-1}\left\|\bar{f}^n\right\|_{L_2(\Omega)}^2+\int_0^{t_m}\left\|\dot{g}_N(t')\right\|_{L_2(\Gamma_N)}^2dt'\Bigg),
\end{aligned}
$$

since $\left\|U_h^0\right\|_V\leq\left\|u_0\right\|_V$ and $\left\|W_h^0\right\|_{L_2(\Omega)}\leq\left\|w_0\right\|_{L_2(\Omega)}$ by (2.3.13) and (2.3.14). □

In Theorem 2.12, we used discrete Grönwall's inequality to prove the stability so the constant C is increasing exponentially with respect to time. However, we can improve the stability bound without discrete Grönwall's inequality. We will introduce the maximum with respect to time steps which is a discrete analogue of L_∞ norm.

Theorem 2.13. *There exists a positive constant C such that*

$$
\frac{\rho}{2}\max_{0\leq n\leq N}\left\|W_h^n\right\|_{L_2(\Omega)}^2+\frac{\varphi_0}{4}\max_{0\leq n\leq N}\left\|U_h^n\right\|_V^2+\sum_{q=1}^{N_\varphi}\frac{\varphi_0}{2\varphi_q^2N_\varphi+\varphi_q\varphi_0}\left\|\Psi_{hq}^m\right\|_V^2
$$

$$
\begin{aligned}
&+\sum_{q=1}^{N_\varphi}\sum_{n=0}^{m-1}\frac{2\tau_q}{\Delta t\varphi_q}\left\|\Psi_{hq}^{n+1}-\Psi_{hq}^{n}\right\|_V^2\\
\leq & C\Bigg(\|w_0\|_{L_2(\Omega)}^2+\|u_0\|_V^2+\|f\|_{L_\infty(0,T;L_2(\Omega))}^2+\|\dot{g}_N\|_{L_2(0,T;L_2(\Gamma_N))}^2\\
&+\|g_N\|_{L_\infty(0,T;L_2(\Gamma_N))}^2\Bigg),
\end{aligned}
$$

for any $m=1,\ldots,N$. Here, C is independent of Δt and numerical solutions but depends on the final time T, indeed $C\propto T^2$.

Proof. Recall (2.3.15) in the proof of Theorem 2.12 such that

$$
\begin{aligned}
&\Delta t\sum_{n=0}^{m-1}\bar{F}_d^n\left(W_h^{n+1}+W_h^n\right)\\
\leq & \Delta t\sum_{n=0}^{m-1}\left\|\bar{f}^n\right\|_{L_2(\Omega)}\left\|W_h^{n+1}+W_h^n\right\|_{L_2(\Omega)}+2C\left\|\bar{g}_N^{m-1}\right\|_{L_2(\Gamma_N)}\left\|U_h^m\right\|_V\\
&+2C\left\|\bar{g}_N^0\right\|_{L_2(\Gamma_N)}\left\|U_h^0\right\|_V+C\sum_{n=1}^{m-1}\int_{t_{n-1}}^{t_{n+1}}\left\|\dot{g}_N(t')\right\|_{L_2(\Gamma_N)}\left\|U_h^n\right\|_V dt'.
\end{aligned}
$$

Triangular inequality, Cauchy-Schwarz inequality and Young's inequality lead us to obtain

$$
\begin{aligned}
&\Delta t\sum_{n=0}^{m-1}\bar{F}_d^n\left(W_h^{n+1}+W_h^n\right)\\
\leq & \frac{\Delta t}{2\epsilon_a}\sum_{n=0}^{m-1}\left\|\bar{f}^n\right\|_{L_2(\Omega)}^2+\frac{\Delta t\epsilon_a}{2}\sum_{n=0}^{m-1}\left\|W_h^{n+1}\right\|_{L_2(\Omega)}^2+\frac{\Delta t}{2\epsilon_a}\sum_{n=0}^{m-1}\left\|\bar{f}^n\right\|_{L_2(\Omega)}^2\\
&+\frac{\Delta t\epsilon_a}{2}\sum_{n=0}^{m-1}\left\|W_h^n\right\|_{L_2(\Omega)}^2+\frac{C}{\epsilon_b}\int_0^{t_m}\left\|\dot{g}_N(t')\right\|_{L_2(\Gamma_N)}^2 dt'+C\Delta t\epsilon_b\sum_{n=0}^{m-1}\left\|U_h^n\right\|_V^2\\
&+C\epsilon_b\left\|U_h^m\right\|_V^2+C\left\|U_h^0\right\|_V^2+\frac{C}{\epsilon_b}\left\|\bar{g}_N^{m-1}\right\|_{L_2(\Gamma_N)}^2+C\left\|\bar{g}_N^0\right\|_{L_2(\Gamma_N)}^2\\
= & \frac{\Delta t}{\epsilon_a}\sum_{n=0}^{m-1}\left\|\bar{f}^n\right\|_{L_2(\Omega)}^2+\Delta t\epsilon_a\sum_{n=1}^{m-1}\left\|W_h^n\right\|_{L_2(\Omega)}^2+\frac{C}{\epsilon_b}\int_0^{t_m}\left\|\dot{g}_N(t')\right\|_{L_2(\Gamma_N)}^2 dt'\\
&+C\Delta t\epsilon_b\sum_{n=0}^{m-1}\left\|U_h^n\right\|_V^2+\frac{\Delta t\epsilon_a}{2}\left\|W_h^m\right\|_{L_2(\Omega)}^2+C\epsilon_b\left\|U_h^m\right\|_V^2+\frac{\Delta t\epsilon_a}{2}\left\|W_h^0\right\|_{L_2(\Omega)}^2\\
&+C\left\|U_h^0\right\|_V^2+\frac{C}{\epsilon_b}\left\|\bar{g}_N^{m-1}\right\|_{L_2(\Gamma_N)}^2+C\left\|\bar{g}_N^0\right\|_{L_2(\Gamma_N)}^2,
\end{aligned}
$$

for positive ϵ_a and ϵ_b. Due to the positive definite and the property of maximum, it implies for any m

$$
\begin{aligned}
&\Delta t \sum_{n=0}^{m-1} \bar{F}_d^n \left(W_h^{n+1} + W_h^n\right) \\
\leq &\frac{\Delta t}{\epsilon_a} \sum_{n=0}^{N-1} \left\|\bar{f}^n\right\|_{L_2(\Omega)}^2 + \Delta t \epsilon_a \sum_{n=1}^{N-1} \|W_h^n\|_{L_2(\Omega)}^2 + \frac{C}{\epsilon_b} \int_0^T \left\|\dot{g}_N(t')\right\|_{L_2(\Gamma_N)}^2 dt' \\
&+ C\Delta t \epsilon_b \sum_{n=0}^{N-1} \|U_h^n\|_V^2 + \frac{\Delta t \epsilon_a}{2} \|W_h^m\|_{L_2(\Omega)}^2 + C\epsilon_b \|U_h^m\|_V^2 + \frac{\Delta t \epsilon_a}{2} \left\|W_h^0\right\|_{L_2(\Omega)}^2 \\
&+ C \left\|U_h^0\right\|_V^2 + \frac{C}{\epsilon_b} \left\|\bar{g}_N^{m-1}\right\|_{L_2(\Gamma_N)}^2 + C \left\|\bar{g}_N^0\right\|_{L_2(\Gamma_N)}^2 , \\
\leq &\frac{\Delta t}{\epsilon_a} \sum_{n=0}^{N-1} \left\|\bar{f}^n\right\|_{L_2(\Omega)}^2 + \Delta t \epsilon_a \sum_{j=1}^{N-1} \max_{0 \leq n \leq N} \|W_h^n\|_{L_2(\Omega)}^2 + \frac{C}{\epsilon_b} \|\dot{g}_N\|_{L_2(0,T;L_2(\Gamma_N))}^2 \\
&+ \frac{\Delta t \epsilon_a}{2} \max_{0 \leq n \leq N} \|W_h^n\|_{L_2(\Omega)}^2 + C\epsilon_b T \max_{0 \leq n \leq N} \|U_h^n\|_V^2 + C\epsilon_b \max_{0 \leq n \leq N} \|U_h^n\|_V^2 \\
&+ \frac{\Delta t \epsilon_a}{2} \left\|W_h^0\right\|_{L_2(\Omega)}^2 + C \left\|U_h^0\right\|_V^2 + \left(\frac{C}{\epsilon_b} + C\right) \max_{0 \leq n \leq N-1} \|\bar{g}_N^n\|_{L_2(\Gamma_N)}^2 , \\
\leq &\frac{\Delta t}{\epsilon_a} \sum_{n=0}^{N-1} \left\|\bar{f}^n\right\|_{L_2(\Omega)}^2 + T\epsilon_a \max_{0 \leq n \leq N} \|W_h^n\|_{L_2(\Omega)}^2 + \frac{C}{\epsilon_b} \|\dot{g}_N\|_{L_2(0,T;L_2(\Gamma_N))}^2 \\
&+ C\epsilon_b (T+1) \max_{0 \leq n \leq N} \|U_h^n\|_V^2 + \frac{\Delta t \epsilon_a}{2} \left\|W_h^0\right\|_{L_2(\Omega)}^2 + C \left\|U_h^0\right\|_V^2 \\
&+ \left(\frac{C}{\epsilon_b} + C\right) \max_{0 \leq n \leq N-1} \|\bar{g}_N^n\|_{L_2(\Gamma_N)}^2 ,
\end{aligned}
$$

since $T = N\Delta t$. Hence from the proof of Theorem 2.12 we have

$$
\sum_{q=1}^{N_\varphi} 2a(U_h^m, \Psi_{hq}^m) \leq \sum_{q=1}^{N_\varphi} \epsilon_q \|U_h^m\|_V^2 + \sum_{q=1}^{N_\varphi} \frac{1}{\epsilon_q} \left\|\Psi_{hq}^m\right\|_V^2 ,
$$

with $\epsilon_q = \varphi_q + \varphi_0/(2N_\varphi) > 0$ for each q, so that it yields

$$
\begin{aligned}
&\rho \|W_h^m\|_{L_2(\Omega)}^2 + \frac{\varphi_0}{2} \|U_h^m\|_V^2 + \sum_{q=1}^{N_\varphi} \frac{\varphi_0}{2\varphi_q^2 N_\varphi + \varphi_q \varphi_0} \left\|\Psi_{hq}^m\right\|_V^2 \\
&+ \sum_{q=1}^{N_\varphi} \sum_{n=0}^{m-1} \frac{2\tau_q}{\Delta t \varphi_q} \left\|\Psi_{hq}^{n+1} - \Psi_{hq}^n\right\|_V^2 \\
\leq &\left(1 + \frac{\Delta t \epsilon_a}{2}\right) \left\|W_h^0\right\|_{L_2(\Omega)}^2 + (1+C) \left\|U_h^0\right\|_V^2 + \frac{\Delta t}{\epsilon_a} \sum_{n=0}^{N-1} \left\|\bar{f}^n\right\|_{L_2(\Omega)}^2 \\
&+ \frac{C}{\epsilon_b} \|\dot{g}_N\|_{L_2(0,T;L_2(\Gamma_N))}^2 + \left(\frac{C}{\epsilon_b} + C\right) \max_{0 \leq n \leq N-1} \|\bar{g}_N^n\|_{L_2(\Gamma_N)}
\end{aligned}
$$

$$+ T\epsilon_a \max_{0\le n\le N} \|W_h^n\|^2_{L_2(\Omega)} + C\epsilon_b(T+1) \max_{0\le n\le N} \|U_h^n\|^2_V \,,$$

for any m. Then by the property of supremum (here in a discrete case, supremum is equivalent to maximum) with respect to m we can derive

$$\rho \max_{0\le n\le N} \|W_h^n\|^2_{L_2(\Omega)} + \frac{\varphi_0}{2} \max_{0\le n\le N} \|U_h^n\|^2_V + \sum_{q=1}^{N_\varphi} \frac{\varphi_0}{2\varphi_q^2 N_\varphi + \varphi_q\varphi_0} \left\|\Psi_{hq}^m\right\|^2_V$$
$$+ \sum_{q=1}^{N_\varphi}\sum_{n=0}^{m-1} \frac{2\tau_q}{\Delta t\varphi_q} \left\|\Psi_{hq}^{n+1} - \Psi_{hq}^n\right\|^2_V$$
$$\le 3\Bigg(\left(1 + \frac{\Delta t\epsilon_a}{2}\right) \|W_h^0\|^2_{L_2(\Omega)} + (1+C)\left\|U_h^0\right\|^2_V + \frac{\Delta t}{\epsilon_a}\sum_{n=0}^{N-1} \left\|\bar{f}^n\right\|^2_{L_2(\Omega)}$$
$$+ \frac{C}{\epsilon_b}\|\dot{g}_N\|^2_{L_2(0,T;L_2(\Gamma_N))} \left(\frac{C}{\epsilon_b} + C\right) \max_{0\le n\le N-1} \|\bar{g}_N^n\|^2_{L_2(\Gamma_N)}$$
$$+ T\epsilon_a \max_{0\le n\le N} \|W_h^n\|^2_{L_2(\Omega)} + C\epsilon_b(T+1) \max_{0\le n\le N} \|U_h^n\|^2_V \Bigg),$$

If we take

$$\epsilon_a = \frac{\rho}{6T} \text{ and } \epsilon_b = \frac{\varphi_0}{12C(T+1)},$$

it can be concluded that

$$\frac{\rho}{2} \max_{0\le n\le N} \|W_h^n\|^2_{L_2(\Omega)} + \frac{\varphi_0}{4} \max_{0\le n\le N} \|U_h^n\|^2_V + \sum_{q=1}^{N_\varphi} \frac{\varphi_0}{2\varphi_q^2 N_\varphi + \varphi_q\varphi_0} \left\|\Psi_{hq}^m\right\|^2_V$$
$$+ \sum_{q=1}^{N_\varphi}\sum_{n=0}^{m-1} \frac{2\tau_q}{\Delta t\varphi_q} \left\|\Psi_{hq}^{n+1} - \Psi_{hq}^n\right\|^2_V$$
$$\le C\Bigg(\left\|W_h^0\right\|^2_{L_2(\Omega)} + \left\|U_h^0\right\|^2_V + \Delta t \sum_{n=0}^{N-1} \left\|\bar{f}^n\right\|^2_{L_2(\Omega)} + \|\dot{g}_N\|^2_{L_2(0,T;L_2(\Gamma_N))}$$
$$+ \max_{0\le n\le N-1} \|\bar{g}_N^n\|^2_{L_2(\Gamma_N)} \Bigg),$$
$$\le C\Bigg(\|w_0\|^2_{L_2(\Omega)} + \|u_0\|^2_V + \Delta t \sum_{n=0}^{N-1} \left\|\bar{f}^n\right\|^2_{L_2(\Omega)} + \|\dot{g}_N\|^2_{L_2(0,T;L_2(\Gamma_N))}$$
$$+ \max_{0\le n\le N-1} \|\bar{g}_N^n\|^2_{L_2(\Gamma_N)} \Bigg),$$

by (2.3.13) and (2.3.14) for some positive C, $\forall m \in \{1,\dots,N\}$. Additionally, Cauchy-Schwarz inequalities and definition of L_∞ norm in time lead us to obtain

$$\Delta t \sum_{n=0}^{N-1} \left\|\bar{f}^n\right\|^2_{L_2(\Omega)} = \frac{\Delta t}{2} \sum_{n=0}^{N-1} \|f(t_{n+1}) + f(t_n)\|^2_{L_2(\Omega)}$$

$$
\begin{aligned}
&\leq \Delta t \sum_{n=0}^{N-1} (\|f(t_{n+1})\|_{L_2(\Omega)}^2 + \|f(t_n)\|_{L_2(\Omega)}^2) \\
&\leq 2\Delta t N \max_{0\leq n\leq N} \|f(t_n)\|_{L_2(\Omega)}^2 \\
&\leq 2T \|f\|_{L_\infty(0,T;L_2(\Omega))}^2 .
\end{aligned}
$$

In a similar way, we can replace the maximum term of Neumann boundary by

$$
\max_{0\leq n\leq N-1} \|\bar{g}_N^n\|_{L_2(\Gamma_N)}^2 \leq 2 \|g_N\|_{L_\infty(0,T;L_2(\Gamma_N))}^2 .
$$

As a consequence, we have

$$
\begin{aligned}
&\frac{\rho}{2} \max_{0\leq n\leq N} \|W_h^n\|_{L_2(\Omega)}^2 + \frac{\varphi_0}{4} \max_{0\leq n\leq N} \|U_h^n\|_V^2 + \sum_{q=1}^{N_\varphi} \frac{\varphi_0}{2\varphi_q^2 N_\varphi + \varphi_q \varphi_0} \|\Psi_{hq}^m\|_V^2 \\
&\quad + \sum_{q=1}^{N_\varphi} \sum_{n=0}^{m-1} \frac{2\tau_q}{\Delta t \varphi_q} \left\| \Psi_{hq}^{n+1} - \Psi_{hq}^n \right\|_V^2 \\
&\leq C \Bigg(\|w_0\|_{L_2(\Omega)}^2 + \|u_0\|_V^2 + \|f\|_{L_\infty(0,T;L_2(\Omega))}^2 + \|\dot{g}_N\|_{L_2(0,T;L_2(\Gamma_N))}^2 \\
&\quad + \|g_N\|_{L_\infty(0,T;L_2(\Gamma_N))}^2 \Bigg).
\end{aligned}
$$

□

In Theorem 2.13, C has increased but not exponentially, as the final time T grows. However, Theorem 2.12 shows exponentially increasing C in time, which means $C \propto \exp(T)$.

From the stability bounds in Theorem 2.12 and 2.13, the fully discrete solutions are bounded by the data such as boundary conditions, initial conditions and source terms. It means that if the data is given by zero data, the solution must be zero.

Recall the concept of linear algebra,

$$
A\underline{x} = \underline{b} \text{ is solved uniquely,}
$$

$$
\Updownarrow
$$

$$
A\underline{x} = \underline{0}, \text{ only if } \underline{x} = \underline{0}.
$$

Note that solving the fully discrete formulation is equivalent to solve the linear system (2.3.9)-(2.3.11). In the above, $\underline{x}$ represents our solution and $\underline{b}$ is defined by the given data. Therefore, if the data is given by 0 then the solution should be also zero so that the linear system is solved uniquely. Furthermore, the matrix $\frac{2\rho}{\Delta t^2} M + \mathcal{A}$ is invertible hence it is able to obtain the solution numerically.

In order to see the error bounds for fully discrete formulations, we will follow the proof of error bounds for semidiscrete formulations. At first let us define

$$\begin{aligned}\theta &:= u - Ru,\\ \chi^n &:= U_h^n - Ru^n,\\ \varpi^n &:= W_h^n - R\dot{u}^n,\\ \vartheta_q &:= \psi_q - R\psi_q \ \forall q,\\ \varsigma_q^n &:= \Psi_{hq}^n - R\psi_q^n \ \forall q,\end{aligned}$$

where $u^n = u(t_n)$.

Lemma 2.10. *Suppose* $u \in H^4(0,T;H^s(\Omega)) \cap W^1_\infty(0,T;H^s(\Omega))$. *Then*

$$\max_{0\le j\le N} \left\|\varpi^j\right\|_{L_2(\Omega)} + \max_{0\le j\le N} \left\|\chi^j\right\|_V \le C(h^{\min(k+1,s)-1} + \Delta t^2).$$

If we also assume elliptic regularity,

$$\max_{0\le j\le N} \left\|\varpi^j\right\|_{L_2(\Omega)} + \max_{0\le j\le N} \left\|\chi^j\right\|_V \le C(h^{\min(k+1,s)} + \Delta t^2).$$

Proof. Recall (2.1.9) and (2.3.2). We have

$$\begin{aligned}&\frac{\rho}{2}\left(\ddot{u}^{n+1} + \ddot{u}^n, v\right)_{L_2(\Omega)} + \frac{1}{2}a\left(u^{n+1} + u^n, v\right) - \frac{1}{2}\sum_{q=1}^{N_\varphi} a\left(\psi_q^{n+1} + \psi_q^n, v\right)\\ &=\bar{F}_d^n(v),\\ &\frac{\rho}{\Delta t}\left(W_h^{n+1} - W_h^n, v\right)_{L_2(\Omega)} + \frac{1}{2}a\left(U_h^{n+1} + U_h^n, v\right) - \frac{1}{2}\sum_{q=1}^{N_\varphi} a\left(\Psi_{hq}^{n+1} + \Psi_{hq}^n, v\right)\\ &=\bar{F}_d^n(v)\end{aligned}$$

for any $v \in V^h$. By subtracting the two equations, it yields

$$\begin{aligned}&\left(\frac{\rho}{2}(\ddot{u}^{n+1} + \ddot{u}^n) - \frac{\rho}{\Delta t}(W_h^{n+1} - W_h^n), v\right)_{L_2(\Omega)} + \frac{1}{2}a\left((u^{n+1} + u^n) - (U_h^{n+1} + U_h^n), v\right)\\ &- \frac{1}{2}\sum_{q=1}^{N_\varphi} a\left((\psi_q^{n+1} + \psi_q^n) - (\Psi_{hq}^{n+1} + \Psi_{hq}^n), v\right)\\ &= 0,\end{aligned}$$

and so

$$\frac{\rho}{\Delta t}\left(\varpi^{n+1} - \varpi^n, v\right)_{L_2(\Omega)} + \frac{1}{2}a\left(\chi^{n+1} + \chi^n, v\right) - \frac{1}{2}\sum_{q=1}^{N_\varphi} a\left(\varsigma_q^{n+1} + \varsigma_q^n, v\right)$$

$$
\begin{aligned}
=&\frac{\rho}{\Delta t}\left(\dot{\theta}^{n+1}-\dot{\theta}^{n},v\right)_{L_2(\Omega)}+\frac{1}{2}a\left(\theta^{n+1}+\theta^{n},v\right)-\frac{1}{2}\sum_{q=1}^{N_\varphi}a\left(\vartheta_q^{n+1}+\vartheta_q^{n},v\right)\\
&+\rho\left(\frac{\ddot{u}^{n+1}+\ddot{u}^{n}}{2}-\frac{\dot{u}^{n+1}-\dot{u}^{n}}{\Delta t},v\right)_{L_2(\Omega)},\\
=&\frac{\rho}{\Delta t}\left(\dot{\theta}^{n+1}-\dot{\theta}^{n},v\right)_{L_2(\Omega)}+\frac{1}{2}a\left(\theta^{n+1}+\theta^{n},v\right)-\frac{1}{2}\sum_{q=1}^{N_\varphi}a\left(\vartheta_q^{n+1}+\vartheta_q^{n},v\right)\\
&+\rho\left(\mathcal{E}_1^n,v\right)_{L_2(\Omega)},\\
=&\frac{\rho}{\Delta t}\left(\dot{\theta}^{n+1}-\dot{\theta}^{n},v\right)_{L_2(\Omega)}+\rho\left(\mathcal{E}_1^n,v\right)_{L_2(\Omega)},
\end{aligned}
$$

for $n=0,\ldots,N-1$ by the property of elliptic projections such as Galerkin orthogonality, where

$$
\mathcal{E}_1(t)=\frac{\ddot{u}(t+\Delta t)+\ddot{u}(t)}{2}-\frac{\dot{u}(t+\Delta t)-\dot{u}(t)}{\Delta t}
$$

Note that by (2.3.1)

$$
\begin{aligned}
\frac{\chi^{n+1}-\chi^{n}}{\Delta t}=&\frac{U^{n+1}-U^{n}}{\Delta t}-\frac{Ru^{n+1}-Ru^{n}}{\Delta t}\\
=&\frac{W^{n+1}+W^{n}}{2}-\frac{Ru^{n+1}-Ru^{n}}{\Delta t}\\
=&\frac{W^{n+1}+W^{n}}{2}-\frac{R\dot{u}^{n+1}+R\dot{u}^{n}}{2}+\frac{R\dot{u}^{n+1}+R\dot{u}^{n}}{2}-\frac{Ru^{n+1}-Ru^{n}}{\Delta t}\\
=&\frac{\varpi^{n+1}+\varpi^{n}}{2}+\frac{R\dot{u}^{n+1}+R\dot{u}^{n}}{2}-\frac{Ru^{n+1}-Ru^{n}}{\Delta t}\\
=&\frac{\varpi^{n+1}+\varpi^{n}}{2}+\frac{R\dot{u}^{n+1}+R\dot{u}^{n}}{2}-\frac{Ru^{n+1}-Ru^{n}}{\Delta t}+\frac{u^{n+1}-u^{n}}{\Delta t}-\frac{u^{n+1}-u^{n}}{\Delta t}\\
=&\frac{\varpi^{n+1}+\varpi^{n}}{2}+\frac{R\dot{u}^{n+1}+R\dot{u}^{n}}{2}+\frac{\theta^{n+1}-\theta^{n}}{\Delta t}-\frac{u^{n+1}-u^{n}}{\Delta t}\\
=&\frac{\varpi^{n+1}+\varpi^{n}}{2}+\frac{R\dot{u}^{n+1}+R\dot{u}^{n}}{2}+\frac{\theta^{n+1}-\theta^{n}}{\Delta t}-\frac{u^{n+1}-u^{n}}{\Delta t}\\
&+\frac{\dot{u}^{n+1}+\dot{u}^{n}}{2}-\frac{\dot{u}^{n+1}+\dot{u}^{n}}{2}\\
=&\frac{\varpi^{n+1}+\varpi^{n}}{2}+\frac{\theta^{n+1}-\theta^{n}}{\Delta t}-\frac{\dot{\theta}^{n+1}+\dot{\theta}^{n}}{2}-\frac{u^{n+1}-u^{n}}{\Delta t}+\frac{\dot{u}^{n+1}+\dot{u}^{n}}{2}\\
=&\frac{\varpi^{n+1}+\varpi^{n}}{2}-\mathcal{E}_2^n-\mathcal{E}_3^n, \qquad (2.3.16)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{E}_2(t)&:=\frac{\dot{\theta}(t+\Delta t)+\dot{\theta}(t)}{2}-\frac{\theta(t+\Delta t)-\theta(t)}{\Delta t},\\
\mathcal{E}_3(t)&:=\frac{u(t+\Delta t)-u(t)}{\Delta t}-\frac{\dot{u}(t+\Delta t)+\dot{u}(t)}{2}.
\end{aligned}
$$

With taking into account $v = \frac{\chi^{n+1}-\chi^{n}}{\Delta t}$ with (2.3.16), we can derive

$$\begin{aligned}
&\frac{\rho}{\Delta t}\left(\varpi^{n+1}-\varpi^{n}, \frac{\chi^{n+1}-\chi^{n}}{\Delta t}\right)_{L_2(\Omega)} + \frac{1}{2}a\left(\chi^{n+1}+\chi^{n}, \frac{\chi^{n+1}-\chi^{n}}{\Delta t}\right) \\
&\quad - \frac{1}{2}\sum_{q=1}^{N_\varphi} a\left(\varsigma_q^{n+1}+\varsigma_q^{n}, \frac{\chi^{n+1}-\chi^{n}}{\Delta t}\right), \\
&=\frac{\rho}{\Delta t}\left(\varpi^{n+1}-\varpi^{n}, \frac{\varpi^{n+1}+\varpi^{n}}{2}\right)_{L_2(\Omega)} - \frac{\rho}{\Delta t}\left(\varpi^{n+1}-\varpi^{n}, \mathcal{E}_2^n\right)_{L_2(\Omega)} \\
&\quad - \frac{\rho}{\Delta t}\left(\varpi^{n+1}-\varpi^{n}, \mathcal{E}_3^n\right)_{L_2(\Omega)} + \frac{1}{2}a\left(\chi^{n+1}+\chi^{n}, \frac{\chi^{n+1}-\chi^{n}}{\Delta t}\right) \\
&\quad - \frac{1}{2}\sum_{q=1}^{N_\varphi} a\left(\varsigma_q^{n+1}+\varsigma_q^{n}, \frac{\chi^{n+1}-\chi^{n}}{\Delta t}\right), \\
&=\frac{\rho}{2\Delta t}\left(\left\|\varpi^{n+1}\right\|_{L_2(\Omega)}^2 - \left\|\varpi^{n}\right\|_{L_2(\Omega)}^2\right) - \frac{\rho}{\Delta t}\left(\varpi^{n+1}-\varpi^{n}, \mathcal{E}_2^n\right)_{L_2(\Omega)} \\
&\quad - \frac{\rho}{\Delta t}\left(\varpi^{n+1}-\varpi^{n}, \mathcal{E}_3^n\right)_{L_2(\Omega)} + \frac{1}{2\Delta t}\left(\left\|\chi^{n+1}\right\|_V^2 - \left\|\chi^{n}\right\|_V^2\right) \\
&\quad - \frac{1}{2\Delta t}\sum_{q=1}^{N_\varphi} a\left(\varsigma_q^{n+1}+\varsigma_q^{n}, \chi^{n+1}-\chi^{n}\right), \\
&=\frac{\rho}{2\Delta t}\left(\dot{\theta}^{n+1}-\dot{\theta}^{n}, \varpi^{n+1}+\varpi^{n}\right)_{L_2(\Omega)} - \frac{\rho}{\Delta t}\left(\dot{\theta}^{n+1}-\dot{\theta}^{n}, \mathcal{E}_2^n\right)_{L_2(\Omega)} \\
&\quad - \frac{\rho}{\Delta t}\left(\dot{\theta}^{n+1}-\dot{\theta}^{n}, \mathcal{E}_3^n\right)_{L_2(\Omega)} + \frac{\rho}{2}\left(\mathcal{E}_1^n, \varpi^{n+1}+\varpi^{n}\right)_{L_2(\Omega)} - \rho\left(\mathcal{E}_1^n, \mathcal{E}_2^n\right)_{L_2(\Omega)} \\
&\quad - \rho\left(\mathcal{E}_1^n, \mathcal{E}_3^n\right)_{L_2(\Omega)},
\end{aligned}$$

hence

$$\begin{aligned}
&\frac{\rho}{2\Delta t}\left(\left\|\varpi^{n+1}\right\|_{L_2(\Omega)}^2 - \left\|\varpi^{n}\right\|_{L_2(\Omega)}^2\right) + \frac{1}{2\Delta t}\left(\left\|\chi^{n+1}\right\|_V^2 - \left\|\chi^{n}\right\|_V^2\right) \\
&\quad - \frac{1}{2}\sum_{q=1}^{N_\varphi} a\left(\varsigma_q^{n+1}+\varsigma_q^{n}, \frac{\chi^{n+1}-\chi^{n}}{\Delta t}\right), \\
&=\frac{\rho}{2\Delta t}\left(\dot{\theta}^{n+1}-\dot{\theta}^{n}, \varpi^{n+1}+\varpi^{n}\right)_{L_2(\Omega)} - \frac{\rho}{\Delta t}\left(\dot{\theta}^{n+1}-\dot{\theta}^{n}, \mathcal{E}_2^n\right)_{L_2(\Omega)} \\
&\quad - \frac{\rho}{\Delta t}\left(\dot{\theta}^{n+1}-\dot{\theta}^{n}, \mathcal{E}_3^n\right)_{L_2(\Omega)} + \frac{\rho}{2}\left(\mathcal{E}_1^n, \varpi^{n+1}+\varpi^{n}\right)_{L_2(\Omega)} - \rho\left(\mathcal{E}_1^n, \mathcal{E}_2^n\right)_{L_2(\Omega)} \\
&\quad - \rho\left(\mathcal{E}_1^n, \mathcal{E}_3^n\right)_{L_2(\Omega)} + \frac{\rho}{\Delta t}\left(\varpi^{n+1}-\varpi^{n}, \mathcal{E}_2^n\right)_{L_2(\Omega)} + \frac{\rho}{\Delta t}\left(\varpi^{n+1}-\varpi^{n}, \mathcal{E}_3^n\right)_{L_2(\Omega)}.
\end{aligned} \tag{2.3.17}$$

Moreover, when we apply summation for $n = 0, \ldots, m-1$ where $m \leq N$ to (2.3.17), it is observed that

$$
\begin{aligned}
&\frac{\rho}{2\Delta t}\left\|\varpi^m\right\|_{L_2(\Omega)}^2+\frac{1}{2\Delta t}\left\|\chi^m\right\|_V^2-\frac{1}{2\Delta t}\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi}a\left(\varsigma_q^{n+1}+\varsigma_q^n,\chi^{n+1}-\chi^n\right),\\
&=\frac{\rho}{2\Delta t}\left\|\varpi^0\right\|_{L_2(\Omega)}^2+\frac{1}{2\Delta t}\left\|\chi^0\right\|_V^2+\frac{\rho}{2\Delta t}\sum_{n=0}^{m-1}\left(\dot{\theta}^{n+1}-\dot{\theta}^n,\varpi^{n+1}+\varpi^n\right)_{L_2(\Omega)}\\
&\quad-\frac{\rho}{\Delta t}\sum_{n=0}^{m-1}\left(\dot{\theta}^{n+1}-\dot{\theta}^n,\mathcal{E}_2^n\right)_{L_2(\Omega)}-\frac{\rho}{\Delta t}\sum_{n=0}^{m-1}\left(\dot{\theta}^{n+1}-\dot{\theta}^n,\mathcal{E}_3^n\right)_{L_2(\Omega)}\\
&\quad+\frac{\rho}{2}\sum_{n=0}^{m-1}\left(\mathcal{E}_1^n,\varpi^{n+1}+\varpi^n\right)_{L_2(\Omega)}-\rho\sum_{n=0}^{m-1}\left(\mathcal{E}_1^n,\mathcal{E}_2^n\right)_{L_2(\Omega)}-\rho\sum_{n=0}^{m-1}\left(\mathcal{E}_1^n,\mathcal{E}_3^n\right)_{L_2(\Omega)}\\
&\quad+\frac{\rho}{\Delta t}\sum_{n=0}^{m-1}\left(\varpi^{n+1}-\varpi^n,\mathcal{E}_2^n\right)_{L_2(\Omega)}+\frac{\rho}{\Delta t}\sum_{n=0}^{m-1}\left(\varpi^{n+1}-\varpi^n,\mathcal{E}_3^n\right)_{L_2(\Omega)}. \qquad (2.3.18)
\end{aligned}
$$

In a similar way, let us consider internal variables (2.1.7) and (2.3.3) for each q

$$
\begin{aligned}
&\tau_q a\left(\frac{\dot{\psi}_q^{n+1}+\dot{\psi}_q^n}{2}-\frac{\Psi_{hq}^{n+1}-\Psi_{hq}^n}{\Delta t},v\right)+\frac{1}{2}a\left((\psi_q^{n+1}+\psi_q^n)-(\Psi_{hq}^{n+1}+\Psi_{hq}^n),v\right)\\
&=\frac{\varphi_q}{2}a\left((u^{n+1}+u^n)-(U_h^{n+1}+U_h^n),v\right)
\end{aligned}
$$

so using elliptic projection gives

$$
\begin{aligned}
&\frac{\tau_q}{\Delta t}a\left(\varsigma_q^{n+1}-\varsigma_q^n,v\right)+\frac{1}{2}a\left(\varsigma_q^{n+1}+\varsigma_q^n,v\right)-\frac{\varphi_q}{2}a\left(\chi^{n+1}+\chi^n,v\right)\\
&=\frac{\tau_q}{\Delta t}a\left(\vartheta_q^{n+1}-\vartheta_q^n,v\right)+\frac{1}{2}a\left(\vartheta_q^{n+1}+\vartheta_q^n,v\right)-\frac{\varphi_q}{2}a\left(\theta^{n+1}+\theta^n,v\right)\\
&\quad+\tau_q a\left(\frac{\dot{\psi}_q^{n+1}+\dot{\psi}_q^n}{2}-\frac{\psi_q^{n+1}-\psi_q^n}{\Delta t},v\right)\\
&=\tau_q a\left(E_q^n,v\right)
\end{aligned}
$$

by Galerkin orthogonality, where for each q

$$E_q(t)=\frac{\dot{\psi}_q(t+\Delta t)+\dot{\psi}_q(t)}{2}-\frac{\psi_q(t+\Delta t)-\psi_q(t)}{\Delta t}.$$

Here, let us set $v=\frac{\varsigma_q^{n+1}-\varsigma_q^n}{\Delta t}$ and take summation for $n=0,\ldots,m-1$ then we have

$$
\begin{aligned}
&\frac{\tau_q}{\Delta t^2}\sum_{n=0}^{m-1}\left\|\varsigma_q^{n+1}-\varsigma_q^n\right\|_V^2+\frac{1}{2\Delta t}\left(\left\|\varsigma_q^m\right\|_V^2-\left\|\varsigma_q^0\right\|_V^2\right)-\frac{\varphi_q}{2\Delta t}\sum_{n=0}^{m-1}a\left(\chi^{n+1}+\chi^n,\varsigma_q^{n+1}-\varsigma_q^n\right)\\
&=\frac{\tau_q}{\Delta t}\sum_{n=0}^{m-1}a\left(E_q^n,\varsigma_q^{n+1}-\varsigma_q^n\right).
\end{aligned}
$$

In addition, we will apply summation by parts then

$$\begin{aligned}\sum_{n=0}^{m-1} a\left(\chi^{n+1}+\chi^{n}, \varsigma_q^{n+1}-\varsigma_q^{n}\right) =& \sum_{n=0}^{m-1} a\left(\chi^{n+1}, \varsigma_q^{n+1}-\varsigma_q^{n}\right) + \sum_{n=0}^{m-1} a\left(\chi^{n}, \varsigma_q^{n+1}-\varsigma_q^{n}\right) \\ =& a\left(\chi^{m}, \varsigma_q^{m}\right) - a\left(\chi^{0}, \varsigma_q^{0}\right) - \sum_{n=0}^{m-1} a\left(\chi^{n+1}-\chi^{n}, \varsigma_q^{n}\right) \\ &+ a\left(\chi^{m}, \varsigma_q^{m}\right) - a\left(\chi^{0}, \varsigma_q^{0}\right) - \sum_{n=0}^{m-1} a\left(\chi^{n+1}-\chi^{n}, \varsigma_q^{n+1}\right) \\ =& 2a\left(\chi^{m}, \varsigma_q^{m}\right) - 2a\left(\chi^{0}, \varsigma_q^{0}\right) - \sum_{n=0}^{m-1} a\left(\chi^{n+1}-\chi^{n}, \varsigma_q^{n+1}+\varsigma_q^{n}\right) \\ =& 2a\left(\chi^{m}, \varsigma_q^{m}\right) - \sum_{n=0}^{m-1} a\left(\chi^{n+1}-\chi^{n}, \varsigma_q^{n+1}+\varsigma_q^{n}\right)\end{aligned}$$

and

$$\begin{aligned}\sum_{n=0}^{m-1} a\left(E_q^{n}, \varsigma_q^{n+1}-\varsigma_q^{n}\right) =& a\left(E_q^{m-1}, \varsigma_q^{m}\right) - a\left(E_q^{0}, \varsigma_q^{0}\right) - \sum_{n=0}^{m-2} a\left(E_q^{n+1}-E_q^{n}, \varsigma_q^{n+1}\right) \\ =& a\left(E_q^{m-1}, \varsigma_q^{m}\right) - \sum_{n=0}^{m-1} a\left(E_q^{n+1}-E_q^{n}, \varsigma_q^{n+1}\right)\end{aligned}$$

since $\varsigma_q^0 = 0,\ \forall q \in \{1, \ldots, N_\varphi\}$. Thus these imply

$$\begin{aligned}\frac{\varphi_q}{2\Delta t}\sum_{n=0}^{m-1} a\left(\chi^{n+1}-\chi^{n}, \varsigma_q^{n+1}+\varsigma_q^{n}\right) =& \frac{\varphi_q}{\Delta t} a\left(\chi^{m}, \varsigma_q^{m}\right) - \frac{\tau_q}{\Delta t^2}\sum_{n=0}^{m-1}\left\|\varsigma_q^{n+1}-\varsigma_q^{n}\right\|_V^2 \\ &- \frac{1}{2\Delta t}\left\|\varsigma_q^{m}\right\|_V^2 + \frac{\tau_q}{\Delta t} a\left(E_q^{m-1}, \varsigma_q^{m}\right) \\ &- \frac{\tau_q}{\Delta t}\sum_{n=0}^{m-2} a\left(E_q^{n+1}-E_q^{n}, \varsigma_q^{n+1}\right).\end{aligned}$$

As following this result, (2.3.18) can be rewritten as

$$\begin{aligned}&\frac{\rho}{2\Delta t}\left\|\varpi^{m}\right\|_{L_2(\Omega)}^2 + \frac{1}{2\Delta t}\left\|\chi^{m}\right\|_V^2 + \frac{1}{2\Delta t}\sum_{q=1}^{N_\varphi}\frac{1}{\varphi_q}\left\|\varsigma_q^{m}\right\|_V^2 + \frac{1}{\Delta t^2}\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi}\frac{\tau_q}{\varphi_q}\left\|\varsigma_q^{n+1}-\varsigma_q^{n}\right\|_V^2 \\ =& \frac{\rho}{2\Delta t}\left\|\varpi^{0}\right\|_{L_2(\Omega)}^2 + \frac{1}{2\Delta t}\left\|\chi^{0}\right\|_V^2 + \frac{\rho}{2\Delta t}\sum_{n=0}^{m-1}\left(\dot{\theta}^{n+1}-\dot{\theta}^{n}, \varpi^{n+1}+\varpi^{n}\right)_{L_2(\Omega)} \\ &- \frac{\rho}{\Delta t}\sum_{n=0}^{m-1}\left(\dot{\theta}^{n+1}-\dot{\theta}^{n}, \mathcal{E}_2^{n}\right)_{L_2(\Omega)} - \frac{\rho}{\Delta t}\sum_{n=0}^{m-1}\left(\dot{\theta}^{n+1}-\dot{\theta}^{n}, \mathcal{E}_3^{n}\right)_{L_2(\Omega)}\end{aligned}$$

$$
+\frac{\rho}{2}\sum_{n=0}^{m-1}\left(\mathcal{E}_1^n,\varpi^{n+1}+\varpi^n\right)_{L_2(\Omega)}-\rho\sum_{n=0}^{m-1}\left(\mathcal{E}_1^n,\mathcal{E}_2^n\right)_{L_2(\Omega)}-\rho\sum_{n=0}^{m-1}\left(\mathcal{E}_1^n,\mathcal{E}_3^n\right)_{L_2(\Omega)}
$$
$$
+\frac{\rho}{\Delta t}\sum_{n=0}^{m-1}\left(\varpi^{n+1}-\varpi^n,\mathcal{E}_2^n\right)_{L_2(\Omega)}+\frac{\rho}{\Delta t}\sum_{n=0}^{m-1}\left(\varpi^{n+1}-\varpi^n,\mathcal{E}_3^n\right)_{L_2(\Omega)}
$$
$$
+\frac{1}{\Delta t}\sum_{q=1}^{N_\varphi}a\left(\chi^m,\varsigma_q^m\right)+\frac{1}{\Delta t}\sum_{q=1}^{N_\varphi}\frac{\tau_q}{\varphi_q}a\left(E_q^{m-1},\varsigma_q^m\right)
$$
$$
-\frac{1}{\Delta t}\sum_{n=0}^{m-2}\sum_{q=1}^{N_\varphi}\frac{\tau_q}{\varphi_q}a\left(E_q^{n+1}-E_q^n,\varsigma_q^{n+1}\right),
$$

and multiplying it by Δt allows us to have

$$
\frac{\rho}{2}\left\|\varpi^m\right\|_{L_2(\Omega)}^2+\frac{1}{2}\left\|\chi^m\right\|_V^2+\frac{1}{2}\sum_{q=1}^{N_\varphi}\frac{1}{\varphi_q}\left\|\varsigma_q^m\right\|_V^2+\Delta t\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi}\frac{\tau_q}{\varphi_q}\left\|\frac{\varsigma_q^{n+1}-\varsigma_q^n}{\Delta t}\right\|_V^2
$$
$$
=\frac{\rho}{2}\left\|\varpi^0\right\|_{L_2(\Omega)}^2+\frac{1}{2}\left\|\chi^0\right\|_V^2+\frac{\rho}{2}\sum_{n=0}^{m-1}\left(\dot{\theta}^{n+1}-\dot{\theta}^n,\varpi^{n+1}+\varpi^n\right)_{L_2(\Omega)}
$$
$$
-\rho\sum_{n=0}^{m-1}\left(\dot{\theta}^{n+1}-\dot{\theta}^n,\mathcal{E}_2^n\right)_{L_2(\Omega)}-\rho\sum_{n=0}^{m-1}\left(\dot{\theta}^{n+1}-\dot{\theta}^n,\mathcal{E}_3^n\right)_{L_2(\Omega)}
$$
$$
+\frac{\rho}{2}\Delta t\sum_{n=0}^{m-1}\left(\mathcal{E}_1^n,\varpi^{n+1}+\varpi^n\right)_{L_2(\Omega)}-\rho\Delta t\sum_{n=0}^{m-1}\left(\mathcal{E}_1^n,\mathcal{E}_2^n\right)_{L_2(\Omega)}-\rho\Delta t\sum_{n=0}^{m-1}\left(\mathcal{E}_1^n,\mathcal{E}_3^n\right)_{L_2(\Omega)}
$$
$$
+\rho\sum_{n=0}^{m-1}\left(\varpi^{n+1}-\varpi^n,\mathcal{E}_2^n\right)_{L_2(\Omega)}+\rho\sum_{n=0}^{m-1}\left(\varpi^{n+1}-\varpi^n,\mathcal{E}_3^n\right)_{L_2(\Omega)}+\sum_{q=1}^{N_\varphi}a\left(\chi^m,\varsigma_q^m\right)
$$
$$
+\sum_{q=1}^{N_\varphi}\frac{\tau_q}{\varphi_q}a\left(E_q^{m-1},\varsigma_q^m\right)-\sum_{n=0}^{m-2}\sum_{q=1}^{N_\varphi}\frac{\tau_q}{\varphi_q}a\left(E_q^{n+1}-E_q^n,\varsigma_q^{n+1}\right). \tag{2.3.19}
$$

Now, we will consider each component of (2.3.19) on the right hand side.

- $\left\|\varpi^0\right\|_{L_2(\Omega)}^2$

$$
\begin{aligned}
\left\|\varpi^0\right\|_{L_2(\Omega)}^2 &= \left(W_h^0-Rw_0,W_h^0-Rw_0\right)_{L_2(\Omega)}\\
&= \left(w_0-Rw_0,W_h^0-Rw_0\right)_{L_2(\Omega)}\\
&\quad (\because\ \text{since } (w_0,v)_{L_2(\Omega)}=\left(W_h^0,v\right)_{L_2(\Omega)},\ \forall v\in V^h)\\
&= \left(\dot{\theta}^0,\varpi^0\right)_{L_2(\Omega)}\\
&\quad (\because \text{since } \dot{\theta}^0=w_0-Rw_0)
\end{aligned}
$$

$$\leq \left\|\dot{\theta}^0\right\|_{L_2(\Omega)} \left\|\varpi^0\right\|_{L_2(\Omega)}$$
($\because$ by Cauchy-Schwarz inequality)

so,

$$\left\|\varpi^0\right\|^2_{L_2(\Omega)} \leq \left\|\dot{\theta}^0\right\|^2_{L_2(\Omega)}.$$

- $\left\|\chi^0\right\|^2_V$

$$a\left(U_h^0, v\right) = a\left(u_0, v\right), \ \forall v \in V^h,$$
$$a\left(Ru_0, v\right) = a\left(u_0, v\right), \ \forall v \in V^h,$$

hence

$$a\left(U_h^0 - Ru_0, v\right) = 0, \ \forall v \in V^h,$$

and

$$\left\|\chi^0\right\|^2_V = a\left(U_h^0 - Ru_0, U_h^0 - Ru_0\right) = 0.$$

- $\sum_{n=0}^{m-1}\left(\dot{\theta}^{n+1} - \dot{\theta}^n, \varpi^{n+1} + \varpi^n\right)_{L_2(\Omega)}$

$$\sum_{n=0}^{m-1}\left(\dot{\theta}^{n+1} - \dot{\theta}^n, \varpi^{n+1} + \varpi^n\right)_{L_2(\Omega)}$$
$$= \sum_{n=0}^{m-1}\int_{t_n}^{t_{n+1}}\left(\ddot{\theta}(t'), \varpi^{n+1} + \varpi^n\right)_{L_2(\Omega)} dt'$$
$$\leq \sum_{n=0}^{m-1}\int_{t_n}^{t_{n+1}}\left\|\ddot{\theta}(t')\right\|_{L_2(\Omega)} \left\|\varpi^{n+1} + \varpi^n\right\|_{L_2(\Omega)} dt'$$
($\because$ by Cauchy-Schwarz inequality)
$$\leq \frac{1}{2\epsilon_a}\sum_{n=0}^{m-1}\int_{t_n}^{t_{n+1}}\left\|\ddot{\theta}(t')\right\|^2_{L_2(\Omega)} dt' + \frac{\epsilon_a}{2}\sum_{n=0}^{m-1}\int_{t_n}^{t_{m+1}}\left\|\varpi^{n+1} + \varpi^n\right\|^2_{L_2(\Omega)} dt'$$
($\because$ by Young's inequality for some positive ϵ_a)
$$\leq \frac{1}{2\epsilon_a}\int_0^{t_m}\left\|\ddot{\theta}(t')\right\|^2_{L_2(\Omega)} dt' + \frac{\epsilon_a}{2}\sum_{n=0}^{m-1}\left\|\varpi^{n+1} + \varpi^n\right\|^2_{L_2(\Omega)} \Delta t$$
($\because$ $\left\|\varpi^{n+1} + \varpi^n\right\|_{L_2(\Omega)}$ is independent of s)
$$\leq \frac{1}{2\epsilon_a}\int_0^{t_m}\left\|\ddot{\theta}(t')\right\|^2_{L_2(\Omega)} dt' + 2\epsilon_a\sum_{n=0}^{m-1}\max_{0\leq j\leq N}\left\|\varpi^j\right\|^2_{L_2(\Omega)} \Delta t$$

$(\because\ \left\|\varpi^{n+1}+\varpi^{n}\right\|_{L_2(\Omega)}^2 \le 2\left\|\varpi^{n+1}\right\|_{L_2(\Omega)}^2 + 2\left\|\varpi^{n}\right\|_{L_2(\Omega)}^2 \le 4\max_{0\le j\le N}\left\|\varpi^{j}\right\|_{L_2(\Omega)}^2)$

$$\le \frac{1}{2\epsilon_a}\int_0^T \left\|\ddot{\theta}(t')\right\|_{L_2(\Omega)}^2 dt' + 2\epsilon_a \sum_{n=0}^{N-1} \max_{0\le j\le N}\left\|\varpi^{j}\right\|_{L_2(\Omega)}^2 \Delta t$$

$(\because m \le N\ )$

$$= \frac{1}{2\epsilon_a}\left\|\ddot{\theta}\right\|_{L_2(0,T;L_2(\Omega))}^2 + 2\epsilon_a T \max_{0\le j\le N}\left\|\varpi^{j}\right\|_{L_2(\Omega)}^2 .$$

- $-\sum_{n=0}^{m-1}\left(\dot{\theta}^{n+1}-\dot{\theta}^{n},\mathcal{E}_2^n\right)_{L_2(\Omega)},\ -\sum_{n=0}^{m-1}\left(\dot{\theta}^{n+1}-\dot{\theta}^{n},\mathcal{E}_3^n\right)_{L_2(\Omega)}$

  In the same sense as the above,

$$\begin{aligned}
-\sum_{n=0}^{m-1}\left(\dot{\theta}^{n+1}-\dot{\theta}^{n},\mathcal{E}_2^n\right)_{L_2(\Omega)} &= -\sum_{n=0}^{m-1}\int_{t_n}^{t_{n+1}}\left(\ddot{\theta}(t'),\mathcal{E}_2^n\right)_{L_2(\Omega)} dt' \\
&\le \frac{1}{2}\int_0^{t_m}\left\|\ddot{\theta}(t')\right\|_{L_2(\Omega)}^2 dt' + \frac{1}{2}\sum_{n=0}^{m-1}\left\|\mathcal{E}_2^n\right\|_{L_2(\Omega)}^2 \Delta t \\
&\le \frac{1}{2}\left\|\ddot{\theta}\right\|_{L_2(0,T;L_2(\Omega))}^2 + \frac{T}{2}\max_{0\le j\le N-1}\left\|\mathcal{E}_2^j\right\|_{L_2(\Omega)}^2 .
\end{aligned}$$

  Also,

$$-\sum_{n=0}^{m-1}\left(\dot{\theta}^{n+1}-\dot{\theta}^{n},\mathcal{E}_3^n\right)_{L_2(\Omega)} \le \frac{1}{2}\left\|\ddot{\theta}\right\|_{L_2(0,T;L_2(\Omega))}^2 + \frac{T}{2}\max_{0\le j\le N-1}\left\|\mathcal{E}_3^j\right\|_{L_2(\Omega)}^2 .$$

- $\Delta t\sum_{n=0}^{m-1}\left(\mathcal{E}_1^n,\varpi^{n+1}+\varpi^{n}\right)_{L_2(\Omega)}$

$$\Delta t\sum_{n=0}^{m-1}\left(\mathcal{E}_1^n,\varpi^{n+1}+\varpi^{n}\right)_{L_2(\Omega)}$$

$$\le \frac{\Delta t}{2\epsilon_b}\sum_{n=0}^{m-1}\left\|\mathcal{E}_1^n\right\|_{L_2(\Omega)}^2 + \frac{\epsilon_b\Delta t}{2}\sum_{n=0}^{m-1}\left\|\varpi^{n+1}+\varpi^{n}\right\|_{L_2(\Omega)}^2$$

($\because$ by Cauchy-Schwarz inequality and Young's inequality for some positive ϵ_b)

$$\le \frac{\Delta t}{2\epsilon_b}\sum_{n=0}^{m-1}\left\|\mathcal{E}_1^n\right\|_{L_2(\Omega)}^2 + 2\epsilon_b\Delta t\sum_{n=0}^{m-1}\max_{0\le j\le N}\left\|\varpi^{j}\right\|_{L_2(\Omega)}^2$$

$(\because\ \left\|\varpi^{n+1}+\varpi^{n}\right\|_{L_2(\Omega)}^2 \le 2\left\|\varpi^{n+1}\right\|_{L_2(\Omega)}^2 + 2\left\|\varpi^{n}\right\|_{L_2(\Omega)}^2 \le 4\max_{0\le j\le N}\left\|\varpi^{j}\right\|_{L_2(\Omega)}^2)$

$$\le \frac{T}{2\epsilon_b}\max_{0\le j\le N-1}\left\|\mathcal{E}_1^j\right\|_{L_2(\Omega)}^2 + 2\epsilon_b T\max_{0\le j\le N}\left\|\varpi^{j}\right\|_{L_2(\Omega)}^2 .$$

$(\because\ m\le N,\ T=N\Delta t)$

- $-\Delta t \sum_{n=0}^{m-1} \left(\mathcal{E}_1^n, \mathcal{E}_2^n\right)_{L_2(\Omega)}, \; -\Delta t \sum_{n=0}^{m-1} \left(\mathcal{E}_1^n, \mathcal{E}_3^n\right)_{L_2(\Omega)}$

$$\begin{aligned} -\Delta t \sum_{n=0}^{m-1} \left(\mathcal{E}_1^n, \mathcal{E}_2^n\right)_{L_2(\Omega)} \leq & \frac{\Delta t}{2} \sum_{n=0}^{m-1} \|\mathcal{E}_1^n\|_{L_2(\Omega)}^2 + \frac{\Delta t}{2} \sum_{n=0}^{m-1} \|\mathcal{E}_2^n\|_{L_2(\Omega)}^2 \\ \leq & \frac{\Delta t}{2} \sum_{n=0}^{m-1} \max_{0\leq j\leq N-1} \left\|\mathcal{E}_1^j\right\|_{L_2(\Omega)}^2 + \frac{\Delta t}{2} \sum_{n=0}^{m-1} \max_{0\leq j\leq N-1} \left\|\mathcal{E}_2^j\right\|_{L_2(\Omega)}^2 \\ \leq & \frac{T}{2} \max_{0\leq j\leq N-1} \left\|\mathcal{E}_1^j\right\|_{L_2(\Omega)}^2 + \frac{T}{2} \max_{0\leq j\leq N-1} \left\|\mathcal{E}_2^j\right\|_{L_2(\Omega)}^2 . \end{aligned}$$

  In the same way,

$$-\Delta t \sum_{n=0}^{m-1} \left(\mathcal{E}_1^n, \mathcal{E}_3^n\right)_{L_2(\Omega)} \leq \frac{T}{2} \max_{0\leq j\leq N-1} \left\|\mathcal{E}_1^j\right\|_{L_2(\Omega)}^2 + \frac{T}{2} \max_{0\leq j\leq N-1} \left\|\mathcal{E}_3^j\right\|_{L_2(\Omega)}^2 .$$

- $\sum_{n=0}^{m-1} \left(\varpi^{n+1} - \varpi^n, \mathcal{E}_2^n\right)_{L_2(\Omega)}, \; \sum_{n=0}^{m-1} \left(\varpi^{n+1} - \varpi^n, \mathcal{E}_3^n\right)_{L_2(\Omega)}$

  By summation by parts,

$$\begin{aligned} & \sum_{n=0}^{m-1} \left(\varpi^{n+1} - \varpi^n, \mathcal{E}_2^n\right)_{L_2(\Omega)} \\ & = \left(\varpi^m, \mathcal{E}_2^m\right)_{L_2(\Omega)} - \left(\varpi^0, \mathcal{E}_2^0\right)_{L_2(\Omega)} - \sum_{n=0}^{m-1} \left(\varpi^{n+1}, \mathcal{E}_2^{n+1} - \mathcal{E}_2^n\right)_{L_2(\Omega)} \\ & = \left(\varpi^m, \mathcal{E}_2^m\right)_{L_2(\Omega)} - \left(\varpi^0, \mathcal{E}_2^0\right)_{L_2(\Omega)} - \sum_{n=0}^{m-1} \int_{t_n}^{t_{n+1}} \left(\varpi^{n+1}, \dot{\mathcal{E}}_2(t')\right)_{L_2(\Omega)} dt' \\ & \leq \frac{\epsilon_c}{2} \|\varpi^m\|_{L_2(\Omega)}^2 + \frac{1}{2\epsilon_c} \|\mathcal{E}_2^m\|_{L_2(\Omega)}^2 + \frac{1}{2} \|\varpi^0\|_{L_2(\Omega)}^2 + \frac{1}{2} \|\mathcal{E}_2^0\|_{L_2(\Omega)}^2 \\ & \quad + \frac{\epsilon_d}{2} \sum_{n=0}^{m-1} \|\varpi^{n+1}\|_{L_2(\Omega)}^2 \Delta t + \frac{1}{2\epsilon_d} \int_0^{t_m} \left\|\dot{\mathcal{E}}_2(t')\right\|_{L_2(\Omega)}^2 dt' \\ & \leq \frac{\epsilon_c}{2} \max_{0\leq j\leq N} \|\varpi^j\|_{L_2(\Omega)}^2 + \frac{1}{2\epsilon_c} \max_{0\leq j\leq N-1} \left\|\mathcal{E}_2^j\right\|_{L_2(\Omega)}^2 + \frac{1}{2} \left\|\dot{\theta}^0\right\|_{L_2(\Omega)}^2 \\ & \quad + \frac{1}{2} \max_{0\leq j\leq N-1} \left\|\mathcal{E}_2^j\right\|_{L_2(\Omega)}^2 + \frac{\epsilon_d}{2} \sum_{n=0}^{m-1} \max_{0\leq j\leq N} \|\varpi^j\|_{L_2(\Omega)}^2 \Delta t \\ & \quad + \frac{1}{2\epsilon_d} \int_0^{t_m} \left\|\dot{\mathcal{E}}_2(t')\right\|_{L_2(\Omega)}^2 dt' \\ & \leq \frac{\epsilon_c}{2} \max_{0\leq j\leq N} \|\varpi^j\|_{L_2(\Omega)}^2 + \frac{1}{2\epsilon_c} \max_{0\leq j\leq N-1} \left\|\mathcal{E}_2^j\right\|_{L_2(\Omega)}^2 + \frac{1}{2} \left\|\dot{\theta}^0\right\|_{L_2(\Omega)}^2 \\ & \quad + \frac{1}{2} \max_{0\leq j\leq N-1} \left\|\mathcal{E}_2^j\right\|_{L_2(\Omega)}^2 + \frac{\epsilon_d}{2} T \max_{0\leq j\leq N} \|\varpi^j\|_{L_2(\Omega)}^2 + \frac{1}{2\epsilon_d} \left\|\dot{\mathcal{E}}_2\right\|_{L_2(0,T;L_2(\Omega))}^2 \end{aligned}$$

for positive ϵ_c and ϵ_d. In this manner,

$$\begin{aligned}
&\sum_{n=0}^{m-1} \left(\varpi^{n+1} - \varpi^n, \mathcal{E}_3^n\right)_{L_2(\Omega)} \\
&\quad \leq \frac{\epsilon_c}{2} \max_{0\leq j\leq N} \left\|\varpi^j\right\|_{L_2(\Omega)}^2 + \frac{1}{2\epsilon_c} \max_{0\leq j\leq N-1} \left\|\mathcal{E}_3^j\right\|_{L_2(\Omega)}^2 + \frac{1}{2} \left\|\dot{\theta}^0\right\|_{L_2(\Omega)}^2 \\
&\qquad + \frac{1}{2} \max_{0\leq j\leq N-1} \left\|\mathcal{E}_3^j\right\|_{L_2(\Omega)}^2 + \frac{\epsilon_d}{2} T \max_{0\leq j\leq N} \left\|\varpi^j\right\|_{L_2(\Omega)}^2 + \frac{1}{2\epsilon_d} \left\|\dot{\mathcal{E}}_3\right\|_{L_2(0,T;L_2(\Omega))}^2 .
\end{aligned}$$

- $\sum_{q=1}^{N_\varphi} a\left(\chi^m, \varsigma_q^m\right)$, $\sum_{q=1}^{N_\varphi} \frac{\tau_q}{\varphi_q} a\left(E_q^{m-1}, \varsigma_q^m\right)$

$$\sum_{q=1}^{N_\varphi} a\left(\chi^m, \varsigma_q^m\right) \leq \sum_{q=1}^{N_\varphi} \frac{\epsilon_q}{2} \left\|\chi^m\right\|_V^2 + \sum_{q=1}^{N_\varphi} \frac{1}{2\epsilon_q} \left\|\varsigma_q^m\right\|_V^2 ,$$

for some positive $\{\epsilon_q\}$. And,

$$\sum_{q=1}^{N_\varphi} \frac{\tau_q}{\varphi_q} a\left(E_q^{m-1}, \varsigma_q^m\right) \leq \sum_{q=1}^{N_\varphi} \frac{\tau_q}{\varphi_q} \frac{\tilde{\epsilon}_q}{2} \left\|E_q^{m-1}\right\|_V^2 + \sum_{q=1}^{N_\varphi} \frac{\tau_q}{\varphi_q} \frac{1}{2\tilde{\epsilon}_q} \left\|\varsigma_q^m\right\|_V^2 ,$$

for some positive $\{\tilde{\epsilon}_q\}$.

- $-\sum_{n=0}^{m-2} \sum_{q=1}^{N_\varphi} \frac{\tau_q}{\varphi_q} a\left(E_q^{n+1} - E_q^n, \varsigma_q^{n+1}\right)$

$$\begin{aligned}
&-\sum_{n=0}^{m-2} \sum_{q=1}^{N_\varphi} \frac{\tau_q}{\varphi_q} a\left(E_q^{n+1} - E_q^n, \varsigma_q^{n+1}\right) \\
&= -\sum_{q=1}^{N_\varphi} \frac{\tau_q}{\varphi_q} \sum_{n=0}^{m-2} \int_{t_n}^{t_{n+1}} a\left(\dot{E}_q(t'), \varsigma_q^{n+1}\right) dt' \\
&\leq \sum_{q=1}^{N_\varphi} \frac{\tau_q}{\varphi_q} \frac{\check{\epsilon}_q}{2} \int_0^{t_m} \left\|\dot{E}_q(t')\right\|_V^2 dt' + \sum_{q=1}^{N_\varphi} \frac{\tau_q}{\varphi_q} \frac{1}{2\check{\epsilon}_q} \sum_{n=0}^{m-1} \left\|\varsigma_q^{n+1}\right\|_V^2 \Delta t \\
&\leq \sum_{q=1}^{N_\varphi} \frac{\tau_q}{\varphi_q} \frac{\check{\epsilon}_q}{2} \left\|\dot{E}_q\right\|_{L_2(0,T;V)}^2 + \sum_{q=1}^{N_\varphi} \frac{\tau_q}{\varphi_q} \frac{T}{2\check{\epsilon}_q} \max_{0\leq j\leq N} \left\|\varsigma_q^j\right\|_V^2
\end{aligned}$$

for some positive $\{\check{\epsilon}_q\}$.

Hereafter, let us recall the finite difference method in time, that is Crank-Nicolson method(e.g. see [40] in detail). Since we suppose $u \in H^4(0,T;H^s(\Omega))$, Crank-Nicolson method provides

$$|\mathcal{E}_1^n| = \left|\frac{\ddot{u}^{n+1} + \ddot{u}^n}{2} - \frac{\dot{u}^{n+1} - \dot{u}^n}{\Delta t}\right| \leq C\Delta t^2$$

for some positive constant C, $\forall n$. Hence it implies

$$\begin{aligned}\|\mathcal{E}_1^n\|_{L_2(\Omega)}^2 =& \int_\Omega |\mathcal{E}_1^n|^2 \, d\Omega \\ \leq& \int_\Omega C^2 \Delta t^4 d\Omega \\ =& C^2 |\Omega| \Delta t^4 = C\Delta t^4\end{aligned}$$

for any n so that

$$\max_{0\leq j\leq N-1} \left\|\mathcal{E}_1^j\right\|_{L_2(\Omega)} \leq C\Delta t^2.$$

Indeed, $\|\mathcal{E}_1(t)\|_{L_2(\Omega)} \leq C\Delta t^2$ for any t. Similarly, it is also true that

$$\begin{aligned}&\max_{0\leq j\leq N-1} \left\|\mathcal{E}_2^j\right\|_{L_2(\Omega)}, \max_{0\leq j\leq N-1} \left\|\mathcal{E}_3^j\right\|_{L_2(\Omega)} \leq C\Delta t^2, \\ \text{and} \quad &\max_{0\leq j\leq N-1} \left\|\dot{\mathcal{E}}_2^j\right\|_{L_2(\Omega)}, \max_{0\leq j\leq N-1} \left\|\dot{\mathcal{E}}_3^j\right\|_{L_2(\Omega)} \leq C\Delta t^2.\end{aligned}$$

Moreover,

$$\left\|\dot{\mathcal{E}}_2\right\|_{L_2(0,T;L_2(\Omega))}^2 \leq \int_0^T C\Delta t^4 dt' = CT\Delta t^4$$

ans so is $\left\|\dot{\mathcal{E}}_3\right\|_{L_2(0,T;L_2(\Omega))}$. Due to Leibniz's integral rule, we can obtain

$$\max_{0\leq j\leq N-1} \|E_q^j\|_V \leq C\Delta t^2, \ \left\|\dot{E}_q\right\|_{L_2(0,T;V)} \leq C\Delta t^2$$

for any q.

For our sake, we should set coefficients for Young's inequalities. Set

$$\epsilon_a = \frac{1}{16(3+N_\varphi)T}, \ \epsilon_b = \frac{1}{16(3+N_\varphi)T}, \ \epsilon_c = \frac{1}{8(3+N_\varphi)}, \ \epsilon_d = \frac{1}{8(3+N_\varphi)T},$$

then

$$\rho\left(\epsilon_a T + \epsilon_b T + \frac{\epsilon_c}{2} + \frac{\epsilon_d}{2} T\right) = \frac{\rho}{4(3+N_\varphi)}.$$

Additionally, put $\epsilon_q = \varphi_q + \frac{\varphi_0}{2N_\varphi}$ and $\tilde{\epsilon}_q = \frac{\tau_q(4\varphi_q^2 N_\varphi + 2\varphi_0\varphi_q)}{\varphi_q\varphi_0}$ for each q. So, we have

$$\frac{1}{2} - \sum_{q=1}^{N_\varphi} \frac{\epsilon_q}{2} = \frac{\varphi_0}{4},$$

and

$$\sum_{q=1}^{N_\varphi} \frac{1}{2\varphi_q} - \sum_{q=1}^{N_\varphi} \frac{1}{2\epsilon_q} - \sum_{q=1}^{N_\varphi} \frac{\tau_q}{\varphi_q} \frac{1}{2\tilde{\epsilon}_q} = \sum_{q=1}^{N_\varphi} \frac{\varphi_0}{4\varphi_q^2 N_\varphi + 2\varphi_0\varphi_q} - \sum_{q=1}^{N_\varphi} \frac{\varphi_0}{8\varphi_q^2 N_\varphi + 4\varphi_0\varphi_q}$$

$$= \sum_{q=1}^{N_\varphi} \frac{\varphi_0}{8\varphi_q^2 N_\varphi + 4\varphi_0\varphi_q}$$

furthermore we suppose $\check{\epsilon}_q = (3 + N_\varphi)\frac{\tau_q T}{\varphi_q}\frac{8\varphi_q^2 N_\varphi + 4\varphi_0\varphi_q}{\varphi_0}$ for each q.

In the end, as tidying the results up with (1.4.8), we can obtain that (2.3.19) yields

$$\begin{aligned}
&\frac{\rho}{2}\left\|\varpi^m\right\|_{L_2(\Omega)}^2 + \frac{\varphi_0}{4}\left\|\chi^m\right\|_V^2 + \sum_{q=1}^{N_\varphi}\frac{\varphi_0}{8\varphi_q^2 N_\varphi + 4\varphi_0\varphi_q}\left\|\varsigma_q^m\right\|_V^2 \\
&\quad + \Delta t \sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi}\frac{\tau_q}{\varphi_q}\left\|\frac{\varsigma_q^{n+1} - \varsigma_q^n}{\Delta t}\right\|_V^2 \\
&\leq \frac{\rho}{4(3+N_\varphi)}\max_{0\leq j\leq N}\left\|\varpi^j\right\|_{L_2(\Omega)}^2 + \sum_{q=1}^{N_\varphi}\frac{1}{2(3+N_\varphi)}\frac{\varphi_0}{8\varphi_q^2 N_\varphi + 4\varphi_0\varphi_q}\max_{0\leq j\leq N}\left\|\varsigma_q^j\right\|_V^2 \\
&\quad + C(h^{2(\min(k+1,s)-1)} + \Delta t^4)
\end{aligned} \tag{2.3.20}$$

for some positive C. With taking into account the maximum with respect to $\|\varpi^m\|_{L_2(\Omega)}$, $\|\chi^m\|_V$ and $\|\varsigma_q^m\|_V$ on the left hand side of (2.3.20) it is able to claim that

$$\begin{aligned}
&\frac{\rho}{2}\max_{0\leq j\leq N}\left\|\varpi^j\right\|_{L_2(\Omega)}^2 + \frac{\varphi_0}{4}\max_{0\leq j\leq N}\left\|\chi^j\right\|_V^2 + \sum_{q=1}^{N_\varphi}\frac{\varphi_0}{8\varphi_q^2 N_\varphi + 4\varphi_0\varphi_q}\max_{0\leq j\leq N}\left\|\varsigma_q^j\right\|_V^2 \\
&\quad + \Delta t \sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi}\frac{\tau_q}{\varphi_q}\left\|\frac{\varsigma_q^{n+1} - \varsigma_q^n}{\Delta t}\right\|_V^2 \\
&\leq (3+N_\varphi)\Bigg(\frac{\rho}{4(3+N_\varphi)}\max_{0\leq j\leq N}\left\|\varpi^j\right\|_{L_2(\Omega)}^2 + \sum_{q=1}^{N_\varphi}\frac{1}{2(3+N_\varphi)}\frac{\varphi_0}{8\varphi_q^2 N_\varphi + 4\varphi_0\varphi_q}\max_{0\leq j\leq N}\left\|\varsigma_q^j\right\|_V^2 \\
&\quad + C(h^{2(\min(k+1,s)-1)} + \Delta t^4)\Bigg)
\end{aligned}$$

for $m = 1, \ldots, N$. Thus, we can also say that

$$\begin{aligned}
&\frac{\rho}{4}\max_{0\leq j\leq N}\left\|\varpi^j\right\|_{L_2(\Omega)}^2 + \frac{\varphi_0}{4}\max_{0\leq j\leq N}\left\|\chi^j\right\|_V^2 + \sum_{q=1}^{N_\varphi}\frac{\varphi_0}{16\varphi_q^2 N_\varphi + 8\varphi_0\varphi_q}\max_{0\leq j\leq N}\left\|\varsigma_q^j\right\|_V^2 \\
&\quad + \Delta t \sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi}\frac{\tau_q}{\varphi_q}\left\|\frac{\varsigma_q^{n+1} - \varsigma_q^n}{\Delta t}\right\|_V^2 \\
&\leq C(h^{(\min(k+1,s)-1)} + \Delta t^2),
\end{aligned}$$

therefore, we have

$$\max_{0\leq j\leq N}\left\|\varpi^j\right\|_{L_2(\Omega)} + \max_{0\leq j\leq N}\left\|\chi^j\right\|_V \leq C(h^{\min(k+1,s)-1} + \Delta t^2).$$

If elliptic regularity estimates is satisfied, using (1.4.9) gives us that (2.3.19) yields

$$\frac{\rho}{2}\left\|\varpi^m\right\|_{L_2(\Omega)}^2+\frac{\varphi_0}{4}\left\|\chi^m\right\|_V^2+\sum_{q=1}^{N_\varphi}\frac{\varphi_0}{8\varphi_q^2N_\varphi+4\varphi_0\varphi_q}\left\|\varsigma_q^m\right\|_V^2$$
$$+\Delta t\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi}\frac{\tau_q}{\varphi_q}\left\|\frac{\varsigma_q^{n+1}-\varsigma_q^n}{\Delta t}\right\|_V^2$$
$$\leq\frac{\rho}{4(3+N_\varphi)}\max_{0\leq j\leq N}\left\|\varpi^j\right\|_{L_2(\Omega)}^2+\sum_{q=1}^{N_\varphi}\frac{1}{2(3+N_\varphi)}\frac{\varphi_0}{8\varphi_q^2N_\varphi+4\varphi_0\varphi_q}\max_{0\leq j\leq N}\left\|\varsigma_q^j\right\|_V^2$$
$$+C(h^{2\min(k+1,s)}+\Delta t^4)\tag{2.3.21}$$

hence we can conclude that

$$\max_{0\leq j\leq N}\left\|\varpi^j\right\|_{L_2(\Omega)}+\max_{0\leq j\leq N}\left\|\chi^j\right\|_V\leq C(h^{\min(k+1,s)}+\Delta t^2).$$

□

In Lemma 2.10, we do not use Grönwall's inequality so that the constant C depends on the final time T but not increasing exponentially in time. As seen in semidiscrete problems, Lemma 2.10 implies the following error estimates.

Theorem 2.14. *Suppose $u\in H^4(0,T;H^s(\Omega))\cap W_\infty^1(0,T;H^s(\Omega))$. Then we have*

$$\max_{0\leq j\leq N}\left\|u(t_j)-U_h^j\right\|_V\leq C(h^{\min(k+1,s)-1}+\Delta t^2)$$

and

$$\max_{0\leq j\leq N}\left\|\dot{u}(t_j)-W_h^j\right\|_{L_2(\Omega)}\leq C(h^{\min(k+1,s)-1}+\Delta t^2)$$

where k is a degree of polynomial basis, for some positive C. With elliptic regularity, it is also observed that

$$\max_{0\leq j\leq N}\left\|\dot{u}(t_j)-W_h^j\right\|_{L_2(\Omega)}\leq C(h^{\min(k+1,s)}+\Delta t^2)$$

for some positive C.

Proof. From Lemma 2.10, it is provided that

$$\max_{0\leq j\leq N}\left\|\varpi^j\right\|_{L_2(\Omega)},\ \max_{0\leq j\leq N}\left\|\chi^j\right\|_V\leq C(h^{\min(k+1,s)-1}+\Delta t^2)$$

for some positive C. Hence we have for any $0\leq n\leq N$

$$\left\|e_h^n\right\|_V=\left\|\theta^n-\chi^n\right\|_V$$

$$
\begin{aligned}
&\leq \|\theta^n\|_V + \|\chi^n\|_V \\
&\leq C(h^{\min(k+1,s)-1} + \Delta t^2)
\end{aligned}
$$

for some positive C by (1.4.8). Since n is arbitrary, it is true that

$$\max_{0\leq j\leq N} \left\|e_h^j\right\|_V \leq C(h^{\min(k+1,s)-1} + \Delta t^2).$$

In this same sense, we can derive

$$\max_{0\leq j\leq N} \left\|\tilde{e}_h^j\right\|_{L_2(\Omega)} \leq C(h^{\min(k+1,s)-1} + \Delta t^2)$$

for some positive C. However, elliptic regularity allows us to have

$$\max_{0\leq j\leq N} \left\|\varpi^j\right\|_{L_2(\Omega)} \leq C(h^{\min(k+1,s)} + \Delta t^2) \text{ and } \max_{0\leq j\leq N} \left\|\dot{\theta}^j\right\|_{L_2(\Omega)} \leq Ch^{\min(k+1,s)}.$$

Thus,

$$
\begin{aligned}
\max_{0\leq j\leq N} \left\|\tilde{e}_h^j\right\|_{L_2(\Omega)} &\leq \max_{0\leq j\leq N} \left\|\varpi^j\right\|_{L_2(\Omega)} + \max_{0\leq j\leq N} \left\|\dot{\theta}^j\right\|_{L_2(\Omega)} \\
&\leq C(h^{\min(k+1,s)} + \Delta t^2)
\end{aligned}
$$

for some positive C. □

Corollary 2.1. *Under same conditions as* Theorem 2.14*, there exists a positive constant* C *such that*

$$\max_{0\leq j\leq N} \left\|u(t_j) - U_h^j\right\|_{L_2(\Omega)} \leq C(h^{\min(k+1,s)-1} + \Delta t^2).$$

If elliptic regularity is given, it shows

$$\max_{0\leq j\leq N} \left\|u(t_j) - U_h^j\right\|_{L_2(\Omega)} \leq C(h^{\min(k+1,s)} + \Delta t^2).$$

Proof. In a similar way with the proof of Theorem 2.14,

$$
\begin{aligned}
\|u(t_n) - U_h^n\|_{L_2(\Omega)} &= \|e_h^n\|_{L_2(\Omega)} \\
&\leq \|\theta^n\|_{L_2(\Omega)} + \|\chi^n\|_{L_2(\Omega)},
\end{aligned}
$$

for any $0 \leq n \leq N$. Note that

$$\|\chi^n\|_{L_2(\Omega)} \leq \|\chi^n\|_{H^1(\Omega)} \leq \frac{1}{\sqrt{\kappa}} \|\chi^n\|_V$$

by coercivity. Hence (1.4.8) and Theorem 2.14 give us

$$\|u(t_n) - U_h^n\|_{L_2(\Omega)} \leq \|\theta^n\|_{L_2(\Omega)} + \|\chi^n\|_{L_2(\Omega)}$$

$$
\begin{aligned}
&\leq \|\theta^n\|_{L_2(\Omega)} + \frac{1}{\sqrt{\kappa}} \|\chi^n\|_V \\
&\leq C h^{\min(k+1,s)-1} + C(h^{\min(k+1,s)-1} + \Delta t^2) \\
&\leq C(h^{\min(k+1,s)-1} + \Delta t^2),
\end{aligned}
$$

for some positive C. Since n is arbitrary, it is concluded that

$$
\max_{0\leq j\leq N} \left\| u(t_j) - U_h^j \right\|_{L_2(\Omega)} \leq C(h^{\min(k+1,s)-1} + \Delta t^2)
$$

for some positive C. Moreover, if elliptic regularity is satisfied,

$$
\begin{aligned}
\|\theta^n\|_{L_2(\Omega)} &\leq C h^{\min(k+1,s)}, \\
\|\chi^n\|_V &\leq C(h^{\min(k+1,s)} + \Delta t^2),
\end{aligned}
$$

$\forall n$, therefore,

$$
\max_{0\leq j\leq N} \left\| u(t_j) - U_h^j \right\|_{L_2(\Omega)} \leq C(h^{\min(k+1,s)} + \Delta t^2)
$$

for some positive C. □

As shown in Theorem 2.14 and Corollary 2.1, we have L_2 estimates and energy estimates of U_h and W_h for the displacement form. Here, the energy norm is equivalent to H^1 norm and hence we have also H^1 estimates for the numerical solutions.

In a similar way with the displacement form, we can seek a fully discrete formulation for **(P2)** and show stability bounds and error bounds without Grönwall's inequality.

2.3.2 Velocity Form

(P2) Find u and $\{\zeta_q\}_{q=1}^{N_\varphi}$ such that for all $v \in V$

$$
\begin{aligned}
(\rho \ddot{u}(t), v)_{L_2(\Omega)} + \varphi_0 a(u(t), v) + \sum_{q=1}^{N_\varphi} a(\zeta_q(t), v) &= F_v(t; v), \\
\tau_q a(\dot{\zeta}_q(t), v) + a(\zeta_q(t), v) &= \tau_q \varphi_q a(\dot{u}(t), v), \ \forall q = 1, \ldots, N_\varphi,
\end{aligned}
$$

with $u(0) = u_0$, $\dot{u}(0) = w_0$ and $\zeta_q(0) = 0$, $\forall q = 1, \ldots, N_\varphi$.

The fully discrete formulation for **(P2)** is introduced with Crank-Nicolson method as find U_h^n, W_h^n and $\mathcal{S}_{hq}^n \in V^h$ for $n = 0, \ldots, N$, $\forall q \in \{1, \ldots, N_\varphi\}$ such that for any $v \in V^h$ for $n = 0, \ldots, N-1$,

$$
\left(\rho \frac{W_h^{n+1} - W_h^n}{\Delta t}, v \right)_{L_2(\Omega)} + \varphi_0 a \left(\frac{U_h^{n+1} + U_h^n}{2}, v \right) + \sum_{q=1}^{N_\varphi} a \left(\frac{\mathcal{S}_{hq}^{n+1} + \mathcal{S}_{hq}^n}{2}, v \right)
$$

$$= \bar{F}_v^n(v), \tag{2.3.22}$$

and for $n = 0, \dots, N-1$,

$$\tau_q a\left(\frac{\mathcal{S}_{hq}^{n+1} - \mathcal{S}_{hq}^n}{\Delta t}, v\right) + a\left(\frac{\mathcal{S}_{hq}^{n+1} + \mathcal{S}_{hq}^n}{2}, v\right) = \tau_q \varphi_q a\left(\frac{W_h^{n+1} + W_h^n}{2}, v\right) \ \forall q, \tag{2.3.23}$$

$$a(U_h^0, v) = a(u_0, v)\ , \tag{2.3.24}$$

$$\left(W_h^0, v\right)_{L_2(\Omega)} = (w_0, v)_{L_2(\Omega)}\ , \tag{2.3.25}$$

with the relation (2.3.1) where we have

$$U_h^n = \sum_{i=1}^{N_{Vh}} \mathfrak{u}_i^n \Phi_i,\ W_h^n = \sum_{i=1}^{N_{Vh}} \mathfrak{w}_i^n \Phi_i,\ \text{and } \mathcal{S}_{hq}^n = \sum_{i=1}^{N_{Vh}} S_{hq,i}^n \Phi_i \text{ for each } q$$

From (2.3.24) and (2.3.25), $\underline{\mathfrak{u}}^0$ and $\underline{\mathfrak{w}}^0$ is obtained. Then we can have the linear system which is equivalent to (2.3.22)-(2.3.25). Let us recall

$$(\underline{\tilde{F}}^n)_i = F_v(t_n; \Phi_i),$$

for $i = 1, \dots, N_{V^h}$ and $n = 0, \dots, N-1$. Note that we have $\underline{S}_{hq}^0 = \underline{0},\ \forall q \in \{1, \dots, N_\varphi\}$. Then (2.3.1) and (2.3.24) provide

$$\underline{\mathfrak{w}}^{n+1} = \frac{2}{\Delta t}(\underline{\mathfrak{u}}^{n+1} - \underline{\mathfrak{u}}^n) - \underline{\mathfrak{w}}^n, \tag{2.3.26}$$

and this yields

$$\begin{aligned}
\underline{S}_{hq}^{n+1} &= \left(\frac{\tau_q}{\Delta t} + \frac{1}{2}\right)^{-1} \frac{\tau_q \varphi_q}{2}(\underline{\mathfrak{w}}^{n+1} + \underline{\mathfrak{w}}^n) + \left(\frac{\tau_q}{\Delta t} + \frac{1}{2}\right)^{-1} \left(\frac{\tau_q}{\Delta t} - \frac{1}{2}\right) \underline{S}_{hq}^n,\ \forall q, \\
&= \left(\frac{\tau_q}{\Delta t} + \frac{1}{2}\right)^{-1} \frac{\tau_q \varphi_q}{\Delta t}(\underline{\mathfrak{u}}^{n+1} - \underline{\mathfrak{u}}^n) + \left(\frac{\tau_q}{\Delta t} + \frac{1}{2}\right)^{-1} \left(\frac{\tau_q}{\Delta t} - \frac{1}{2}\right) \underline{S}_{hq}^n,\ \forall q, \qquad (2.3.27)
\end{aligned}$$

respectively. As a result, (2.3.22) implies

$$\frac{\rho}{\Delta t} M\left(\frac{2}{\Delta t}(\underline{\mathfrak{u}}^{n+1} - \underline{\mathfrak{u}}^n) - 2\underline{\mathfrak{w}}^n\right) + \frac{\varphi_0}{2} A(\underline{\mathfrak{u}}^{n+1} + \underline{\mathfrak{u}}^n) + \sum_{q=1}^{N_\varphi} \frac{\tau_q \varphi_q}{2\tau_q + \Delta t} A(\underline{\mathfrak{u}}^{n+1} - \underline{\mathfrak{u}}^n)$$
$$+ \sum_{q=1}^{N_\varphi} \frac{2\tau_q}{2\tau_q + \Delta t} A \underline{S}_{hq}^n = \frac{1}{2}(\underline{\tilde{F}}^{n+1} + \underline{\tilde{F}}^n).$$

Let us set a matrix by

$$\mathcal{B} = \sum_{q=1}^{N_\varphi} \frac{\tau_q \varphi_q}{2\tau_q + \Delta t} A.$$

Then we have

$$\left(\frac{2\rho}{\Delta t^2}M+\frac{\varphi_0}{2}A+\mathcal{B}\right)\underline{\mathfrak{u}}^{n+1}=\frac{2\rho}{\Delta t}M\underline{\mathfrak{w}}^n+\left(\frac{2\rho}{\Delta t^2}M-\frac{\varphi_0}{2}A+\mathcal{B}\right)\underline{\mathfrak{u}}^n$$
$$-\sum_{q=1}^{N_\varphi}\frac{2\tau_q}{2\tau_q+\Delta t}A\underline{S}_{hq}^n+\frac{1}{2}(\underline{\tilde{F}}^{n+1}+\underline{\tilde{F}}^n),$$

and hence

$$\underline{\mathfrak{u}}^{n+1}=\left(\frac{2\rho}{\Delta t^2}M+\frac{\varphi_0}{2}A+\mathcal{B}\right)^{-1}\left[\frac{2\rho}{\Delta t}M\underline{\mathfrak{w}}^n+\left(\frac{2\rho}{\Delta t^2}M-\frac{\varphi_0}{2}A+\mathcal{B}\right)\underline{\mathfrak{u}}^n\right.$$
$$\left.-\sum_{q=1}^{N_\varphi}\frac{2\tau_q}{2\tau_q+\Delta t}A\underline{S}_{hq}^n+\frac{1}{2}(\underline{\tilde{F}}^{n+1}+\underline{\tilde{F}}^n)\right]. \tag{2.3.28}$$

In the end the linear system (2.3.26), (2.3.27) and (2.3.28) can be solved uniquely for $n=0,\ldots,N-1$ if there exist stability bounds. Thus, we shall show the stability bounds for the fully discrete formula of **(P2)**.

Note that a linear form F_v consists of only data such as initial data, boundary data and f.

Lemma 2.11. *For any $m\in\mathbb{N}$ such that $1\le m\le N$,*

$$\rho\left\|W_h^m\right\|_{L_2(\Omega)}^2+\varphi_0\left\|U_h^m\right\|_V^2+\frac{\Delta t}{2}\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi}a(\mathcal{S}_{hq}^{n+1}+\mathcal{S}_{hq}^n,W_h^{n+1}+W_h^n)$$
$$=\rho\left\|W_h^0\right\|_{L_2(\Omega)}^2+\varphi_0\left\|U_h^0\right\|_V^2+\sum_{n=0}^{m-1}\Delta t\bar{F}_v^n\left(W_h^{n+1}+W_h^n\right).$$

Proof. Let $v=W_h^{n+1}+W_h^n$ for $0\le n\le m-1$ and put it into (2.3.22). Then we have

$$\frac{\rho}{\Delta t}\left(\left\|W_h^{n+1}\right\|_{L_2(\Omega)}^2-\left\|W_h^n\right\|_{L_2(\Omega)}^2\right)+\frac{\varphi_0}{2}a(U_h^{n+1}+U_h^n,W_h^{n+1}+W_h^n)$$
$$+\frac{1}{2}\sum_{q=1}^{N_\varphi}a(\mathcal{S}_{hq}^{n+1}+\mathcal{S}_{hq}^n,W_h^{n+1}+W_h^n)=\bar{F}_v^n\left(W_h^{n+1}+W_h^n\right).$$

By the relation (2.3.1),

$$\frac{\rho}{\Delta t}\left(\left\|W_h^{n+1}\right\|_{L_2(\Omega)}^2-\left\|W_h^n\right\|_{L_2(\Omega)}^2\right)+\frac{\varphi_0}{\Delta t}(\left\|U_h^{n+1}\right\|_V^2-\left\|U_h^n\right\|_V^2)$$
$$+\frac{1}{2}\sum_{q=1}^{N_\varphi}a(\mathcal{S}_{hq}^{n+1}+\mathcal{S}_{hq}^n,W_h^{n+1}+W_h^n)=\bar{F}_v^n\left(W_h^{n+1}+W_h^n\right).$$

With taking into account summation from $n = 0$ to $n = m - 1$ and multiplication by Δt, it yields

$$\begin{aligned}&\rho \left\|W_h^m\right\|_{L_2(\Omega)}^2 + \varphi_0 \left\|U_h^m\right\|_V^2 + \frac{\Delta t}{2}\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi} a(\mathcal{S}_{hq}^{n+1} + \mathcal{S}_{hq}^n, W_h^{n+1} + W_h^n)\\ =&\rho \left\|W_h^0\right\|_{L_2(\Omega)}^2 + \varphi_0 \left\|U_h^0\right\|_V^2 + \Delta t \bar{F}_v^n \left(W_h^{n+1} + W_h^n\right).\end{aligned}$$

□

Lemma 2.12. *For each q and for any $m \in \mathbb{N}$ such that $1 \leq m \leq N$,*

$$\sum_{n=0}^{m-1} a(W_h^{n+1} + W_h^n, \mathcal{S}_{hq}^{n+1} + \mathcal{S}_{hq}^n) = \frac{2}{\varphi_q \Delta t}\left\|\mathcal{S}_{hq}^m\right\|_V^2 + \sum_{n=0}^{m-1}\frac{1}{\tau_q\varphi_q}\left\|\mathcal{S}_{hq}^{n+1} + \mathcal{S}_{hq}^n\right\|_V^2.$$

Proof. It is easy to check Lemma 2.12. Consider $v = \mathcal{S}_{hq}^{n+1} + \mathcal{S}_{hq}^n$ for $0 \leq n \leq m-1$ with (2.3.23). We have

$$\frac{\tau_q}{\Delta t}\left(\left\|\mathcal{S}_{hq}^{n+1}\right\|_V^2 - \left\|\mathcal{S}_{hq}^n\right\|_V^2\right) + \frac{1}{2}\left\|\mathcal{S}_{hq}^{n+1} + \mathcal{S}_{hq}^n\right\|_V^2 = \frac{\tau_q\varphi_q}{2} a(W_h^{n+1} + W_h^n, \mathcal{S}_{hq}^{n+1} + \mathcal{S}_{hq}^n).$$

Summing with respect to n,

$$\frac{\tau_q}{\Delta t}\left\|\mathcal{S}_{hq}^m\right\|_V^2 + \sum_{n=0}^{m-1}\frac{1}{2}\left\|\mathcal{S}_{hq}^{n+1} + \mathcal{S}_{hq}^n\right\|_V^2 = \sum_{n=0}^{m-1}\frac{\tau_q\varphi_q}{2} a(W_h^{n+1} + W_h^n, \mathcal{S}_{hq}^{n+1} + \mathcal{S}_{hq}^n),$$

since $\mathcal{S}_{hq}^0 = 0$ for any q, thus it is observed that

$$\sum_{n=0}^{m-1} a(W_h^{n+1} + W_h^n, \mathcal{S}_{hq}^{n+1} + \mathcal{S}_{hq}^n) = \frac{2}{\varphi_q \Delta t}\left\|\mathcal{S}_{hq}^m\right\|_V^2 + \sum_{n=0}^{m-1}\frac{1}{\tau_q\varphi_q}\left\|\mathcal{S}_{hq}^{n+1} + \mathcal{S}_{hq}^n\right\|_V^2.$$

□

Theorem 2.15. *For any $m \in \mathbb{N}$ such that $1 \leq m \leq N$, there exists a positive constant C such that*

$$\begin{aligned}&\frac{\rho}{2}\left\|W_h^m\right\|_{L_2(\Omega)}^2 + \frac{\varphi_0}{2}\left\|U_h^m\right\|_V^2 + \sum_{q=1}^{N_\varphi}\frac{1}{\varphi_q}\left\|\mathcal{S}_{hq}^m\right\|_V^2 + \sum_{q=1}^{N_\varphi}\sum_{n=0}^{m-1}\frac{\Delta t}{2\tau_q\varphi_q}\left\|\mathcal{S}_{hq}^{n+1} + \mathcal{S}_{hq}^n\right\|_V^2\\ \leq& C\Bigg(\left\|w_0\right\|_{L_2(\Omega)}^2 + \left\|u_0\right\|_V^2 + \Delta t\sum_{n=0}^{m-1}\left\|\bar{f}^n\right\|_{L_2(\Omega)}^2 + \left\|\bar{g}_N^{m-1}\right\|_{L_2(\Gamma_N)}^2\\ &+ \left\|\bar{g}_N^0\right\|_{L_2(\Gamma_N)}^2 + \int_0^{t_m}\left\|\dot{g}_N(t')\right\|_{L_2(\Gamma_N)}^2 dt'\Bigg).\end{aligned}$$

Proof. Combining Lemma 2.11 and 2.12, we can have for $m = 1, \dots, N$

$$\begin{aligned}
&\rho \left\| W_h^m \right\|_{L_2(\Omega)}^2 + \varphi_0 \left\| U_h^m \right\|_V^2 + \sum_{q=1}^{N_\varphi} \frac{1}{\varphi_q} \left\| \mathcal{S}_{hq}^m \right\|_V^2 + \sum_{q=1}^{N_\varphi} \sum_{n=0}^{m-1} \frac{\Delta t}{2\tau_q \varphi_q} \left\| \mathcal{S}_{hq}^{n+1} + \mathcal{S}_{hq}^n \right\|_V^2 \\
=&\rho \left\| W_h^0 \right\|_{L_2(\Omega)}^2 + \varphi_0 \left\| U_h^0 \right\|_V^2 + \sum_{n=0}^{m-1} \Delta t \bar{F}_v^n \left(W_h^{n+1} + W_h^n \right). \qquad (2.3.29)
\end{aligned}$$

Now, let us consider $\Delta t \sum_{n=0}^{m-1} \bar{F}_v^n \left(W_h^{n+1} + W_h^n \right)$. By the definition,

$$\begin{aligned}
&\Delta t \sum_{n=0}^{m-1} \bar{F}_v^n \left(W_h^{n+1} + W_h^n \right) \\
=&\Delta t \sum_{n=0}^{m-1} \left(\bar{f}^n, W_h^{n+1} + W_h^n \right)_{L_2(\Omega)} + \Delta t \sum_{n=0}^{m-1} \left(\bar{g}_N^n, W_h^{n+1} + W_h^n \right)_{L_2(\Gamma_N)} \\
&+ \Delta t \sum_{n=0}^{m-1} \sum_{q=1}^{N_\varphi} \varphi_q \frac{e^{-t_{n+1}/\tau_q} + e^{-t_n/\tau_q}}{2} a \left(u_0, W_h^{n+1} + W_h^n \right),
\end{aligned}$$

and so by (2.3.1)

$$\begin{aligned}
&\Delta t \sum_{n=0}^{m-1} \bar{F}_v^n \left(W_h^{n+1} + W_h^n \right) \\
=&\Delta t \sum_{n=0}^{m-1} \left(\bar{f}^n, W_h^{n+1} + W_h^n \right)_{L_2(\Omega)} + 2 \sum_{n=0}^{m-1} \left(\bar{g}_N^n, U_h^{n+1} - U_h^n \right)_{L_2(\Gamma_N)} \\
&+ 2 \sum_{n=0}^{m-1} \sum_{q=1}^{N_\varphi} \varphi_q \frac{e^{-t_{n+1}/\tau_q} + e^{-t_n/\tau_q}}{2} a \left(u_0, U_h^{n+1} - U_h^n \right) \\
=&\Delta t \sum_{n=0}^{m-1} \left(\bar{f}^n, W_h^{n+1} + W_h^n \right)_{L_2(\Omega)} + 2 \sum_{n=0}^{m-1} \left(\bar{g}_N^n, U_h^{n+1} - U_h^n \right)_{L_2(\Gamma_N)} \\
&+ 2 \sum_{q=1}^{N_\varphi} \varphi_q \sum_{n=0}^{m-1} a \left(\bar{R}_q^n u_0, U_h^{n+1} - U_h^n \right),
\end{aligned}$$

where $\bar{R}_q^n := (e^{-t_{n+1}/\tau_q} + e^{-t_n/\tau_q})/2$. By summation by parts, this yields, with the definition of integration,

$$\begin{aligned}
&\Delta t \sum_{n=0}^{m-1} \bar{F}_v^n \left(W_h^{n+1} + W_h^n \right) \\
=&\Delta t \sum_{n=0}^{m-1} \left(\bar{f}^n, W_h^{n+1} + W_h^n \right)_{L_2(\Omega)} + 2 \left(\bar{g}_N^{m-1}, U_h^m \right)_{L_2(\Gamma_N)}
\end{aligned}$$

$$
\begin{aligned}
&- 2\left(\bar{g}_N^0, U_h^0\right)_{L_2(\Gamma_N)} - \sum_{n=1}^{m-1} \int_{t_{n-1}}^{t_{n+1}} \left(\dot{g}_N(t'), U_h^n\right)_{L_2(\Gamma_N)} dt' \\
&+ 2\sum_{q=1}^{N_\varphi} \varphi_q a\left(\bar{R}_q^{m-1} u_0, U_h^m\right) - 2\sum_{q=1}^{N_\varphi} \varphi_q a\left(\bar{R}_q^0 u_0, U_h^0\right) \\
&- \sum_{q=1}^{N_\varphi} \varphi_q \sum_{n=1}^{m-1} \int_{t_{n-1}}^{t_{n+1}} a\left(-\frac{e^{-t'/\tau_q}}{\tau_q} u_0, U_h^n\right) dt'.
\end{aligned}
$$

In the same sense in the proof of Theorem 2.12, we will apply Cauchy-Schwarz inequality and (2.1.13) as

$$
\begin{aligned}
&\Delta t \sum_{n=0}^{m-1} \bar{F}_v^n\left(W_h^{n+1} + W_h^n\right) \\
\leq &\Delta t \sum_{n=0}^{m-1} \left\|\bar{f}^n\right\|_{L_2(\Omega)} \left\|W_h^{n+1}\right\|_{L_2(\Omega)} + \Delta t \sum_{n=0}^{m-1} \left\|\bar{f}^n\right\|_{L_2(\Omega)} \left\|W_h^n\right\|_{L_2(\Omega)} \\
&+ 2C\left\|\bar{g}_N^{m-1}\right\|_{L_2(\Gamma_N)} \left\|U_h^m\right\|_V + 2C\left\|\bar{g}_N^0\right\|_{L_2(\Gamma_N)} \left\|U_h^0\right\|_V \\
&+ C\sum_{n=1}^{m-1} \int_{t_{n-1}}^{t_{n+1}} \left\|\dot{g}_N(t')\right\|_{L_2(\Gamma_N)} \left\|U_h^n\right\|_V dt' + 2\sum_{q=1}^{N_\varphi} \varphi_q \bar{R}_q^{m-1} \left\|u_0\right\|_V \left\|U_h^m\right\|_V \\
&+ 2\sum_{q=1}^{N_\varphi} \varphi_q \bar{R}_q^0 a\left(u_0, U_h^0\right) + \sum_{q=1}^{N_\varphi} \varphi_q \sum_{n=1}^{m-1} \int_{t_{n-1}}^{t_{n+1}} \left\|\frac{e^{-t'/\tau_q}}{\tau_q} u_0\right\|_V \left\|U_h^n\right\|_V dt' \\
\leq &\Delta t \sum_{n=0}^{m-1} \left\|\bar{f}^n\right\|_{L_2(\Omega)} \left\|W_h^{n+1}\right\|_{L_2(\Omega)} + \Delta t \sum_{n=0}^{m-1} \left\|\bar{f}^n\right\|_{L_2(\Omega)} \left\|W_h^n\right\|_{L_2(\Omega)} \\
&+ 2C\left\|\bar{g}_N^{m-1}\right\|_{L_2(\Gamma_N)} \left\|U_h^m\right\|_V + 2C\left\|\bar{g}_N^0\right\|_{L_2(\Gamma_N)} \left\|U_h^0\right\|_V \\
&+ C\sum_{n=1}^{m-1} \int_{t_{n-1}}^{t_{n+1}} \left\|\dot{g}_N(t')\right\|_{L_2(\Gamma_N)} \left\|U_h^n\right\|_V dt' + 2\left\|u_0\right\|_V \left\|U_h^m\right\|_V + 2\left\|U_h^0\right\|_V^2 \\
&+ \sum_{q=1}^{N_\varphi} \varphi_q \sum_{n=1}^{m-1} \int_{t_{n-1}}^{t_{n+1}} \left\|\frac{e^{-t'/\tau_q}}{\tau_q} u_0\right\|_V \left\|U_h^n\right\|_V dt' \\
= &\Delta t \left\|\bar{f}^{m-1}\right\|_{L_2(\Omega)} \left\|W_h^m\right\|_{L_2(\Omega)} + \Delta t \sum_{n=0}^{m-2} \left\|\bar{f}^n\right\|_{L_2(\Omega)} \left\|W_h^{n+1}\right\|_{L_2(\Omega)} \\
&+ \Delta t \sum_{n=0}^{m-1} \left\|\bar{f}^n\right\|_{L_2(\Omega)} \left\|W_h^n\right\|_{L_2(\Omega)} + 2C\left\|\bar{g}_N^{m-1}\right\|_{L_2(\Gamma_N)} \left\|U_h^m\right\|_V \\
&+ 2C\left\|\bar{g}_N^0\right\|_{L_2(\Gamma_N)} \left\|U_h^0\right\|_V + C\sum_{n=1}^{m-1} \int_{t_{n-1}}^{t_{n+1}} \left\|\dot{g}_N(t')\right\|_{L_2(\Gamma_N)} \left\|U_h^n\right\|_V dt'
\end{aligned}
$$

$$+ 2\left\|u_0\right\|_V\left\|U_h^m\right\|_V + 2\left\|U_h^0\right\|_V^2 + \sum_{q=1}^{N_\varphi}\varphi_q\sum_{n=1}^{m-1}\int_{t_{n-1}}^{t_{n+1}}\left\|\frac{e^{-t'/\tau_q}}{\tau_q}u_0\right\|_V\left\|U_h^n\right\|_V dt', \quad (2.3.30)$$

for a positive constant C, since

$$|\bar{R}_q^n|,\ |e^{-t/\tau_q}| \le 1,\ \forall n,\ \forall q \in \{1,\ldots,N_\varphi\}, \forall t \ge 0, \qquad \sum_{q=1}^{N_\varphi}\varphi_q < 1,$$

and (2.3.24) implies

$$a\left(u_0, U_h^0\right) = a\left(U_h^0, U_h^0\right) = \left\|U_h^0\right\|_V^2.$$

Hence with Young's inequalities we have

$$\begin{aligned}
&\Delta t\sum_{n=0}^{m-1}\bar{F}_v^n\left(W_h^{n+1} + W_h^n\right)\\
\le&\frac{\Delta t}{2\epsilon_a}\left\|\bar{f}^{m-1}\right\|_{L_2(\Omega)}^2 + \frac{\Delta t\epsilon_a}{2}\left\|W_h^m\right\|_{L_2(\Omega)}^2 + \frac{\Delta t}{2}\sum_{n=0}^{m-2}\left\|\bar{f}^n\right\|_{L_2(\Omega)}^2\\
&+\frac{\Delta t}{2}\sum_{n=0}^{m-2}\left\|W_h^{n+1}\right\|_{L_2(\Omega)}^2 + \frac{\Delta t}{2}\sum_{n=0}^{m-1}\left\|\bar{f}^n\right\|_{L_2(\Omega)}^2 + \frac{\Delta t}{2}\sum_{n=0}^{m-1}\left\|W_h^n\right\|_{L_2(\Omega)}^2\\
&+\frac{C}{\epsilon_b}\left\|\bar{g}_N^{m-1}\right\|_{L_2(\Gamma_N)}^2 + C\epsilon_b\left\|U_h^m\right\|_V^2 + C\left\|\bar{g}_N^0\right\|_{L_2(\Gamma_N)}^2 + C\left\|U_h^0\right\|_V^2\\
&+C\int_0^{t_m}\left\|\dot{g}_N(t')\right\|_{L_2(\Gamma_N)}^2 dt' + C\Delta t\sum_{n=0}^{m-1}\left\|U_h^n\right\|_V^2 + \frac{1}{\epsilon_c}\left\|u_0\right\|_V^2 + \epsilon_c\left\|U_h^m\right\|_V^2 + 2\left\|U_h^0\right\|_V^2\\
&+\sum_{q=1}^{N_\varphi}\frac{\varphi_q}{2}\int_0^{t_m}\left|\frac{e^{-t'/\tau_q}}{\tau_q}\right|^2\left\|u_0\right\|_V^2 dt' + \sum_{q=1}^{N_\varphi}\frac{\varphi_q}{2}\Delta t\sum_{n=0}^{m-1}\left\|U_h^n\right\|_V^2\\
\le&\left(\Delta t + \frac{\Delta t}{2\epsilon_a}\right)\sum_{n=0}^{m-1}\left\|\bar{f}^n\right\|_{L_2(\Omega)}^2 + \frac{\Delta t\epsilon_a}{2}\left\|W_h^m\right\|_{L_2(\Omega)} + \Delta t\sum_{n=0}^{m-1}\left\|W_h^n\right\|_{L_2(\Omega)}^2\\
&+\frac{C}{\epsilon_b}\left\|\bar{g}_N^{m-1}\right\|_{L_2(\Gamma_N)}^2 + C\epsilon_b\left\|U_h^m\right\|_V^2 + C\left\|\bar{g}_N^0\right\|_{L_2(\Gamma_N)}^2 + C\left\|U_h^0\right\|_V^2\\
&+C\int_0^{t_m}\left\|\dot{g}_N(t')\right\|_{L_2(\Gamma_N)}^2 dt' + C\Delta t\sum_{n=0}^{m-1}\left\|U_h^n\right\|_V^2 + \frac{1}{\epsilon_c}\left\|u_0\right\|_V^2 + \epsilon_c\left\|U_h^m\right\|_V^2 + 2\left\|U_h^0\right\|_V^2\\
&+\sum_{q=1}^{N_\varphi}\frac{1}{4\tau_q}\left\|u_0\right\|_V^2 + \frac{\Delta t}{2}\sum_{n=0}^{m-1}\left\|U_h^n\right\|_V^2,
\end{aligned}$$

for positive ϵ_a, ϵ_b and ϵ_c. Note that

$$\sum_{q=1}^{N_\varphi}\varphi_q < 1,$$

and for each q and any positive t_m

$$\begin{aligned}\int_0^{t_m} \left|\frac{e^{-t'/\tau_q}}{\tau_q}\right|^2 dt' =& -\frac{1}{2\tau_q} e^{-2t'/\tau_q}\Big|_{t'=0}^{t'=t_m} \\ =& \frac{1}{2\tau_q}(1 - e^{-2t_m/\tau_q}) \\ <& \frac{1}{2\tau_q}.\end{aligned}$$

In the end, if we take $\epsilon_a = \frac{\rho}{\Delta t} > 0$, $\epsilon_b = \frac{\varphi_0}{4C}$ and $\epsilon_c = \frac{\varphi_0}{4}$, (2.3.29) yields

$$\begin{aligned}&\frac{\rho}{2}\left\|W_h^m\right\|^2_{L_2(\Omega)} + \frac{\varphi_0}{2}\left\|U_h^m\right\|_V^2 + \sum_{q=1}^{N_\varphi}\frac{1}{\varphi_q}\left\|\mathcal{S}_{hq}^m\right\|_V^2 + \sum_{q=1}^{N_\varphi}\sum_{n=0}^{m-1}\frac{\Delta t}{2\tau_q\varphi_q}\left\|\mathcal{S}_{hq}^{n+1} + \mathcal{S}_{hq}^n\right\|_V^2 \\ \le& \rho\left\|W_h^0\right\|^2_{L_2(\Omega)} + (C+2)\left\|U_h^0\right\|_V^2 + \left(\frac{4}{\varphi_0} + \sum_{q=1}^{N_\varphi}\frac{1}{4\tau_q}\right)\left\|u_0\right\|_V^2 \\ &+ \left(\Delta t + \frac{\Delta t^2}{2\rho}\right)\sum_{n=0}^{m-1}\left\|\bar{f}^n\right\|^2_{L_2(\Omega)} + \frac{4C^2}{\varphi_0}\left\|\bar{g}_N^{m-1}\right\|^2_{L_2(\Gamma_N)} + C\left\|\bar{g}_N^0\right\|^2_{L_2(\Gamma_N)} \\ &+ C\int_0^{t_m}\left\|\dot{g}_N(t')\right\|^2_{L_2(\Gamma_N)} dt' + \Delta t\sum_{n=0}^{m-1}\left\|W_h^n\right\|^2_{L_2(\Omega)} + (C+1/2)\Delta t\sum_{n=0}^{m-1}\left\|U_h^n\right\|_V^2.\end{aligned}$$

Finally, Grönwall's inequality allows us to have for some positive C

$$\begin{aligned}&\frac{\rho}{2}\left\|W_h^m\right\|^2_{L_2(\Omega)} + \frac{\varphi_0}{2}\left\|U_h^m\right\|_V^2 + \sum_{q=1}^{N_\varphi}\frac{1}{\varphi_q}\left\|\mathcal{S}_{hq}^m\right\|_V^2 + \sum_{q=1}^{N_\varphi}\sum_{n=0}^{m-1}\frac{\Delta t}{2\tau_q\varphi_q}\left\|\mathcal{S}_{hq}^{n+1} + \mathcal{S}_{hq}^n\right\|_V^2 \\ \le& C\Bigg(\left\|W_h^0\right\|^2_{L_2(\Omega)} + \left\|U_h^0\right\|_V^2 + \left\|u_0\right\|_V^2 + \Delta t\sum_{n=0}^{m-1}\left\|\bar{f}^n\right\|^2_{L_2(\Omega)} \\ &+ \left\|\bar{g}_N^{m-1}\right\|^2_{L_2(\Gamma_N)} + \left\|\bar{g}_N^0\right\|^2_{L_2(\Gamma_N)} + \int_0^{t_m}\left\|\dot{g}_N(t')\right\|^2_{L_2(\Gamma_N)} dt'\Bigg).\end{aligned}$$

Furthermore, we also know

$$\left\|U_h^0\right\|_V \le \left\|u_0\right\|_V \text{ and } \left\|W_h^0\right\|_{L_2(\Omega)} \le \left\|w_0\right\|_{L_2(\Omega)}$$

by (2.3.24) and (2.3.25) therefore we have

$$\begin{aligned}&\frac{\rho}{2}\left\|W_h^m\right\|^2_{L_2(\Omega)} + \frac{\varphi_0}{2}\left\|U_h^m\right\|_V^2 + \sum_{q=1}^{N_\varphi}\frac{1}{\varphi_q}\left\|\mathcal{S}_{hq}^m\right\|_V^2 + \sum_{q=1}^{N_\varphi}\sum_{n=0}^{m-1}\frac{\Delta t}{2\tau_q\varphi_q}\left\|\mathcal{S}_{hq}^{n+1} + \mathcal{S}_{hq}^n\right\|_V^2 \\ \le& C\Bigg(\left\|w_0\right\|^2_{L_2(\Omega)} + \left\|u_0\right\|_V^2 + \Delta t\sum_{n=0}^{m-1}\left\|\bar{f}^n\right\|^2_{L_2(\Omega)} + \left\|\bar{g}_N^{m-1}\right\|^2_{L_2(\Gamma_N)}\end{aligned}$$

$$+ \|\bar{g}_N^0\|^2_{L_2(\Gamma_N)} + \int_0^{t_m} \|\dot{g}_N(t')\|^2_{L_2(\Gamma_N)}\, dt' \bigg).$$

□

Similarly, it is able to show the stability based on maximum valued without using Grönwall's inequality.

Theorem 2.16. *There exists a positive constant C such that*

$$\begin{aligned}
&\frac{\rho}{2}\max_{0\le n\le N}\|W_h^n\|^2_{L_2(\Omega)} + \frac{\varphi_0}{2}\max_{0\le n\le N}\|U_h^n\|^2_V + \sum_{q=1}^{N_\varphi}\frac{1}{\varphi_q}\left\|\mathcal{S}_{hq}^m\right\|^2_V \\
&+ \sum_{q=1}^{N_\varphi}\sum_{n=0}^{m-1}\frac{\Delta t}{2\tau_q\varphi_q}\left\|\mathcal{S}_{hq}^{n+1}+\mathcal{S}_{hq}^n\right\|^2_V \\
\le& C\bigg(\|w_0\|^2_{L_2(\Omega)} + \|u_0\|^2_V + \|f\|^2_{L_\infty(0,T;L_2(\Omega))} + \|g_N\|^2_{L_\infty(0,T;L_2(\Gamma_N))} \\
&+ \|\dot{g}_N\|^2_{L_2(0,T;L_2(\Gamma_N))}\bigg),
\end{aligned}$$

for $m = 1, \ldots, N$.

Proof. Recall (2.3.29) and (2.3.30) from the proof of Theorem 2.15. For any $1 \le m \le N$, applying Young's inequality leads us to have

$$\begin{aligned}
&\rho\|W_h^m\|^2_{L_2(\Omega)} + \varphi_0\|U_h^m\|^2_V + \sum_{q=1}^{N_\varphi}\frac{1}{\varphi_q}\left\|\mathcal{S}_{hq}^m\right\|^2_V + \sum_{q=1}^{N_\varphi}\sum_{n=0}^{m-1}\frac{\Delta t}{2\tau_q\varphi_q}\left\|\mathcal{S}_{hq}^{n+1}+\mathcal{S}_{hq}^n\right\|^2_V \\
\le& \rho\left\|W_h^0\right\|^2_{L_2(\Omega)} + (C+2)\left\|U_h^0\right\|^2_V + \frac{\Delta t}{\epsilon_a}\sum_{n=0}^{m-1}\left\|\bar{f}^n\right\|^2_{L_2(\Omega)} + \frac{\Delta t\epsilon_a}{2}\|W_h^m\|_{L_2(\Omega)} \\
&+ \Delta t\epsilon_a\sum_{n=0}^{m-1}\|W_h^n\|^2_{L_2(\Omega)} + \frac{C}{\epsilon_b}\left\|\bar{g}_N^{m-1}\right\|^2_{L_2(\Gamma_N)} + C\left\|\bar{g}_N^0\right\|^2_{L_2(\Gamma_N)} \\
&+ \frac{C}{\epsilon_d}\int_0^{t_m}\|\dot{g}_N(t')\|^2_{L_2(\Gamma_N)}\,dt' + \big(C\epsilon_d + \frac{\epsilon_e}{2}\big)\Delta t\sum_{n=0}^{m-1}\|U_h^n\|^2_V + \big(\frac{1}{\epsilon_c} + \sum_{q=1}^{N_\varphi}\frac{1}{4\tau_q\epsilon_e}\big)\|u_0\|^2_V \\
&+ (C\epsilon_b + \epsilon_c)\|U_h^m\|^2_V,
\end{aligned}$$

for any positive ϵ_a, ϵ_b, ϵ_c, ϵ_d and ϵ_e where C is a positive constant from (2.1.13). Also, by the property of maximum and the positive definiteness of norms,

$$\rho\|W_h^m\|^2_{L_2(\Omega)} + \varphi_0\|U_h^m\|^2_V + \sum_{q=1}^{N_\varphi}\frac{1}{\varphi_q}\left\|\mathcal{S}_{hq}^m\right\|^2_V + \sum_{q=1}^{N_\varphi}\sum_{n=0}^{m-1}\frac{\Delta t}{2\tau_q\varphi_q}\left\|\mathcal{S}_{hq}^{n+1}+\mathcal{S}_{hq}^n\right\|^2_V$$

$$
\begin{aligned}
\leq&\rho \left\|W_h^0\right\|_{L_2(\Omega)}^2 + (C+2)\left\|U_h^0\right\|_V^2 + \frac{\Delta t}{\epsilon_a}\sum_{n=0}^{N-1}\left\|\bar{f}^n\right\|_{L_2(\Omega)}^2 + \frac{\Delta t\epsilon_a}{2}\left\|W_h^m\right\|_{L_2(\Omega)} \\
&+ \Delta t\epsilon_a \sum_{n=0}^{N-1}\left\|W_h^n\right\|_{L_2(\Omega)}^2 + \frac{C}{\epsilon_b}\left\|\bar{g}_N^{m-1}\right\|_{L_2(\Gamma_N)}^2 + C\left\|\bar{g}_N^0\right\|_{L_2(\Gamma_N)}^2 + \frac{C}{\epsilon_d}\int_0^T \left\|\dot{g}_N(t')\right\|_{L_2(\Gamma_N)}^2 dt' \\
&+ \big(C\epsilon_d + \frac{\epsilon_e}{2}\big)\Delta t\sum_{n=0}^{m-1}\left\|U_h^n\right\|_V^2 + \big(\frac{1}{\epsilon_c} + \sum_{q=1}^{N_\varphi}\frac{1}{4\tau_q\epsilon_e}\big)\left\|u_0\right\|_V^2 + (C\epsilon_b + \epsilon_c)\left\|U_h^m\right\|_V^2 \\
\leq&\rho \left\|W_h^0\right\|_{L_2(\Omega)}^2 + (C+2)\left\|U_h^0\right\|_V^2 + \frac{\Delta t}{\epsilon_a}\sum_{n=0}^{N-1}\left\|\bar{f}^n\right\|_{L_2(\Omega)}^2 \\
&+ \left(\frac{C}{\epsilon_b} + C\right)\max_{0\leq n\leq N-1}\left\|\bar{g}_N^n\right\|_{L_2(\Gamma_N)}^2 + C\epsilon_d\left\|\dot{g}_N\right\|_{L_2(0,T;L_2(\Gamma_N))}^2 + \big(\frac{1}{\epsilon_c} + \sum_{q=1}^{N_\varphi}\frac{1}{4\tau_q\epsilon_e}\big)\left\|u_0\right\|_V^2 \\
&+ \Delta t\epsilon_a\left(N + \frac{1}{2}\right)\max_{0\leq n\leq N}\left\|W_h^n\right\|_{L_2(\Omega)}^2 + \left(C\epsilon_b + \epsilon_c + CT\epsilon_d + T\frac{\epsilon_e}{2}\right)\max_{0\leq n\leq N}\left\|U_h^n\right\|_V^2 .
\end{aligned}
$$

If we consider the property of maximum, then we have

$$
\begin{aligned}
&\rho\max_{0\leq n\leq N}\left\|W_h^n\right\|_{L_2(\Omega)}^2 + \varphi_0\max_{0\leq n\leq N}\left\|U_h^n\right\|_V^2 + \sum_{q=1}^{N_\varphi}\frac{1}{\varphi_q}\left\|\mathcal{S}_{hq}^m\right\|_V^2 \\
&+ \sum_{q=1}^{N_\varphi}\sum_{n=0}^{m-1}\frac{\Delta t}{2\tau_q\varphi_q}\left\|\mathcal{S}_{hq}^{n+1} + \mathcal{S}_{hq}^n\right\|_V^2 \\
\leq&3\Bigg(\rho\left\|W_h^0\right\|_{L_2(\Omega)}^2 + (C+2)\left\|U_h^0\right\|_V^2 + \frac{\Delta t}{\epsilon_a}\sum_{n=0}^{N-1}\left\|\bar{f}^n\right\|_{L_2(\Omega)}^2 \\
&+ \left(\frac{C}{\epsilon_b} + C\right)\max_{0\leq n\leq N-1}\left\|\bar{g}_N^n\right\|_{L_2(\Gamma_N)}^2 + C\epsilon_d\left\|\dot{g}_N\right\|_{L_2(0,T;L_2(\Gamma_N))}^2 + \big(\frac{1}{\epsilon_c} + \sum_{q=1}^{N_\varphi}\frac{1}{4\tau_q\epsilon_e}\big)\left\|u_0\right\|_V^2 \\
&+ \Delta t\epsilon_a\left(N + \frac{1}{2}\right)\max_{0\leq n\leq N}\left\|W_h^n\right\|_{L_2(\Omega)}^2 + \left(C\epsilon_b + \epsilon_c + CT\epsilon_d + T\frac{\epsilon_e}{2}\right)\max_{0\leq n\leq N}\left\|U_h^n\right\|_V^2\Bigg).
\end{aligned}
$$

Set ϵ_a, ϵ_b, ϵ_c, ϵ_d and ϵ_e by

$$
\epsilon_a = \frac{\rho}{\Delta t(6N+3)} = \frac{\rho}{6T + 3\Delta t} > 0,
$$

$$
\epsilon_b = \frac{\varphi_0}{24C} > 0,\ \epsilon_c = \frac{\varphi_0}{24} > 0,\ \epsilon_d = \frac{\varphi_0}{24CT} > 0,\ \text{and}\ \epsilon_e = \frac{\varphi_0}{12T} > 0,
$$

then

$$
\begin{aligned}
3\Delta t\epsilon_a\left(N + \frac{1}{2}\right) &= \frac{\rho}{2}, \\
3\left(C\epsilon_b + \epsilon_c + CT\epsilon_d + T\epsilon_e/2\right) &= 3\left(\frac{\varphi_0}{24} + \frac{\varphi_0}{24} + \frac{\varphi_0}{24} + \frac{\varphi_0}{24}\right) = \frac{\varphi_0}{2}.
\end{aligned}
$$

Thus we have a positive constant C such that

$$
\begin{aligned}
&\frac{\rho}{2}\max_{0\le n\le N}\|W_h^n\|_{L_2(\Omega)}^2+\frac{\varphi_0}{2}\max_{0\le n\le N}\|U_h^n\|_V^2+\sum_{q=1}^{N_\varphi}\frac{1}{\varphi_q}\left\|\mathcal{S}_{hq}^m\right\|_V^2\\
&\quad+\sum_{q=1}^{N_\varphi}\sum_{n=0}^{m-1}\frac{\Delta t}{2\tau_q\varphi_q}\left\|\mathcal{S}_{hq}^{n+1}+\mathcal{S}_{hq}^n\right\|_V^2\\
&\le C\Bigg(\left\|W_h^0\right\|_{L_2(\Omega)}^2+\left\|U_h^0\right\|_V^2+\Delta t\sum_{n=0}^{N-1}\|\bar{f}^n\|_{L_2(\Omega)}^2+\max_{0\le n\le N-1}\|\bar{g}_N^n\|_{L_2(\Gamma_N)}^2\\
&\quad+\|\dot{g}_N\|_{L_2(0,T;L_2(\Gamma_N))}^2+\|u_0\|_V^2\Bigg).
\end{aligned}
$$

Therefore, it is concluded that

$$
\begin{aligned}
&\frac{\rho}{2}\max_{0\le n\le N}\|W_h^n\|_{L_2(\Omega)}^2+\frac{\varphi_0}{2}\max_{0\le n\le N}\|U_h^n\|_V^2+\sum_{q=1}^{N_\varphi}\frac{1}{\varphi_q}\left\|\mathcal{S}_{hq}^m\right\|_V^2\\
&\quad+\sum_{q=1}^{N_\varphi}\sum_{n=0}^{m-1}\frac{\Delta t}{2\tau_q\varphi_q}\left\|\mathcal{S}_{hq}^{n+1}+\mathcal{S}_{hq}^n\right\|_V^2\\
&\le C\Bigg(\|w_0\|_{L_2(\Omega)}^2+\|u_0\|_V^2+\|f\|_{L_\infty(0,T;L_2(\Omega))}^2+\|g_N\|_{L_\infty(0,T;L_2(\Gamma_N))}^2\\
&\quad+\|\dot{g}_N\|_{L_2(0,T;L_2(\Gamma_N))}^2\Bigg),
\end{aligned}
$$

when we use L_∞ norms of f and g_N in time and initial conditions,

$$
\left\|U_h^0\right\|_V\le\|u_0\|_V \text{ and } \left\|W_h^0\right\|_{L_2(\Omega)}\le\|w_0\|_{L_2(\Omega)}.
$$

□

Theorem 2.15 and 2.16 provide the stability bounds for fully discrete formula of **(P2)** in terms of data so that the governing linear system (2.3.26)-(2.3.28) can be solved uniquely.

In a similar way with Theorem 2.14, error estimates for **(P2)** would be introduced and shown. First of all, let us define the following notations,

$$
\begin{aligned}
&\theta:=u-Ru, &&\chi^n:=U_h^n-Ru^n, &&\varpi^n:=W_h^n-R\dot{u}^n,\\
&\nu_q:=\zeta_q-R\zeta_q, &&\Upsilon_q^n:=\mathcal{S}_{hq}^n-R\zeta_q^n,\ \forall q\in\{1,\dots,N_\varphi\},
\end{aligned}
$$

where $u^n=u(t_n)$.

Lemma 2.13. *Suppose $u \in H^4(0,T;H^s(\Omega)) \cap W^1_\infty(0,T;H^s(\Omega))$. Then there exists a positive constant C such that*

$$\max_{0\le j\le N} \left\|\varpi^j\right\|_{L_2(\Omega)} + \max_{0\le j\le N} \left\|\chi^j\right\|_V \le C(h^{\min(k+1,s)-1} + \Delta t^2),$$

Furthermore, if our domain fulfils the condition of elliptic regularity, we have

$$\max_{0\le j\le N} \left\|\varpi^j\right\|_{L_2(\Omega)} + \max_{0\le j\le N} \left\|\chi^j\right\|_V \le C(h^{\min(k+1,s)} + \Delta t^2).$$

Proof. This proof will follow the similar steps with those of Lemma 2.7 and 2.10. By subtraction of (2.2.11) from (2.3.22), we have for any $v \in V^h$

$$\begin{aligned}
&\rho\left(\frac{\ddot{u}^{n+1}+\ddot{u}^n}{2} - \frac{W_h^{n+1}-W_h^n}{\Delta t}, v\right)_{L_2(\Omega)} + \varphi_0 a\left(\frac{u^{n+1}+u^n}{2} - \frac{U_h^{n+1}+U_h^n}{2}, v\right)\\
&+\sum_{q=1}^{N_\varphi} a\left(\frac{\zeta_q^{n+1}+\zeta_q^n}{2} - \frac{\mathcal{S}_{hq}^{n+1}+\mathcal{S}_{hq}^n}{2}, v\right) = 0,\\
\Rightarrow &\rho\left(\frac{\dot{u}^{n+1}-\dot{u}^n}{\Delta t} - \frac{W_h^{n+1}-W_h^n}{\Delta t}, v\right)_{L_2(\Omega)} + \varphi_0 a\left(\frac{u^{n+1}+u^n}{2} - \frac{U_h^{n+1}+U_h^n}{2}, v\right)\\
&+\sum_{q=1}^{N_\varphi} a\left(\frac{\zeta_q^{n+1}+\zeta_q^n}{2} - \frac{\mathcal{S}_{hq}^{n+1}+\mathcal{S}_{hq}^n}{2}, v\right) = \rho\left(\frac{\dot{u}^{n+1}-\dot{u}^n}{\Delta t} - \frac{\ddot{u}^{n+1}+\ddot{u}^n}{2}, v\right)_{L_2(\Omega)}.
\end{aligned}$$

So adding zeros leads

$$\begin{aligned}
&\frac{\rho}{\Delta t}\left(\varpi^{n+1}-\varpi^n, v\right)_{L_2(\Omega)} + \frac{\varphi_0}{2} a\left(\chi^{n+1}+\chi^n, v\right) + \frac{1}{2}\sum_{q=1}^{N_\varphi} a\left(\Upsilon_q^{n+1}+\Upsilon_q^n, v\right)\\
=&\frac{\rho}{\Delta t}\left(\dot{\theta}^{n+1}-\dot{\theta}^n, v\right)_{L_2(\Omega)} + \frac{\varphi_0}{2} a\left(\theta^{n+1}+\theta^n, v\right) + \frac{1}{2}\sum_{q=1}^{N_\varphi} a\left(\nu_q^{n+1}+\nu_q^n, v\right)\\
&+\frac{\rho}{\Delta t}\left(\mathcal{E}_1^n, v\right)_{L_2(\Omega)}\\
=&\frac{\rho}{\Delta t}\left(\dot{\theta}^{n+1}-\dot{\theta}^n, v\right)_{L_2(\Omega)} + \rho\left(\mathcal{E}_1^n, v\right)_{L_2(\Omega)}
\end{aligned}$$

for any $v \in V^h$, where $\mathcal{E}_1(t) := \frac{\ddot{u}(t+\Delta t)+\ddot{u}(t)}{2} - \frac{\dot{u}(t+\Delta t)-\dot{u}(t)}{\Delta t}$, since Galerkin orthogonality gives

$$a(\theta, v) = 0 \text{ and } a(\nu_q, v) = 0,\ \forall q \in \{1,\dots,N_\varphi\}.$$

If we put $v = \frac{\chi^{n+1}-\chi^n}{\Delta t}$ with (2.3.16) here, we can obtain

$$\frac{\rho}{\Delta t}\left(\varpi^{n+1}-\varpi^n, \frac{\varpi^{n+1}+\varpi^n}{2}\right)_{L_2(\Omega)} + \frac{\varphi_0}{2} a\left(\chi^{n+1}+\chi^n, \frac{\chi^{n+1}-\chi^n}{\Delta t}\right)$$

$$
\begin{aligned}
&+\frac{1}{2}\sum_{q=1}^{N_\varphi} a\left(\Upsilon_q^{n+1}+\Upsilon_q^n,\frac{\chi^{n+1}-\chi^n}{\Delta t}\right)\\
=&\frac{\rho}{2\Delta t}\left(\|\varpi^{n+1}\|^2_{L_2(\Omega)}-\|\varpi^n\|^2_{L_2(\Omega)}\right)+\frac{\varphi_0}{2\Delta t}\left(\|\chi^{n+1}\|_V^2-\|\chi^n\|_V^2\right)\\
&+\frac{1}{2\Delta t}\sum_{q=1}^{N_\varphi} a\left(\Upsilon_q^{n+1}+\Upsilon_q^n,\chi^{n+1}-\chi^n\right)\\
=&\frac{\rho}{2\Delta t}\left(\dot\theta^{n+1}-\dot\theta^n,\varpi^{n+1}+\varpi^n\right)_{L_2(\Omega)}+\frac{\rho}{2}\left(\mathcal{E}_1^n,\varpi^{n+1}+\varpi^n\right)_{L_2(\Omega)}\\
&-\frac{\rho}{\Delta t}\left(\dot\theta^{n+1}-\dot\theta^n,\mathcal{E}_2^n\right)_{L_2(\Omega)}-\frac{\rho}{\Delta t}\left(\dot\theta^{n+1}-\dot\theta^n,\mathcal{E}_3^n\right)_{L_2(\Omega)}\\
&-\rho\left(\mathcal{E}_1^n,\mathcal{E}_2^n\right)_{L_2(\Omega)}-\rho\left(\mathcal{E}_1^n,\mathcal{E}_3^n\right)_{L_2(\Omega)},
\end{aligned}
\tag{2.3.31}
$$

where

$$
\begin{aligned}
\mathcal{E}_2(t):=&\frac{\dot\theta(t+\Delta t)+\dot\theta(t)}{2}-\frac{\theta(t+\Delta t)-\theta(t)}{\Delta t},\\
\mathcal{E}_3(t):=&\frac{u(t+\Delta t)-u(t)}{\Delta t}-\frac{\dot u(t+\Delta t)+\dot u(t)}{2}.
\end{aligned}
$$

On the other hand, (2.1.21) and (2.3.23) yield for any $v\in V^h$

$$
\begin{aligned}
&\tau_q a\left(\frac{\dot\zeta_q^{n+1}+\dot\zeta_q^n}{2}-\frac{\mathcal{S}_{hq}^{n+1}-\mathcal{S}_{hq}^n}{\Delta t},v\right)+a\left(\frac{\zeta_q^{n+1}+\zeta_q^n}{2}-\frac{\mathcal{S}_{hq}^{n+1}+\mathcal{S}_{hq}^n}{2},v\right)\\
=&\tau_q\varphi_q a\left(\frac{\dot u^{n+1}+\dot u^n}{2}-\frac{U_h^{n+1}-U_h^n}{\Delta t},v\right)
\end{aligned}
$$

for any q hence

$$
\begin{aligned}
&\frac{\tau_q}{\Delta t}a\left(\Upsilon_q^{n+1}-\Upsilon_q^n,v\right)+\frac{1}{2}a\left(\Upsilon_q^{n+1}+\Upsilon_q^n,v\right)-\frac{\tau_q\varphi_q}{\Delta t}a\left(\chi^{n+1}-\chi^n,v\right)\\
&=\frac{\tau_q}{\Delta t}a\left(\nu_q^{n+1}-\nu_q^n,v\right)+\frac{1}{2}a\left(\nu_q^{n+1}+\nu_q^n,v\right)-\frac{\tau_q\varphi_q}{\Delta t}a\left(\theta^{n+1}-\theta^n,v\right)\\
&\quad+\tau_q a\left(E_q^n,v\right)-\tau_q\varphi_q a\left(\mathcal{E}_3^n,v\right)\\
&=\tau_q a\left(E_q^n,v\right)-\tau_q\varphi_q a\left(\mathcal{E}_3^n,v\right)
\end{aligned}
$$

where for each q

$$
E_q(t):=\frac{\dot\zeta_q(t+\Delta t)+\dot\zeta_q(t)}{2}-\frac{\zeta_q(t+\Delta t)-\zeta_q(t)}{\Delta t}.
$$

Here, we will set $v=\frac{\Upsilon_q^{n+1}+\Upsilon_q^n}{2}$, then

$$
\frac{1}{2\Delta t}a\left(\chi^{n+1}-\chi^n,\Upsilon_q^{n+1}+\Upsilon_q^n\right)=\frac{1}{2\Delta t\varphi_q}\left(\|\Upsilon_q^{n+1}\|_V^2-\|\Upsilon_q^n\|_V^2\right)+\frac{1}{2\tau_q\varphi_q}\|\Upsilon_q^{n+1}+\Upsilon_q^n\|_V^2
$$

$$-\frac{1}{2\varphi_q}a\left(E_q^n,\Upsilon_q^{n+1}+\Upsilon_q^n\right)+\frac{1}{2}a\left(\mathcal{E}_3^n,\Upsilon_q^{n+1}+\Upsilon_q^n\right). \tag{2.3.32}$$

By substitution of (2.3.32) into (2.3.31) and multiplication by Δt, it implies

$$\begin{aligned}
&\frac{\rho}{2}\left(\|\varpi^{n+1}\|_{L_2(\Omega)}^2-\|\varpi^n\|_{L_2(\Omega)}^2\right)+\frac{\varphi_0}{2}\left(\|\chi^{n+1}\|_V^2-\|\chi^n\|_V^2\right)\\
&\quad+\frac{1}{2}\sum_{q=1}^{N_\varphi}\frac{1}{\varphi_q}\left(\|\Upsilon_q^{n+1}\|_V^2-\|\Upsilon_q^n\|_V^2\right)+\Delta t\sum_{q=1}^{N_\varphi}\frac{1}{2\tau_q\varphi_q}\|\Upsilon_q^{n+1}+\Upsilon_q^n\|_V^2\\
=&\frac{\rho}{2}\left(\dot\theta^{n+1}-\dot\theta^n,\varpi^{n+1}+\varpi^n\right)_{L_2(\Omega)}+\frac{\rho}{2}\Delta t\left(\mathcal{E}_1^n,\varpi^{n+1}+\varpi^n\right)_{L_2(\Omega)}\\
&\quad-\rho\left(\dot\theta^{n+1}-\dot\theta^n,\mathcal{E}_2^n\right)_{L_2(\Omega)}-\rho\left(\dot\theta^{n+1}-\dot\theta^n,\mathcal{E}_3^n\right)_{L_2(\Omega)}-\rho\Delta t\left(\mathcal{E}_1^n,\mathcal{E}_2^n\right)_{L_2(\Omega)}\\
&\quad-\rho\Delta t\left(\mathcal{E}_1^n,\mathcal{E}_3^n\right)_{L_2(\Omega)}+\frac{\Delta t}{2}\sum_{q=1}^{N_\varphi}\frac{1}{\varphi_q}a\left(E_q^n,\Upsilon_q^{n+1}+\Upsilon_q^n\right)-\frac{\Delta t}{2}\sum_{q=1}^{N_\varphi}a\left(\mathcal{E}_3^n,\Upsilon_q^{n+1}+\Upsilon_q^n\right).
\end{aligned} \tag{2.3.33}$$

Consider the summation of (2.3.33) for $n=0,1,\dots,m-1$ when $1\le m\le N$. Then we have

$$\begin{aligned}
&\frac{\rho}{2}\|\varpi^m\|_{L_2(\Omega)}^2+\frac{\varphi_0}{2}\|\chi^m\|_V^2+\frac{1}{2}\sum_{q=1}^{N_\varphi}\frac{1}{\varphi_q}\|\Upsilon_q^m\|_V^2+\Delta t\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi}\frac{1}{2\tau_q\varphi_q}\|\Upsilon_q^{n+1}+\Upsilon_q^n\|_V^2\\
=&\frac{\rho}{2}\|\varpi^0\|_{L_2(\Omega)}^2+\frac{\varphi_0}{2}\|\chi^0\|_V^2+\frac{1}{2}\sum_{q=1}^{N_\varphi}\frac{1}{\varphi_q}\|\Upsilon_q^0\|_V^2+\frac{\rho}{2}\sum_{n=0}^{m-1}\left(\dot\theta^{n+1}-\dot\theta^n,\varpi^{n+1}+\varpi^n\right)_{L_2(\Omega)}\\
&\quad+\frac{\rho}{2}\Delta t\sum_{n=0}^{m-1}\left(\mathcal{E}_1^n,\varpi^{n+1}+\varpi^n\right)_{L_2(\Omega)}-\rho\sum_{n=0}^{m-1}\left(\dot\theta^{n+1}-\dot\theta^n,\mathcal{E}_2^n\right)_{L_2(\Omega)}\\
&\quad-\rho\sum_{n=0}^{m-1}\left(\dot\theta^{n+1}-\dot\theta^n,\mathcal{E}_3^n\right)_{L_2(\Omega)}-\rho\Delta t\sum_{n=0}^{m-1}\left(\mathcal{E}_1^n,\mathcal{E}_2^n\right)_{L_2(\Omega)}-\rho\Delta t\sum_{n=0}^{m-1}\left(\mathcal{E}_1^n,\mathcal{E}_3^n\right)_{L_2(\Omega)}\\
&\quad+\frac{\Delta t}{2}\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi}\frac{1}{\varphi_q}a\left(E_q^n,\Upsilon_q^{n+1}+\Upsilon_q^n\right)-\frac{\Delta t}{2}\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi}a\left(\mathcal{E}_3^n,\Upsilon_q^{n+1}+\Upsilon_q^n\right).
\end{aligned} \tag{2.3.34}$$

As following the proof of Lemma 2.10, we can consider each component of the right hand side of (2.3.34). Let $r=\min(k+1,s)$.

- $\|\varpi^0\|_{L_2(\Omega)}$

$$\|\varpi^0\|_{L_2(\Omega)}=O(h^{r-1}).$$

- $\|\chi^0\|_V$

$$\|\chi^0\|_V=0.$$

- $\left\|\Upsilon_q^0\right\|_V$ for each q

$$\left\|\Upsilon_q^0\right\|_V = 0 \text{ by initial condition.}$$

- $\sum_{n=0}^{m-1}\left(\dot\theta^{n+1}-\dot\theta^n,\varpi^{n+1}+\varpi^n\right)_{L_2(\Omega)}$

$$\begin{aligned}\sum_{n=0}^{m-1}\left(\dot\theta^{n+1}-\dot\theta^n,\varpi^{n+1}+\varpi^n\right)_{L_2(\Omega)} \leq & \frac{1}{2\epsilon_a}\left\|\ddot\theta\right\|^2_{L_2(0,T;L_2(\Omega))}+2\epsilon_a T\max_{0\leq j\leq N}\left\|\varpi^j\right\|^2_{L_2(\Omega)}\\ \leq & \frac{1}{2\epsilon_a}O(h^{2(r-1)})+2\epsilon_a T\max_{0\leq j\leq N}\left\|\varpi^j\right\|^2_{L_2(\Omega)}\end{aligned}$$

for any positive ϵ_a.

- $\Delta t\sum_{n=0}^{m-1}\left(\mathcal{E}_1^n,\varpi^{n+1}+\varpi^n\right)_{L_2(\Omega)}$

$$\begin{aligned}\Delta t\sum_{n=0}^{m-1}\left(\mathcal{E}_1^n,\varpi^{n+1}+\varpi^n\right)_{L_2(\Omega)} \leq & \frac{T}{2\epsilon_b}\max_{0\leq j\leq N-1}\left\|\mathcal{E}_1^j\right\|^2_{L_2(\Omega)}+2\epsilon_b T\max_{0\leq j\leq N}\left\|\varpi^j\right\|^2_{L_2(\Omega)}\\ \leq & \frac{T}{2\epsilon_b}O(\Delta t^4)+2\epsilon_b T\max_{0\leq j\leq N}\left\|\varpi^j\right\|^2_{L_2(\Omega)}\end{aligned}$$

for any positive ϵ_b.

- $-\sum_{n=0}^{m-1}\left(\dot\theta^{n+1}-\dot\theta^n,\mathcal{E}_2^n\right)_{L_2(\Omega)}$, $-\sum_{n=0}^{m-1}\left(\dot\theta^{n+1}-\dot\theta^n,\mathcal{E}_3^n\right)_{L_2(\Omega)}$

$$\begin{aligned}-\sum_{n=0}^{m-1}\left(\dot\theta^{n+1}-\dot\theta^n,\mathcal{E}_2^n\right)_{L_2(\Omega)} \leq & \frac{1}{2}\left\|\ddot\theta\right\|^2_{L_2(0,T;L_2(\Omega))}+\frac{T}{2}\max_{0\leq j\leq N-1}\left\|\mathcal{E}_2^j\right\|^2_{L_2(\Omega)}\\ \leq & O(h^{2(r-1)}+\Delta t^4),\end{aligned}$$

and

$$\begin{aligned}-\sum_{n=0}^{m-1}\left(\dot\theta^{n+1}-\dot\theta^n,\mathcal{E}_3^n\right)_{L_2(\Omega)} \leq & \frac{1}{2}\left\|\ddot\theta\right\|^2_{L_2(0,T;L_2(\Omega))}+\frac{T}{2}\max_{0\leq j\leq N-1}\left\|\mathcal{E}_3^j\right\|^2_{L_2(\Omega)}\\ \leq & O(h^{2(r-1)}+\Delta t^4).\end{aligned}$$

- $-\Delta t\sum_{n=0}^{m-1}\left(\mathcal{E}_1^n,\mathcal{E}_2^n\right)_{L_2(\Omega)}$, $-\Delta t\sum_{n=0}^{m-1}\left(\mathcal{E}_1^n,\mathcal{E}_3^n\right)_{L_2(\Omega)}$

$$\begin{aligned}-\Delta t\sum_{n=0}^{m-1}\left(\mathcal{E}_1^n,\mathcal{E}_2^n\right)_{L_2(\Omega)} \leq & \frac{T}{2}\max_{0\leq j\leq N-1}\left\|\mathcal{E}_1^j\right\|^2_{L_2(\Omega)}+\frac{T}{2}\max_{0\leq j\leq N-1}\left\|\mathcal{E}_2^j\right\|^2_{L_2(\Omega)},\\ \leq & O(\Delta t^4),\end{aligned}$$

and

$$
\begin{aligned}
-\Delta t \sum_{n=0}^{m-1} \left(\mathcal{E}_1^n, \mathcal{E}_3^n\right)_{L_2(\Omega)} \leq & \frac{T}{2} \max_{0\leq j\leq N-1} \left\|\mathcal{E}_1^j\right\|_{L_2(\Omega)}^2 + \frac{T}{2} \max_{0\leq j\leq N-1} \left\|\mathcal{E}_3^j\right\|_{L_2(\Omega)}^2, \\
\leq & O(\Delta t^4).
\end{aligned}
$$

- $\Delta t \sum\limits_{n=0}^{m-1} \sum\limits_{q=1}^{N_\varphi} \frac{1}{\varphi_q} a\left(E_q^n, \Upsilon_q^{n+1} + \Upsilon_q^n\right)$, $-\Delta t \sum\limits_{n=0}^{m-1} \sum\limits_{q=1}^{N_\varphi} a\left(\mathcal{E}_3^n, \Upsilon_q^{n+1} + \Upsilon_q^n\right)$

$$
\begin{aligned}
\Delta t \sum_{n=0}^{m-1} \sum_{q=1}^{N_\varphi} \frac{1}{\varphi_q} a\left(E_q^n, \Upsilon_q^{n+1} + \Upsilon_q^n\right) \leq & \Delta t \sum_{n=0}^{m-1} \sum_{q=1}^{N_\varphi} \frac{1}{\varphi_q} \left\|E_q^n\right\|_V \left\|\Upsilon_q^{n+1} + \Upsilon_q^n\right\|_V \\
& \text{(by Cauchy-Schwarz inequality)} \\
\leq & \Delta t \sum_{n=0}^{m-1} \sum_{q=1}^{N_\varphi} \frac{1}{2\varphi_q^2 \epsilon_q} \left\|E_q^n\right\|_V^2 \\
& + \Delta t \sum_{n=0}^{m-1} \sum_{q=1}^{N_\varphi} \frac{\epsilon_q}{2} \left\|\Upsilon_q^{n+1} + \Upsilon_q^n\right\|_V^2 \\
& \text{(by Young's inequality)} \\
\leq & \Delta t \sum_{n=0}^{N-1} \sum_{q=1}^{N_\varphi} \frac{1}{\varphi_q^2} \frac{1}{2\epsilon_q} \left\|E_q^n\right\|_V^2 \\
& + \Delta t \sum_{n=0}^{m-1} \sum_{q=1}^{N_\varphi} \frac{\epsilon_q}{2} \left\|\Upsilon_q^{n+1} + \Upsilon_q^n\right\|_V^2 \\
& (\because m \leq N \text{ for the first term}) \\
\leq & T \sum_{q=1}^{N_\varphi} \frac{1}{\varphi_q^2} \frac{1}{2\epsilon_q} O(\Delta t^4) \\
& + \Delta t \sum_{n=0}^{m-1} \sum_{q=1}^{N_\varphi} \frac{\epsilon_q}{2} \left\|\Upsilon_q^{n+1} + \Upsilon_q^n\right\|_V^2 \\
& (\because \ \left\|E_q^n\right\|_V = O(\Delta t^2)).
\end{aligned}
$$

In the same sense

$$
\begin{aligned}
-\Delta t \sum_{n=0}^{m-1} \sum_{q=1}^{N_\varphi} a\left(E_3^n, \Upsilon_q^{n+1} + \Upsilon_q^n\right) \leq & T \sum_{q=1}^{N_\varphi} \frac{1}{2\epsilon_q} O(\Delta t^4) \\
& + \Delta t \sum_{n=0}^{m-1} \sum_{q=1}^{N_\varphi} \frac{\epsilon_q}{2} \left\|\Upsilon_q^{n+1} + \Upsilon_q^n\right\|_V^2 \\
& (\because \ \|E_3^n\|_V = O(\Delta t^2)).
\end{aligned}
$$

While we choose $\epsilon_a = \frac{\rho}{24T}$, $\epsilon_b = \frac{\rho}{24T}$ and $\epsilon_q = \frac{1}{2\tau_q\varphi_q}$ for each q, (2.3.34) yields

$$\begin{aligned}&\frac{\rho}{2}\|\varpi^m\|^2_{L_2(\Omega)} + \frac{\varphi_0}{2}\|\chi^m\|^2_V + \frac{1}{2}\sum_{q=1}^{N_\varphi}\frac{1}{\varphi_q}\left\|\Upsilon_q^m\right\|^2_V + \Delta t\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi}\frac{1}{2\tau_q\varphi_q}\left\|\Upsilon_q^{n+1}+\Upsilon_q^n\right\|^2_V\\ \leq&\frac{\rho}{12}\max_{0\leq j\leq N}\left\|\varpi^j\right\|^2_{L_2(\Omega)} + \Delta t\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi}\frac{1}{4\tau_q\varphi_q}\left\|\Upsilon_q^{n+1}+\Upsilon_q^n\right\|^2_V + O(h^{2(r-1)}+\Delta t^4)\end{aligned}$$

so that

$$\begin{aligned}&\frac{\rho}{2}\|\varpi^m\|^2_{L_2(\Omega)} + \frac{\varphi_0}{2}\|\chi^m\|^2_V + \frac{1}{2}\sum_{q=1}^{N_\varphi}\frac{1}{\varphi_q}\left\|\Upsilon_q^m\right\|^2_V + \Delta t\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi}\frac{1}{4\tau_q\varphi_q}\left\|\Upsilon_q^{n+1}+\Upsilon_q^n\right\|^2_V\\ \leq&\frac{\rho}{12}\max_{0\leq j\leq N}\left\|\varpi^j\right\|^2_{L_2(\Omega)} + O(h^{2(r-1)}+\Delta t^4).\end{aligned}$$

Taking into account maximum,

$$\begin{aligned}&\frac{\rho}{2}\max_{0\leq j\leq N}\left\|\varpi^j\right\|^2_{L_2(\Omega)} + \frac{\varphi_0}{2}\max_{0\leq j\leq N}\left\|\chi^j\right\|^2_V + \frac{1}{2}\sum_{q=1}^{N_\varphi}\frac{1}{\varphi_q}\left\|\Upsilon_q^m\right\|^2_V\\ &+ \Delta t\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi}\frac{1}{4\tau_q\varphi_q}\left\|\Upsilon_q^{n+1}+\Upsilon_q^n\right\|^2_V\\ \leq&3\left(\frac{\rho}{12}\max_{0\leq j\leq N}\left\|\varpi^j\right\|^2_{L_2(\Omega)} + O(h^{2(r-1)}+\Delta t^4)\right).\end{aligned}$$

Then we have

$$\begin{aligned}&\frac{\rho}{4}\max_{0\leq j\leq N}\left\|\varpi^j\right\|^2_{L_2(\Omega)} + \frac{\varphi_0}{2}\max_{0\leq j\leq N}\left\|\chi^j\right\|^2_V + \frac{1}{2}\sum_{q=1}^{N_\varphi}\frac{1}{\varphi_q}\left\|\Upsilon_q^m\right\|^2_V\\ &+ \Delta t\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi}\frac{1}{4\tau_q\varphi_q}\left\|\Upsilon_q^{n+1}+\Upsilon_q^n\right\|^2_V\\ \leq&O(h^{2(r-1)}+\Delta t^4),\end{aligned}$$

therefore,

$$\max_{0\leq j\leq N}\left\|\varpi^j\right\|_{L_2(\Omega)} + \max_{0\leq j\leq N}\left\|\chi^j\right\|_V \leq C(h^{(r-1)}+\Delta t^2),$$

for some positive C.

Additionally, if we have elliptic regularity, (1.4.9) allows us to obtain

$$\left\|\dot{\theta}^0\right\|_{L_2(\Omega)}, \left\|\ddot{\theta}\right\|_{L_2(0,T;L_2(\Omega))} \leq Ch^r$$

for some positive C, thus

$$\max_{0\le j\le N}\left\|\varpi^j\right\|_{L_2(\Omega)}+\max_{0\le j\le N}\left\|\chi^j\right\|_V \le C(h^r+\Delta t^2),$$

for some positive C. □

From this lemma, we can observe L_2 estimates of the velocity and energy estimates of the displacement in a similar way with Theorem 2.14.

Theorem 2.17. *Suppose $u\in H^4(0,T;H^s(\Omega))\cap L_\infty(0,T;H^s(\Omega))$. Then we have*

$$\max_{0\le j\le N}\left\|u(t_j)-U_h^j\right\|_V\le C(h^{\min(k+1,s)-1}+\Delta t^2)$$

and

$$\max_{0\le j\le N}\left\|\dot{u}(t_j)-W_h^j\right\|_{L_2(\Omega)}\le C(h^{\min(k+1,s)-1}+\Delta t^2)$$

for some positive C. In addition,

$$\max_{0\le j\le N}\left\|\dot{u}(t_j)-W_h^j\right\|_{L_2(\Omega)}\le C(h^{\min(k+1,s)}+\Delta t^2)$$

if elliptic regularity is provided.

Proof. From Lemma 2.13, it is provided that

$$\max_{0\le j\le N}\left\|\varpi^j\right\|_{L_2(\Omega)},\ \max_{0\le j\le N}\left\|\chi^j\right\|_V\le C(h^{\min(k+1,s)-1}+\Delta t^2)$$

for some positive C. Hence, as following the proof of Theorem 2.14, it is given that

$$\max_{0\le j\le N}\left\|u(t_j)-U_h^j\right\|_V \le C(h^{\min(k+1,s)-1}+\Delta t^2)$$

and

$$\max_{0\le j\le N}\left\|\dot{u}(t_j)-W_h^j\right\|_{L_2(\Omega)} \le C(h^{\min(k+1,s)-1}+\Delta t^2)$$

for some positive C. Also,

$$\max_{0\le j\le N}\left\|\dot{u}(t_j)-W_h^j\right\|_{L_2(\Omega)} \le C(h^{\min(k+1,s)}+\Delta t^2)$$

with elliptic regularity. Thus our claim is shown. □

Corollary 2.2. *Under same conditions as* Theorem 2.17*, there exists a positive constant C such that*

$$\max_{0\le j\le N}\left\|u(t_j)-U_h^j\right\|_{L_2(\Omega)}\le C(h^{\min(k+1,s)-1}+\Delta t^2).$$

If elliptic regularity is given, it shows

$$\max_{0\le j\le N}\left\|u(t_j)-U_h^j\right\|_{L_2(\Omega)}\le C(h^{\min(k+1,s)}+\Delta t^2).$$

Proof. The proof parallels to that of Corollary 2.1 but instead of using the result from Theorem 2.14, here Theorem 2.17 is applied. □

By Theorem 2.17 and Corollary 2.2, we can observe the error estimates of the displacement and the velocity for **(P2)** with respect to L_2 norm as well as the energy norm in space. Indeed, since the energy norm is equivalent to H^1 norm in space, we gain H^1 error estimates too.

Regardless of the form of internal variables, error estimation theorems for **(P1)** and **(P2)** describe same error convergence rates. Hence we will check this result by numerical experiments.

2.4 Numerical Experiments

Before carrying out our experiments, let the exact solution u be

$$u(x, y, t) = e^{-t} \sin(xy)$$

on the unit square $(x, y) \in [0, 1] \times [0, 1]$ and $t \in [0, T]$ where $T = 1$. Hence our domain Ω satisfies the condition for elliptic regularity since Ω is a polytopic domain [11] so elliptic regularity is given on our domain. The Dirichlet boundary condition is given by

$$u = 0 \text{ if } x = 0 \text{ or } y = 0, \ \forall t.$$

While we set

$$\varphi_0 = 0.5, \ \varphi_1 = 0.1, \ \varphi_2 = 0.4,$$
$$\tau_1 = 0.5, \ \tau_2 = 1.5,$$

and we suppose $\rho = 1$ and $D = 1$, internal variables ψ_q and ζ_q for $q = 1, 2$, the source term f and the Neumann boundary condition g_n are governed by the our primal problem. Then our exact solutions satisfy all conditions for stability bounds and error bounds. In other words, our exact solutions are sufficiently smooth in time and with respect to the domain Ω. Furthermore, our domain, the unit square, gives elliptic regularity. Finally, the code implementation is constructed by the finite element library FEniCS which allows us to get the powerful and useful computing platform.

(Test 1) As a first stage, we shall check the exactness for our code implementation. Let us define

$$e_h^n := u(t_n) - U_h^n \text{ and } \tilde{e}_h^n := \dot{u}(t_n) - W_h^n$$

where $N \in \mathbb{N}$, $\Delta t = T/N$ and $t_n = n\Delta t$ for $n = 0, \ldots, N$. Regardless of V^h, if we set $u = x + y + t^2$ without internal variables, that is $\varphi_q = 0, \forall q = 1, 2$, the error should be zero since our error estimates theorems indicate more than first order accuracy with respect to the spatial mesh size h and second order accuracy in time step Δt. As shown in Table 2.1, the errors are sufficiently small even if h and Δt are quite large so that we can conclude that our codes are equipped with exactness.

h	Δt	$\|e_h^N\|_V$	$\|\tilde{e}_h^N\|_{L_2(\Omega)}$	$\|e_h^N\|_{L_2(\Omega)}$
1/2	1/2	4.3147×10^{-14}	5.3779×10^{-14}	2.0207×10^{-14}
1/4	1/4	8.2422×10^{-14}	9.2163×10^{-14}	3.7541×10^{-14}
1/8	1/8	3.9651×10^{-13}	2.8720×10^{-13}	1.7180×10^{-13}

Table 2.1: Errors at $k = 1$ for $u = x + y + t^2$

(Test 2) If we set $u = e^{-t}(x + y)$ with $\varphi_1 = 0.1$ and $\varphi_2 = 0.4$, due to the exactness the errors should have second order convergence rate in time steps but it is also independent of h. This can be observed in Table 2.2 and 2.3. More precisely, in Table 2.2, the errors has decreased to quarter when Δt has halved, which implies the convergence order is 2. On the other hand, though h becomes smaller, the convergence of error is not shown. Table 2.3 indicates that even if the time step is significantly small, the domain mesh size has no effect on the errors. Therefore in this case we can conclude that $\|e_h^N\|_V, \|\tilde{e}_h^N\|_{L_2(\Omega)}, \|e_h^N\|_{L_2(\Omega)} = O(\Delta t^2)$.

h	Δt	$\|e_h^N\|_V$		$\|\tilde{e}_h^N\|_{L_2(\Omega)}$		$\|e_h^N\|_{L_2(\Omega)}$	
1/4	1/4	3.7885×10^{-3}		8.8124×10^{-3}		1.6380×10^{-3}	
	1/8	1.0240×10^{-3}	(1.89)	2.2092×10^{-3}	(2.00)	4.4722×10^{-4}	(1.87)
	1/16	2.6274×10^{-4}	(1.96)	5.5361×10^{-4}	(2.00)	1.1278×10^{-4}	(1.99)
1/8	1/4	3.8149×10^{-3}		8.8375×10^{-3}		1.6187×10^{-3}	
	1/8	1.0173×10^{-3}	(1.91)	2.2033×10^{-3}	(2.00)	4.4155×10^{-4}	(1.87)
	1/16	2.5549×10^{-4}	(1.99)	5.5225×10^{-4}	(2.00)	1.1278×10^{-4}	(1.99)
1/16	1/4	3.8474×10^{-3}		8.8428×10^{-3}		1.6138×10^{-3}	
	1/8	1.0252×10^{-3}	(1.91)	2.2021×10^{-3}	(2.00)	4.4015×10^{-4}	(1.87)
	1/16	2.5661×10^{-4}	(2.00)	5.5081×10^{-4}	(2.00)	1.1240×10^{-4}	(1.99)

Table 2.2: Errors at $k = 1$ for $u = e^{-t}(x + y)$

h	$\|e_h^N\|_V$	$\|\tilde{e}_h^N\|_{L_2(\Omega)}$	$\|e_h^N\|_{L_2(\Omega)}$
1/4	6.9856×10^{-8}	1.4221×10^{-7}	3.0189×10^{-8}
1/8	6.8304×10^{-8}	1.4207×10^{-7}	2.9703×10^{-8}
1/16	6.7354×10^{-8}	1.4179×10^{-7}	2.9581×10^{-8}
1/32	6.7048×10^{-8}	1.4162×10^{-7}	2.9573×10^{-8}

Table 2.3: Errors at $k = 1$ and $\Delta t = 1/1000$ for $u = e^{-t}(x + y)$

(Test 3) However, when we consider $u = t\sin(xy)$ without internal variables, the error convergent rates depend only on h. For example, as following the theorems, $\|e_h^N\|_V = O(h)$, $\|\tilde{e}_h^N\|_{L_2(\Omega)} = O(h^2)$ and $\|e_h^N\|_{L_2(\Omega)} = O(h^2)$ where $k = 1$. The numerical convergence order of h is shown in Table 2.4. But it is not seen that the errors has been reduced as Δt has changed. To be specific, if we suppose h is sufficiently smaller than

Δt, the error convergence is not observed with respect to change of Δt in Table 2.5. As a result the error estimates are given as $\left\|e_h^N\right\|_V = O(h)$, $\left\|\tilde{e}_h^N\right\|_{L_2(\Omega)} = O(h^2)$ and $\left\|e_h^N\right\|_{L_2(\Omega)}$ for $u = t\sin(xy)$ where $k = 1$. Furthermore, if we do not assume $\varphi_q = 0$, errors regarding time steps occur. By the definition of internal variables, integration of exponential functions is included so that it is required to use numerical approximations for calculation of these exponential functions. In other words, it loses the exactness for this case. It can be seen in Table 2.6. For $\left\|e_h^N\right\|_V$, since h is not negligible, it is not clear to find the convergence with respect to Δt, however the second order convergent rates are observed for $\left\|e_h^N\right\|_{L_2(\Omega)}$ and $\left\|\tilde{e}_h^N\right\|_{L_2(\Omega)}$ on Table 2.6.

Δt	h	$\left\|e_h^N\right\|_V$		$\left\|\tilde{e}_h^N\right\|_{L_2(\Omega)}$		$\left\|e_h^N\right\|_{L_2(\Omega)}$	
	1/4	1.2029×10^{-1}		1.0202×10^{-2}		7.1642×10^{-3}	
1/4	1/8	6.0817×10^{-2}	(0.98)	2.7633×10^{-3}	(1.88)	1.8611×10^{-3}	(1.94)
	1/16	3.0509×10^{-2}	(1.00)	7.0892×10^{-4}	(1.96)	4.7085×10^{-4}	(1.98)
	1/4	1.2027×10^{-1}		9.8821×10^{-3}		7.1406×10^{-3}	
1/8	1/8	6.0817×10^{-2}	(0.98)	2.6471×10^{-3}	(1.90)	1.8726×10^{-3}	(1.93)
	1/16	3.0509×10^{-2}	(1.00)	6.8225×10^{-4}	(1.96)	4.7460×10^{-4}	(1.98)
	1/4	1.2026×10^{-1}		9.8599×10^{-3}		7.1117×10^{-3}	
1/8	1/8	6.0816×10^{-2}	(0.98)	2.5786×10^{-3}	(1.90)	1.8694×10^{-3}	(1.93)
	1/16	3.0509×10^{-2}	(1.00)	6.6630×10^{-4}	(1.96)	4.7497×10^{-4}	(1.98)

Table 2.4: Errors at $k = 1$ for $u = t\sin(xy)$

Δt	$\left\|e_h^N\right\|_V$	$\left\|\tilde{e}_h^N\right\|_{L_2(\Omega)}$	$\left\|e_h^N\right\|_{L_2(\Omega)}$
1/4	1.5275×10^{-3}	1.7893×10^{-6}	1.1830×10^{-6}
1/8	1.5275×10^{-3}	1.7342×10^{-6}	1.1929×10^{-6}
1/16	1.5275×10^{-3}	1.6955×10^{-6}	1.1944×10^{-6}
1/32	1.5275×10^{-3}	1.6794×10^{-6}	1.1945×10^{-6}

Table 2.5: Errors at $k = 1$ and $h = 1/320$ for $u = t\sin(xy)$ without internal variables

Δt	$\left\|e_h^N\right\|_V$		$\left\|\tilde{e}_h^N\right\|_{L_2(\Omega)}$		$\left\|e_h^N\right\|_{L_2(\Omega)}$	
1/2	2.6616×10^{-3}		2.0240×10^{-3}		9.9014×10^{-4}	
1/4	1.0350×10^{-3}	(1.36)	5.2150×10^{-4}	(1.96)	2.8293×10^{-4}	(1.81)
1/8	7.8432×10^{-4}	(0.40)	1.3114×10^{-4}	(1.99)	7.3287×10^{-5}	(1.95)
1/16	7.6509×10^{-4}	(0.04)	3.3054×10^{-5}	(1.99)	1.8603×10^{-5}	(1.98)

Table 2.6: Errors at $k = 1$ and $h = 1/640$ for $u = t\sin(xy)$ with internal variables

Turning to the main experiment, let us consider $u(x, y, t) = e^{-t}\sin(xy)$. As following the above experiments and the theorems, $\left\|e_h^N\right\|_V$, $\left\|\tilde{e}_h^N\right\|_{L_2(\Omega)}$ and $\left\|e_h^N\right\|_{L_2(\Omega)}$ become

$O(h^k + \Delta t^2)$, $O(h^{k+1} + \Delta t^2)$ and $O(h^{k+1} + \Delta t^2)$ respectively, since $s = \infty$. As analysed the following Tables 2.7 and 2.10, the convergent orders are given as

$$\|e_h^N\|_V = O(h + \Delta t^2),\ \|\tilde{e}_h^N\|_{L_2(\Omega)} = O(h^2 + \Delta t^2),\ \text{and}\ \|e_h^N\|_{L_2(\Omega)} = O(h^2 + \Delta t^2),$$

for the both formulations. On the other hand, if we assume our test function space such that be a set of piecewise quadratic polynomials then the orders of the convergence rates would increase with respect to h but the orders for Δt are same as 2. As seen in Tables 2.8 and 2.11, the diagonals indicate the second order convergent rates. To see higher order in space, a significantly small Δt allows us to observe that

$$\|e_h^N\|_V = O(h^2),\ \|\tilde{e}_h^N\|_{L_2(\Omega)} = O(h^3),\ \|e_h^N\|_{L_2(\Omega)} = O(h^3),$$

on Tables 2.9 and 2.12. With combining all results, we have

$$\|e_h^N\|_V = O(h^2 + \Delta t^2),\ \|\tilde{e}_h^N\|_{L_2(\Omega)} = O(h^3 + \Delta t^2),\ \|e_h^N\|_{L_2(\Omega)} = O(h^3 + \Delta t^2),$$

for $k = 2$.

In conclusion, our two formulations **(P1)** and **(P2)** provide appropriate numerical results with respect to error estimates theorems. To be described in details, the energy estimates are given by $O(h^{\min(k+1,s)-1} + \Delta t^2)$ and L_2 estimates with elliptic regularity follow $O(h^{\min(k+1,s)} + \Delta t^2)$. All numerical experiments are important evidence and examples of Theorem 2.14, 2.17, Corollary 2.1 and 2.2.

$\|e_h^N\|_V$

h \ Δt	1/4	1/8	1/16	1/32
1/10	1.8081×10^{-2}	1.7944×10^{-2}	1.7936×10^{-2}	1.7935×10^{-2}
1/20	9.2785×10^{-3}	9.0035×10^{-3}	8.9854×10^{-3}	8.9844×10^{-3}
1/40	5.0622×10^{-3}	4.5348×10^{-3}	4.4971×10^{-3}	4.4948×10^{-3}
1/80	3.2388×10^{-3}	2.3279×10^{-3}	2.2528×10^{-3}	2.2480×10^{-3}
1/160	2.5893×10^{-3}	1.2772×10^{-3}	1.1342×10^{-3}	1.1245×10^{-3}

$\|\tilde{e}_h^N\|_{L_2(\Omega)}$

h \ Δt	1/4	1/8	1/16	1/32
1/10	4.3081×10^{-3}	1.7857×10^{-3}	1.1475×10^{-3}	9.9722×10^{-4}
1/20	3.6276×10^{-3}	1.0830×10^{-3}	4.4689×10^{-4}	2.8818×10^{-4}
1/40	3.4599×10^{-3}	9.0966×10^{-4}	2.7070×10^{-4}	1.1158×10^{-4}
1/80	3.4182×10^{-3}	8.6689×10^{-4}	2.2749×10^{-4}	6.7658×10^{-5}
1/160	3.4078×10^{-3}	8.5625×10^{-4}	2.1681×10^{-4}	5.6878×10^{-5}

$\left\|e_h^N\right\|_{L_2(\Omega)}$

h \ Δt	1/4	1/8	1/16	1/32
1/10	4.7709×10^{-4}	3.6395×10^{-4}	5.1613×10^{-4}	5.5879×10^{-4}
1/20	8.0626×10^{-4}	1.3246×10^{-4}	9.0652×10^{-5}	1.2971×10^{-4}
1/40	9.0416×10^{-4}	2.1667×10^{-4}	3.3979×10^{-5}	2.2650×10^{-5}
1/80	9.2915×10^{-4}	2.4084×10^{-4}	5.5125×10^{-5}	8.5486×10^{-6}
1/160	9.3543×10^{-4}	2.4695×10^{-4}	6.1149×10^{-5}	1.3841×10^{-5}

Table 2.7: Numerical errors of **(P1)**; $u(x,y,t) = e^{-t}\sin(xy)$ where $k = 1$

$\left\|e_h^N\right\|_V$

h \ Δt	1/4	1/8	1/16	1/32
1/10	2.3664×10^{-3}	7.2192×10^{-4}	4.1995×10^{-4}	3.9285×10^{-4}
1/20	2.3351×10^{-3}	6.1532×10^{-4}	1.8307×10^{-4}	1.0735×10^{-4}
1/40	2.3330×10^{-3}	6.0759×10^{-4}	1.5525×10^{-4}	4.6005×10^{-5}
1/80	2.3329×10^{-3}	6.0708×10^{-4}	1.5330×10^{-4}	3.8900×10^{-5}
1/160	2.3329×10^{-3}	6.0705×10^{-4}	1.5317×10^{-4}	3.8406×10^{-5}

$\left\|\tilde{e}_h^N\right\|_{L_2(\Omega)}$

h \ Δt	1/4	1/8	1/16	1/32
1/10	3.4051×10^{-3}	8.5355×10^{-4}	2.1414×10^{-4}	5.4382×10^{-5}
1/20	3.4044×10^{-3}	8.5276×10^{-4}	2.1332×10^{-4}	5.3381×10^{-5}
1/40	3.4043×10^{-3}	8.5271×10^{-4}	2.1327×10^{-4}	5.3329×10^{-5}
1/80	3.4043×10^{-3}	8.5271×10^{-4}	2.1327×10^{-4}	5.3326×10^{-5}
1/160	3.4043×10^{-3}	8.5271×10^{-4}	2.1327×10^{-4}	5.3325×10^{-5}

$\left\|e_h^N\right\|_{L_2(\Omega)}$

h \ Δt	1/4	1/8	1/16	1/32
1/10	9.3703×10^{-4}	2.4859×10^{-4}	6.2878×10^{-5}	1.6085×10^{-5}
1/20	9.3750×10^{-4}	2.4901×10^{-4}	6.3163×10^{-5}	1.5837×10^{-5}
1/40	9.3753×10^{-4}	2.4904×10^{-4}	6.3190×10^{-5}	1.5854×10^{-5}
1/80	9.3753×10^{-4}	2.4904×10^{-4}	6.3191×10^{-5}	1.5856×10^{-5}
1/160	9.3753×10^{-4}	2.4904×10^{-4}	6.3191×10^{-5}	1.5856×10^{-5}

Table 2.8: Numerical errors of **(P1)**; $u(x,y,t) = e^{-t}\sin(xy)$ where $k = 2$

h	$\left\|e_h^N\right\|_V$		$\left\|\tilde{e}_h^N\right\|_{L_2(\Omega)}$		$\left\|e_h^N\right\|_{L_2(\Omega)}$	
1/4	2.2557×10^{-3}		8.1101×10^{-5}		6.9417×10^{-5}	
1/8	6.0301×10^{-4}	(1.90)	1.0491×10^{-5}	(2.95)	9.2260×10^{-6}	(2.91)
1/16	1.5566×10^{-4}	(1.95)	1.2803×10^{-6}	(3.00)	1.1954×10^{-6}	(2.95)
1/32	3.9526×10^{-5}	(1.98)	1.6466×10^{-7}	(3.00)	1.5241×10^{-7}	(2.97)

Table 2.9: Numerical errors of **(P1)**; $u(x, y, t) = e^{-t} \sin(xy)$ where $k = 2$ and $\Delta t = 1/1200$

$\left\|e_h^N\right\|_V$

h \ Δt	1/4	1/8	1/16	1/32
1/10	1.8442×10^{-2}	1.7990×10^{-2}	1.7939×10^{-2}	1.7936×10^{-2}
1/20	1.0000×10^{-2}	9.0901×10^{-3}	8.9912×10^{-3}	8.9847×10^{-3}
1/40	6.3006×10^{-3}	4.7033×10^{-3}	4.5091×10^{-3}	4.4955×10^{-3}
1/80	4.9599×10^{-3}	2.6410×10^{-3}	2.2769×10^{-3}	2.2495×10^{-3}
1/160	4.5635×10^{-3}	1.7852×10^{-3}	1.1815×10^{-3}	1.1277×10^{-3}

$\left\|\tilde{e}_h^N\right\|_{L_2(\Omega)}$

h \ Δt	1/4	1/8	1/16	1/32
1/10	1.1624×10^{-2}	3.6501×10^{-3}	1.6242×10^{-3}	1.1075×10^{-3}
1/20	1.0928×10^{-2}	2.9619×10^{-3}	9.1773×10^{-4}	4.0630×10^{-4}
1/40	1.0756×10^{-2}	2.7927×10^{-3}	7.4330×10^{-4}	2.2934×10^{-4}
1/80	1.0714×10^{-2}	2.7507×10^{-3}	7.0032×10^{-4}	1.8590×10^{-4}
1/160	1.0703×10^{-2}	2.7402×10^{-3}	6.8964×10^{-4}	1.7518×10^{-4}

$\left\|e_h^N\right\|_{L_2(\Omega)}$

h \ Δt	1/4	1/8	1/16	1/32
1/10	1.0124×10^{-3}	2.4818×10^{-4}	4.4425×10^{-4}	5.3869×10^{-4}
1/20	1.4149×10^{-3}	4.1863×10^{-4}	6.2463×10^{-5}	1.1092×10^{-4}
1/40	1.5225×10^{-3}	5.1433×10^{-4}	1.1584×10^{-4}	1.5761×10^{-5}
1/80	1.5497×10^{-3}	5.3921×10^{-4}	1.3961×10^{-4}	2.9672×10^{-5}
1/160	1.5566×10^{-3}	5.4548×10^{-4}	1.4574×10^{-4}	3.5603×10^{-5}

Table 2.10: Numerical errors of **(P2)**; $u(x, y, t) = e^{-t} \sin(xy)$ where $k = 1$

$\|e_h^N\|_V$

h \ Δt	1/4	1/8	1/16	1/32
1/10	4.4414×10^{-3}	1.4420×10^{-3}	5.3452×10^{-4}	4.0173×10^{-4}
1/20	4.4246×10^{-3}	1.3908×10^{-3}	3.7811×10^{-4}	1.3620×10^{-4}
1/40	4.4235×10^{-3}	1.3874×10^{-3}	3.6542×10^{-4}	9.5601×10^{-5}
1/80	4.4234×10^{-3}	1.3871×10^{-3}	3.6459×10^{-4}	9.2395×10^{-5}
1/160	4.4234×10^{-3}	1.3871×10^{-3}	3.6454×10^{-4}	9.2188×10^{-5}

$\|\tilde{e}_h^N\|_{L_2(\Omega)}$

h \ Δt	1/4	1/8	1/16	1/32
1/10	1.0700×10^{-2}	2.7375×10^{-3}	6.8693×10^{-4}	1.7252×10^{-4}
1/20	1.0699×10^{-2}	2.7367×10^{-3}	6.8614×10^{-4}	1.7168×10^{-4}
1/40	1.0699×10^{-2}	2.7367×10^{-3}	6.8609×10^{-4}	1.7163×10^{-5}
1/80	1.0699×10^{-2}	2.7367×10^{-3}	6.8608×10^{-4}	1.7163×10^{-5}
1/160	1.0699×10^{-2}	2.7367×10^{-3}	6.8608×10^{-4}	1.7163×10^{-5}

$\|e_h^N\|_{L_2(\Omega)}$

h \ Δt	1/4	1/8	1/16	1/32
1/10	1.5582×10^{-3}	5.4708×10^{-4}	1.4738×10^{-4}	3.7453×10^{-5}
1/20	1.5588×10^{-3}	5.4754×10^{-4}	1.4777×10^{-4}	3.7617×10^{-5}
1/40	1.5588×10^{-3}	5.4757×10^{-4}	1.4780×10^{-4}	3.7641×10^{-5}
1/80	1.5588×10^{-3}	5.4757×10^{-4}	1.4780×10^{-4}	3.7643×10^{-5}
1/160	1.5588×10^{-3}	5.4757×10^{-4}	1.4780×10^{-4}	3.7643×10^{-5}

Table 2.11: Numerical errors of **(P2)**; $u(x,y,t) = e^{-t}\sin(xy)$ where $k = 2$

h	$\|e_h^N\|_V$		$\|\tilde{e}_h^N\|_{L_2(\Omega)}$		$\|e_h^N\|_{L_2(\Omega)}$	
1/4	2.2557×10^{-3}		8.1098×10^{-5}		6.9419×10^{-5}	
1/8	6.0301×10^{-4}	(1.90)	1.0489×10^{-5}	(2.95)	9.2266×10^{-6}	(2.91)
1/16	1.5566×10^{-4}	(1.95)	1.2794×10^{-6}	(3.04)	1.1957×10^{-6}	(2.95)
1/32	3.9526×10^{-5}	(1.98)	1.6269×10^{-7}	(2.98)	1.5226×10^{-7}	(2.97)

Table 2.12: Numerical errors of **(P2)**; $u(x,y,t) = e^{-t}\sin(xy)$ where $k = 2$ and $\Delta t = 1/1200$

Summary

In this chapter, we used CGFEM for scalar wave equations with two types of internal variables; **(P1)** and **(P2)**. The semidiscrete formulations and the fully discrete formulations have been introduced, and the existence and uniqueness of the solutions have been shown by stability bounds. In particular, using the concept of L_∞ norm in time and maximum with respect to time rather than using Grönwall's inequality, we have the constant bounds increasing in the final time but not exponentially. In the error estimates theorems, we can observe L_2 estimates and H^1(or energy) estimates with respect to the mesh size h and the time step Δt. It is also verified in a number of numerical experiments. The convergence rates of time are fixed as Δt^2 whence Crank-Nicolson finite difference method is applied to time discretisation. On the other hand, elliptic projection leads us to have the optimal error convergence rates with respect to the spatial mesh size. Hereafter, we will use DGFEM for spatial discretisation. Most techniques are almost same as CGFEM to prove stability bounds and error bounds but details are slightly different.

Chapter 3

DGFEM to Scalar Wave Equation with Memory

3.1 Model Problems with DGFEM

In the previous chapter, we have focused on CGFEM to solve the wave equation with internal variables in two ways. From now on, we are going to deal with the same model problems but use DGFEM. In the same sense with CGFEM, we will define variational problems with respect to the displacement form and the velocity form and derive semi-discrete formulations and fully discrete formulation, respectively. At the same time, stability bounds and error estimates would be also introduced and proven.

For a variational form, let us suppose $s > 3/2$ for $s \in \mathbb{N}$ and $\mathcal{E}_h$ is a quasi-uniform subdivision of Ω. Then we can define DG bilinear forms $a_\epsilon : H^s(\mathcal{E}_h) \times H^s(\mathcal{E}_h) \mapsto \mathbb{R}$ by for any $v, w \in H^s(\mathcal{E}_h)$

$$\begin{aligned} a_\epsilon(v,w) = &\sum_{E\in\mathcal{E}_h} \int_E D\nabla v \cdot \nabla w \, dE - \sum_{e\subset\Gamma_h\cup\Gamma_D} \int_e \{D\nabla v \cdot \underline{n}_e\}[w] de \\ &+ \underbrace{\epsilon \sum_{e\subset\Gamma_h\cup\Gamma_D} \int_e \{D\nabla w \cdot \underline{n}_e\}[v] de}_{\text{interior penalty}} + \underbrace{J_0^{\alpha_0,\beta_0}(v,w)}_{\text{jump penalty}}, \end{aligned}$$

where $\epsilon \in \{-1, 0, 1\}$ and $J_0^{\alpha_0,\beta_0}(v,w) = \sum_{e\subset\Gamma_h\cup\Gamma_D} \frac{\alpha_0}{|e|^{\beta_0}} \int_e [v][w] de$. If $\epsilon = -1$, the bilinear form is symmetric called SIPG, otherwise, it is nonsymmetric which is called NIPG for $\epsilon = 1$ or IIPG for $\epsilon = 0$. Then we can denote $a_1(\cdot,\cdot)$, $a_0(\cdot,\cdot)$, and $a_{-1}(\cdot,\cdot)$, respectively.

Remark In 1970s, Wheeler [20] introduced SIPG with large enough α_0 for stability. We refer to estimation of penalty parameters in [58]. NIPG was used for elliptic problems in [21]. More applications of NIPG for hyperbolic problems were seen in [23]. On the other hand, we can observe IIPG for transport equation by Dawson, Sun and Wheeler [59]. For unified analysis of DG, see [60].

3.1.1 Displacement Form

Recall the model problem (2.1.1)-(2.1.5) and (2.1.7).

$$\begin{aligned} \rho\ddot{u} - \nabla \cdot \underline{\sigma} &= f && \text{in } (0,T] \times \Omega, \\ u &= 0 && \text{on } [0,T] \times \Gamma_D, \\ \underline{\sigma} \cdot \underline{n} &= g_N && \text{on } [0,T] \times \Gamma_N, \\ u &= u_0 && \text{on } \{0\} \times \Omega, \\ \dot{u} &= w_0 && \text{on } \{0\} \times \Omega, \end{aligned}$$

and

$$\underline{\sigma} = D\nabla \left(u - \sum_{q=1}^{N_\varphi} \psi_q \right).$$

We assume the strong solution satisfies

$$u \in H^2(0,T;L_2(\Omega)) \cap H^1(0,T;C^2(\Omega)).$$

We will use DG bilinear forms to derive the variational formula corresponding to (2.1.1). Also, the linear form will be given as

$$F_d(t;v) = \int_\Omega f(t)v \; d\Omega + \sum_{e \subset \Gamma_N} \int_e v g_N(t) \; de$$

for any $v \in H^s(\mathcal{E}_h)$. Then we could obtain the following variational problem. When we suppose

$$\tau_q \dot{\psi}_q(t) + \psi_q(t) = u(t), \qquad \psi_q(0) = 0, \forall q \in \{1, \dots, N_\varphi\}, \ \forall t,$$

the weak problem is given by:
(Q1) Find u and $\{\psi_q\}_{q=1}^{N_\varphi}$ such that for all $v \in H^s(\mathcal{E}_h)$

$$(\rho\ddot{u}(t), v)_{L_2(\Omega)} + a_1(u(t), v) - \sum_{q=1}^{N_\varphi} a_{-1}(\psi_q(t), v) + J_0^{\alpha_0,\beta_0}(\dot{u}(t), v) = F_d(t;v), \tag{3.1.1}$$

$$a_{-1}\left(\tau_q \dot{\psi}_q(t) + \psi_q(t), v\right) = \varphi_q a_{-1}(u(t), v), \tag{3.1.2}$$

$$u(0) = u_0, \tag{3.1.3}$$

$$\dot{u}(0) = w_0, \tag{3.1.4}$$

where $\psi_q(0) = 0, \forall q \in \{1, \dots, N_\varphi\}$. In order to verify our claim, let us prove that **(Q1)** is the variational problem of (2.1.1)-(2.1.5).

Theorem 3.1. *Let $s > 3/2$ for $s \in \mathbb{N}$. Suppose $u(t)$ and $\{\psi_q(t)\}_{q=1}^{N_\varphi}$ are the solution satisfying* (2.1.9), (2.1.2), (2.1.10), (2.1.4) *and* (2.1.5) *which belong to $H^s(\mathcal{E}_h)$ for any t. Then the solution satisfies* (3.1.1)-(3.1.4).

Proof. Let $z(t) = u(t) - \sum\limits_{q=1}^{N_\varphi} \psi_q(t)$. For an element $E \in \mathcal{E}_h$, it satisfies

$$-(\nabla \cdot D\nabla z(t), v)_{L_2(E)} = (D\nabla z(t), \nabla v)_{L_2(E)} - \sum_{e \subset \partial E} \int_e v D\nabla z(t) \cdot \underline{n}_e \; de.$$

If $e \subset \Gamma_N$,

$$\int_e v D\nabla z(t) \cdot \underline{n}_e \; de = \int_e v g_N(t) \; de.$$

For faced elements E_1 and E_2 with common side $e_{1,2}$ and the normal vector $\underline{n}_{e_{1,2}}$ from E_1 to E_2,

$$\begin{aligned}
-\int_{E_1 \cup E_2} \nabla \cdot D\nabla z(t) v \; dE &= \int_{E_1} D\nabla z(t) \cdot \nabla v \; dE + \int_{E_2} D\nabla z(t) \cdot \nabla v \; dE \\
&\quad - \sum_{e \subset \partial(E_1 \cup E_2)} \int_e v D\nabla z(t) \cdot \underline{n}_e \; de \\
&\quad - \int_{e_{1,2}} v|_{E_1} D\nabla z(t) \cdot \underline{n}_{e_{1,2}} \; de \\
&\quad - \int_{e_{1,2}} v|_{E_2} D\nabla z(t) \cdot (-\underline{n}_{e_{1,2}}) \; de \\
&= \int_{E_1} D\nabla z(t) \cdot \nabla v \; dE + \int_{E_2} D\nabla z(t) \cdot \nabla v \; dE \\
&\quad - \sum_{e \subset \partial(E_1 \cup E_2)} \int_e v D\nabla z(t) \cdot \underline{n}_e \; de - \int_{e_{1,2}} [v] D\nabla z(t) \cdot \underline{n}_{e_{1,2}} \; de
\end{aligned}$$

by the definition of jump. Moreover, note that for continuous u on $E_1 \cup E_2$ with common side e,

$$[u](\boldsymbol{x}) = (u|_{E_1})(\boldsymbol{x}) - (u|_{E_2})(\boldsymbol{x}) = 0 \qquad \text{for } \boldsymbol{x} \in e$$

and

$$\{u\}(\boldsymbol{x}) = \frac{(u|_{E_1})(\boldsymbol{x}) + (u|_{E_2})(\boldsymbol{x})}{2} = u|_e(\boldsymbol{x}) \qquad \text{for } \boldsymbol{x} \in e.$$

Since $u(t) \in C^2(\Omega)$, so is $\psi_q(t)$ $\forall q \in \{1, \ldots, N_\varphi\}$, $\forall t$. This yields

$$-\int_{E_1 \cup E_2} \nabla \cdot D\nabla z(t) v \; dE = \int_{E_1} D\nabla z(t) \cdot \nabla v \; dE + \int_{E_2} D\nabla z(t) \cdot \nabla v \; dE$$

$$
\begin{aligned}
&- \sum_{e \subset \partial(E_1 \cup E_2)} \int_e v D \nabla z(t) \cdot \underline{n}_e \, de \\
&- \int_{e_{1,2}} [v]\{D\nabla z(t) \cdot \underline{n}_{e_{1,2}}\} \, de \\
&+ \int_{e_{1,2}} [u(t)]\{D\nabla v \cdot \underline{n}_{e_{1,2}}\} \, de \\
&+ \sum_{q=1}^{N_\varphi} \int_{e_{1,2}} [\psi_q(t)]\{D\nabla v \cdot \underline{n}_{e_{1,2}}\} \, de,
\end{aligned}
$$

since

$$
\{D\nabla z(t) \cdot \underline{n}_{e_{1,2}}\} = D\nabla z(t) \cdot \underline{n}_{e_{1,2}} \text{ and } [u(t)] = 0 = [\psi_q(t)], \ \forall q \in \{1, \ldots, N_\varphi\},
$$

for any $\boldsymbol{x} \in e_{1,2}$. As reminding the definition of Γ_h, it is indicated that

$$
\partial \left(\bigcup_{E \in \mathcal{E}_h} \overline{E} \right) = \partial\Omega = \Gamma_D \cup \Gamma_N,
$$

and the set of common side elements is Γ_h. From the above results, (2.1.9) can be rewritten as, by multiplying by v and integrating,

$$
\begin{aligned}
&(\rho\ddot{u}(t), v)_{L_2(\Omega)} + \sum_{E \in \mathcal{E}_h} \int_E D\nabla z(t) \cdot \nabla v \, dE - \sum_{e \subset \Gamma_D} \int_e v D\nabla z(t) \cdot \underline{n}_e \, de \\
&- \sum_{e \subset \Gamma_N} \int_e v D\nabla z(t) \cdot \underline{n}_e \, de - \sum_{e \subset \Gamma_h} \int_e \{D\nabla z(t) \cdot \underline{n}_e\}[v] \, de \\
= &(\rho\ddot{u}(t), v)_{L_2(\Omega)} + \sum_{E \in \mathcal{E}_h} \int_E D\nabla z(t) \cdot \nabla v \, dE - \sum_{e \subset \Gamma_D} \int_e v D\nabla z(t) \cdot \underline{n}_e \, de \\
&- \sum_{e \subset \Gamma_N} \int_e v D\nabla z(t) \cdot \underline{n}_e \, de - \sum_{e \subset \Gamma_h} \int_e \{D\nabla z(t) \cdot \underline{n}_e\}[v] \, de + \sum_{e \subset \Gamma_h} \int_e [u(t)]\{D\nabla v \cdot \underline{n}_e\} \, de \\
&+ \sum_{q=1}^{N_\varphi} \sum_{e \subset \Gamma_h} \int_e [\psi_q(t)]\{D\nabla v \cdot \underline{n}_e\} \, de \\
&(\because \ [u(t)] = 0, \qquad [\psi_q(t)] = 0, \ \forall q) \\
= &(\rho\ddot{u}(t), v)_{L_2(\Omega)} + \sum_{E \in \mathcal{E}_h} \int_E D\nabla z(t) \cdot \nabla v \, dE - \sum_{e \subset \Gamma_D} \int_e \{D\nabla z(t) \cdot \underline{n}_e\}[v] \, de \\
&- \sum_{e \subset \Gamma_N} \int_e v g_N(t) \, de - \sum_{e \subset \Gamma_h} \int_e \{D\nabla z(t) \cdot \underline{n}_e\}[v] \, de + \sum_{e \subset \Gamma_h} \int_e [u(t)]\{D\nabla v \cdot \underline{n}_e\} \, de \\
&+ \sum_{q=1}^{N_\varphi} \sum_{e \subset \Gamma_h} \int_e [\psi_q(t)]\{D\nabla v \cdot \underline{n}_e\} \, de
\end{aligned}
$$

$(\because$ definition of jump and average on Γ_D and $D\nabla z(t)\cdot \underline{n}_e = g_N(t)$ on $e\subset\Gamma_N)$

$$
\begin{aligned}
=&(\rho\ddot{u}(t),v)_{L_2(\Omega)} + \sum_{E\in\mathcal{E}_h}\int_E D\nabla z(t)\cdot\nabla v\, dE \\
&-\sum_{e\subset\Gamma_N}\int_e v g_N(t)\, de - \sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e \{D\nabla z(t)\cdot\underline{n}_e\}[v]\, de \\
&+\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e [u(t)]\{D\nabla v\cdot\underline{n}_e\}\, de + \sum_{q=1}^{N_\varphi}\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e [\psi_q(t)]\{D\nabla v\cdot\underline{n}_e\}\, de
\end{aligned}
$$

$(\because$ Dirichlet boundary condition.)

$$
=(f(t),v)_{L_2(\Omega)}.
$$

So, we obtain

$$
\begin{aligned}
&(\rho\ddot{u}(t),v)_{L_2(\Omega)} + \sum_{E\in\mathcal{E}_h}\int_E D\nabla z(t)\cdot\nabla v\, dE - \sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e \{D\nabla z(t)\cdot\underline{n}_e\}[v]\, de \\
&+\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e [u(t)]\{D\nabla v\cdot\underline{n}_e\}\, de + \sum_{q=1}^{N_\varphi}\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e [\psi_q(t)]\{D\nabla v\cdot\underline{n}_e\}\, de \\
=&(f(t),v)_{L_2(\Omega)} + (g_N(t),v)_{\Gamma_N}.
\end{aligned}
$$

Furthermore, by the definition of the jump penalty operators,

$$
\begin{aligned}
J_0^{\alpha_0,\beta_0}(\dot{u}(t),v) &= \sum_{e\subset\Gamma_h\cup\Gamma_D}\frac{\alpha_0}{|e|^{\beta_0}}\int_e [\dot{u}(t)][v]\, de = 0, \\
J_0^{\alpha_0,\beta_0}(z(t),v) &= \sum_{e\subset\Gamma_h\cup\Gamma_D}\frac{\alpha_0}{|e|^{\beta_0}}\int_e [z(t)][v]\, de = 0,
\end{aligned}
$$

since $\dot{u}(t), z(t)$ and $\nabla z(t)$ are continuous on Ω and so $[\dot{u}(t)], [z(t)] = 0$ on $\forall e\subset\Gamma_h\cup\Gamma_D$, $[D\nabla z(t)\cdot\underline{n}_e] = 0$ on $\forall e\subset\Gamma_h$. Therefore we have

$$
\begin{aligned}
&(\rho\ddot{u}(t),v)_{L_2(\Omega)} + a_1(u(t),v) - \sum_{q=1}^{N_\varphi} a_{-1}(\psi_q(t),v) + J_0^{\alpha_0,\beta_0}(\dot{u}(t),v) \\
=&(\rho\ddot{u}(t),v)_{L_2(\Omega)} + \sum_{E\in\mathcal{E}_h}\int_E D\nabla z(t)\cdot\nabla v\, dE - \sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e \{D\nabla z(t)\cdot\underline{n}_e\}[v]\, de \\
&+\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e [u(t)]\{D\nabla v\cdot\underline{n}_e\}\, de + \sum_{q=1}^{N_\varphi}\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e [\psi_q(t)]\{D\nabla v\cdot\underline{n}_e\}\, de \\
&+J_0^{\alpha_0,\beta_0}(\dot{u}(t),v) + J_0^{\alpha_0,\beta_0}(z(t),v) \\
=&(f(t),v)_{L_2(\Omega)} + (g_N(t),v)_{\Gamma_N} = F_d(t;v).
\end{aligned}
$$

Hence (3.1.1) is satisfied.

On the other hand, since $a_{-1}(\cdot,\cdot)$ is a well-defined bilinear form if $u_1 = u_2$

$$a_{-1}(u_1, v) = a_{-1}(u_2, v), \ \forall v \in H^s(\mathcal{E}_h).$$

By the definition of ψ_q, (2.1.8) always holds for each q so that it is also true that

$$a_{-1}\left(\tau_q \dot{\psi}_q(t) + \psi_q(t), v\right) = a_{-1}(\varphi_q u(t), v).$$

□

(3.1.1) is the main equation to solve the variational problem **(Q1)** but (3.1.2) is an auxiliary equation which is governed by the definition of displacement internal variables. Note that DG bilinear form deals with only discontinuity of u and internal variables, i.e. we cannot control discontinuity of $\dot{u}$ over interior edges. However, use of jump penalty of $\dot{u}$ resolves this issue and will manage non-symmetric part of the bilinear form for stability and error bounds later. Also, we consider NIPG for u and SIPG for internal variables. This imposes challenging difficulty on our weak problems such as non-symmetry. Indeed, using SIPG and the strong form of auxiliary ODEs allows us to show stability and error analysis more easily but we restrict ourselves to prove it in a more difficult manner.

3.1.2 Velocity Form

We can also introduce an alternative formulation of **(Q1)** by using ζ_q as

$$\psi_q(t) = \varphi_q u(t) - \varphi_q e^{-t/\tau_q} u_0 - \zeta_q(t), \ \forall q \in \{1, \ldots, N_\varphi\}.$$

Note that the definition of ζ_q gives $\tau_q \dot{\zeta}_q(t) + \zeta_q(t) = \varphi_q \tau_q \dot{u}(t)$ by integration by parts. Replacing ψ_q by ζ_q, the velocity form is given as
(Q2) Find u and $\{\zeta_q\}_{q=1}^{N_\varphi}$ such that for all $v \in H^s(\mathcal{E}_h)$

$$(\rho \ddot{u}(t), v)_{L_2(\Omega)} + \varphi_0 a_1(u(t), v) + \sum_{q=1}^{N_\varphi} a_{-1}(\zeta_q(t), v) + J_0^{\alpha_0, \beta_0}(\dot{u}(t), v) = F_v(t; v), \quad (3.1.5)$$

$$a_{-1}\left(\tau_q \dot{\zeta}_q(t) + \zeta_q(t), v\right) = a_{-1}(\tau_q \varphi_q \dot{u}(t), v), \quad (3.1.6)$$

$$u(0) = u_0, \quad (3.1.7)$$

$$\dot{u}(0) = w_0, \quad (3.1.8)$$

where $\zeta_q(0) = 0, \forall q \in \{1, \ldots, N_\varphi\}$ and

$$F_v(t; v) = (f(t), v)_{L_2(\Omega)} + (g_N(t), v)_{\Gamma_N} - \sum_{q=1}^{N_\varphi} \varphi_q e^{-t/\tau_q} a_1(u_0, v).$$

It is easy to check **(Q2)** is the weak problem for our primal problem. By replacing ψ_q by ζ_q for all q in the strong form, we have

$$\rho \ddot{u}(t) - \nabla \cdot D \nabla \left(\varphi_0 u(t) + \sum_{q=1}^{N_\varphi} (\varphi_q e^{-t/\tau_q} u_0 + \zeta_q(t))\right) = f(t)$$

so that

$$\rho\ddot{u}(t) - \nabla\cdot D\nabla\left(\varphi_0 u(t) + \sum_{q=1}^{N_\varphi}\zeta_q(t)\right) = f(t) + \sum_{q=1}^{N_\varphi}\nabla\cdot D\nabla\varphi_q e^{-t/\tau_q}u_0. \tag{3.1.9}$$

Theorem 3.2. *Let $s > 3/2$. Suppose $u(t)$ and $\{\zeta_q(t)\}_{q=1}^{N_\varphi}$ are the solutions which fulfil* (2.1.9), (2.1.2), (2.1.10), (2.1.4) *and* (2.1.5) *belonging to $H^s(\mathcal{E}_h)$ for any t with*

$$\psi_q(t) = \varphi_q u(t) - \varphi_q e^{-t/\tau_q}u_0 - \zeta_q(t),\ \forall q.$$

Then the solution satisfies (3.1.5)-(3.1.8).

Proof. As shown in Theorem 3.1, we can use integration by parts on (3.1.9) with respect to the spatial domain and adding zeros, which (3.1.9) implies (3.1.5). On the other hand, since

$$\tau_q\dot{\zeta}_q(t) + \zeta_q(t) = \varphi_q\tau_q\dot{u}(t),\ \forall q \in \{1,\ldots,N_\varphi\}$$

for any $v \in H^s(\mathcal{E}_h)$, $a_{-1}\left(\tau_q\dot{\zeta}_q(t) + \zeta_q(t), v\right) = a_{-1}\left(\dot{u}(t), v\right)\ \forall q$. Hence our claim holds. □

From the both formulations **(Q1)** and **(Q2)**, we define the DG bilinear form and the linear form so that we can consider the variational problems instead of the given wave equation. However, the existence and uniqueness of solution is not seen here. In order to show it, it is required to have coercivity on the bilinear form and continuity on the bilinear form and the linear form. In the next section, we will take our test space as the finite dimensional space $\mathcal{D}_k(\mathcal{E}_h)$. Then we can obtain the semidiscrete formulations with respect to the displacement form and the velocity form. Furthermore, we will observe the stability bounds and error bounds.

3.2 Semidiscrete Formulation for DGFEM

Consider a finite dimensional space of polynomials of degree of k $\mathcal{D}_k(\mathcal{E}_h)$ as our test space $\mathcal{D}_k(\mathcal{E}_h) \subset H^s(\mathcal{E}_h)$. Since $\mathcal{D}_k(\mathcal{E}_h)$ is the finite dimensional space, we can denote the set of global basis functions by $\{\phi_i^E \mid 1 \le i \le N_{\text{loc}}, E \in \mathcal{E}_h\}$ where N_{loc} is the number of local basis functions on each element. To compute and consider it easily, the set of global basis functions can be rewritten as $\{\phi_j \mid 1 \le j \le N_{\text{loc}}N_{\text{el}}\}$ where N_{el} is the number of elements. Then our semidiscrete solution can be expressed as

$$\begin{aligned}\forall t \in [0,T],\ \forall \boldsymbol{x} \in \Omega,\ u_h(\boldsymbol{x},t) &= \sum_{E\in\mathcal{E}_h}\sum_{i=1}^{N_{\text{loc}}}\mathfrak{u}_i^E(t)\phi_i^E(\boldsymbol{x}),\\ &= \sum_{j=1}^{N_{\text{loc}}N_{\text{el}}}\mathfrak{u}_j(t)\phi_j(\boldsymbol{x}).\end{aligned}$$

Without loss of generality, let us define

$$N_{V^h} = N_{\text{loc}} N_{\text{el}},$$

and so for any $v \in \mathcal{D}_k(\mathcal{E}_h)$ we can denote

$$v(\boldsymbol{x}) = \sum_{i=1}^{N_{V^h}} v_i \phi_i(\boldsymbol{x}) \text{ for } v_i \in \mathbb{R}, \ \forall i.$$

We will observe our DG bilinear forms to be equipped with coercivity and continuity.

Let $\|\cdot\|_{\mathcal{V}}$ be a norm defined by for any $v \in H^s(\mathcal{E}_h)$

$$\|v\|_{\mathcal{V}} = \left(\sum_{E \in \mathcal{E}_h} \int_E D\nabla v \cdot \nabla v \; dE + J_0^{\alpha_0,\beta_0}(v,v) \right)^{1/2}.$$

It is easy to show $\|\cdot\|_{\mathcal{V}}$ is a norm since

$$\|v\|_{\mathcal{V}}^2 = \sum_{E \in \mathcal{E}_h} \left\| D^{1/2} \nabla v \right\|_{L_2(E)}^2 + \sum_{e \subset \Gamma_h \cup \Gamma_D} \frac{\alpha_0}{|e|^{\beta_0}} \|[v]\|_{L_2(e)}^2 .$$

Moreover, let us denote L_∞ norm in time with respect to the norm $\|\cdot\|_{\mathcal{V}}$ by

$$\|v\|_{L_\infty(0,T;\mathcal{V})} = \text{ess} \sup_{0 \le t \le T} \|v(t)\|_{\mathcal{V}} .$$

Theorem 3.3. *The bilinear form $a_\epsilon(\cdot,\cdot)$ is coercive on $\mathcal{D}_k(\mathcal{E}_h)$. That is, there exists a positive constant $\kappa > 0$ such that*

$$a_\epsilon(v,v) \ge \kappa \|v\|_{\mathcal{V}}^2 , \ \forall v \in \mathcal{D}_k(\mathcal{E}_h)$$

under certain conditions such as $\beta_0(d-1) \ge 1$ and large enough α_0 if $\epsilon = 0, -1$. Since h is bounded above by h_Ω, without loss of generality, we assume $h \le 1$.

Proof. By the definition, for any $v \in \mathcal{D}_k(\mathcal{E}_h)$,

$$a_\epsilon(v,v) = \sum_{E \in \mathcal{E}_h} \left\| D^{1/2} \nabla v \right\|_{L_2(E)}^2 + \sum_{e \subset \Gamma_h \cup \Gamma_D} \frac{\alpha_0}{|e|^{\beta_0}} \|[v]\|_{L_2(e)}^2$$
$$+ (\epsilon - 1) \sum_{e \subset \Gamma_h \cup \Gamma_D} \int_e \{D\nabla v \cdot \underline{n}_e\}[v] de.$$

For $\epsilon = 1$, it gives clearly

$$a_\epsilon(v,v) = \|v\|_{\mathcal{V}}^2 .$$

For other cases, we consider

$$\sum_{e \subset \Gamma_h \cup \Gamma_D} \int_e \{D\nabla v \cdot \underline{n}_e\}[v] de \le \sum_{e \subset \Gamma_h \cup \Gamma_D} \|\{D\nabla v \cdot \underline{n}_e\}\|_{L_2(e)} \|[v]\|_{L_2(e)}$$

by Cauchy-Schwarz inequality. For faced elements E_1 and E_2 with shared edge e, we have

$$\begin{aligned}
\|\{D\nabla v\cdot \underline{n}_e\}\|_{L_2(e)} =& \left\|\frac{1}{2}(D\nabla v\cdot \underline{n}_e)|_{E_1} + \frac{1}{2}(D\nabla v\cdot \underline{n}_e)|_{E_2}\right\|_{L_2(e)} \\
\leq& \frac{1}{2}\left\|(D\nabla v\cdot \underline{n}_e)|_{E_1}\right\|_{L_2(e)} + \frac{1}{2}\left\|(D\nabla v\cdot \underline{n}_e)|_{E_2}\right\|_{L_2(e)}
\end{aligned}$$

by triangular inequality. By inverse polynomial trace theorem, this yields

$$\begin{aligned}
\|\{D\nabla v\cdot \underline{n}_e\}\|_{L_2(e)} \leq& \frac{D}{2}\left\|(\nabla v\cdot \underline{n}_e)|_{E_1}\right\|_{L_2(e)} + \frac{D}{2}\left\|(\nabla v\cdot \underline{n}_e)|_{E_2}\right\|_{L_2(e)} \\
\leq& C\frac{D}{2}h_{E_1}^{-1/2}\|\nabla v\|_{L_2(E_1)} + C\frac{D}{2}h_{E_2}^{-1/2}\|\nabla v\|_{L_2(E_2)} \\
\leq& CD\left(h_{E_1}^{-1/2}\|\nabla v\|_{L_2(E_1)} + h_{E_2}^{-1/2}\|\nabla v\|_{L_2(E_2)}\right),
\end{aligned}$$

for some positive constant C. Since

$$\forall e\subset \partial E,\ \forall E\in \mathcal{E}_h,\ |e|\leq h_E^{d-1}\leq h^{d-1},$$

$$\begin{aligned}
\int_e \{D\nabla v\cdot \underline{n}_e\}[v]de \leq& \left\|[v]\right\|_{L_2(e)}\frac{1}{|e|^{\beta_0/2}}|e|^{\beta_0/2}\left\|\{D\nabla v\cdot \underline{n}_e\}\right\|_{L_2(e)} \\
\leq& CD\left\|[v]\right\|_{L_2(e)}\frac{1}{|e|^{\beta_0/2}}|e|^{\beta_0/2}\Big(h_{E_1}^{-1/2}\|\nabla v\|_{L_2(E_1)} \\
&+ h_{E_2}^{-1/2}\|\nabla v\|_{L_2(E_2)}\Big) \\
\leq& CD\frac{1}{|e|^{\beta_0/2}}\left\|[v]\right\|_{L_2(e)}\Big(h_{E_1}^{\beta_0(d-1)/2-1/2}\|\nabla v\|_{L_2(E_1)} \\
&+ h_{E_2}^{\beta_0(d-1)/2-1/2}\|\nabla v\|_{L_2(E_2)}\Big).
\end{aligned}$$

Here, we want to introduce discrete Cauchy-Schwarz inequality, also known as Cauchy-Buniakowsky-Schwarz inequality, such that for positive real numbers it holds

$$(a_1^2+a_2^2+\cdots+a_n^2)(b_1^2+b_2^2+\cdots+b_n^2)\geq (a_1b_1+a_2b_2+\cdots+a_nb_n)^2.$$

When we apply this to our result, we have

$$\begin{aligned}
\int_e \{D\nabla v\cdot \underline{n}_e\}[v]de \leq& CD\frac{1}{|e|^{\beta_0/2}}\left\|[v]\right\|_{L_2(e)}\left(h_{E_1}^{\beta_0(d-1)-1} + h_{E_2}^{\beta_0(d-1)-1}\right)^{1/2}\Big(\|\nabla v\|_{L_2(E_1)}^2 \\
&+ \|\nabla v\|_{L_2(E_2)}^2\Big)^{1/2}.
\end{aligned}$$

If β_0 satisfies $\beta_0(d-1) \geq 1$ and we assume $h \leq 1$, we have

$$\int_e \{D\nabla v \cdot \underline{n}_e\}[v]de \leq CD\frac{1}{|e|^{\beta_0/2}} \|[v]\|_{L_2(e)} \left(\|\nabla v\|^2_{L_2(E_1)} + \|\nabla v\|^2_{L_2(E_2)} \right)^{1/2}.$$

With using Young's inequality, for $\epsilon_a > 0$, we can obtain

$$\begin{aligned}\int_e \{D\nabla v \cdot \underline{n}_e\}[v]de \leq & CD\frac{1}{2\epsilon_a}\frac{1}{|e|^{\beta_0}} \|[v]\|^2_{L_2(e)} \\ & + C\frac{\epsilon_a}{2}\left(\|D^{1/2}\nabla v\|^2_{L_2(E_1)} + \|D^{1/2}\nabla v\|^2_{L_2(E_2)} \right).\end{aligned}$$

If $e \subset \Gamma_D \cap \partial E$,

$$\begin{aligned}\|\{D\nabla v \cdot \underline{n}_e\}\|_{L_2(e)} =& \|D\nabla v \cdot \underline{n}_e\|_{L_2(e)} \\ \leq & CDh_E^{-1/2} \|\nabla v\|_{L_2(E)}\end{aligned}$$

and so

$$\begin{aligned}\int_e \{D\nabla v \cdot \underline{n}_e\}[v]de \leq & CD \|[v]\|_{L_2(e)} \frac{1}{|e|^{\beta_0/2}} |e|^{\beta_0/2} h_E^{-1/2} \|\nabla v\|_{L_2(E)} \\ \leq & CD \|[v]\|_{L_2(e)} \frac{1}{|e|^{\beta_0/2}} \|\nabla v\|_{L_2(E)} \\ \leq & CD\frac{1}{2\epsilon_a}\frac{1}{|e|^{\beta_0}} \|[v]\|^2_{L_2(e)} + C\frac{\epsilon_a}{2} \left\| D^{1/2}\nabla v \right\|^2_{L_2(E)}.\end{aligned}$$

Thus, the summation over Γ_h and Γ_D allows us to have

$$\begin{aligned}\sum_{e\subset\Gamma_h\cup\Gamma_D} \int_e \{D\nabla v \cdot \underline{n}_e\}[v]de \leq & CD\frac{1}{2\epsilon_a} \sum_{e\subset\Gamma_h\cup\Gamma_D} \frac{1}{|e|^{\beta_0}} \|[v]\|^2_{L_2(e)} \\ & + C\frac{\epsilon_a}{2} \sum_{E\in\mathcal{E}_h} \left\| D^{1/2}\nabla v \right\|^2_{L_2(E)}.\end{aligned}$$

Hence, it gives

$$\begin{aligned}a_\epsilon(v,v) \geq & \sum_{E\in\mathcal{E}_h} \left\| D^{1/2}\nabla v \right\|^2_{L_2(E)} + \sum_{e\subset\Gamma_h\cup\Gamma_D} \frac{\alpha_0}{|e|^{\beta_0}} \|[v]\|^2_{L_2(e)} \\ & - |1-\epsilon| \left(C\frac{\epsilon_a}{2} \sum_{E\in\mathcal{E}_h} \left\| D^{1/2}\nabla v \right\|^2_{L_2(E)} + CD\frac{1}{2\epsilon_a} \sum_{e\subset\Gamma_h\cup\Gamma_D} \frac{1}{|e|^{\beta_0}} \|[v]\|^2_{L_2(e)} \right) \\ \geq & \left(1 - |1-\epsilon| C\frac{\epsilon_a}{2}\right) \sum_{E\in\mathcal{E}_h} \left\| D^{1/2}\nabla v \right\|^2_{L_2(E)} \\ & + \left(1 - |1-\epsilon| CD\frac{1}{2\epsilon_a\alpha_0}\right) \sum_{e\subset\Gamma_h\cup\Gamma_D} \frac{\alpha_0}{|e|^{\beta_0}} \|[v]\|^2_{L_2(e)}.\end{aligned}$$

If we take ϵ_a as

$$0 < \frac{CD|1-\epsilon|}{2\alpha_0} < \epsilon_a < \frac{2}{C|1-\epsilon|}$$

and we have sufficiently large α_0 as

$$\frac{C^2 D|1-\epsilon|^2}{4} < \alpha_0,$$

there exists a positive constant κ such that

$$\begin{aligned} a_\epsilon(v,v) \geq& \kappa \left(\sum_{E \in \mathcal{E}_h} \left\| D^{1/2} \nabla v \right\|^2_{L_2(E)} + \sum_{e \subset \Gamma_h \cup \Gamma_D} \frac{\alpha_0}{|e|^{\beta_0}} \|[v]\|^2_{L_2(e)} \right) \\ =& \kappa \|v\|^2_{\mathcal{V}} \end{aligned}$$

where

$$\kappa = \min\left(1 - |1-\epsilon| C\frac{\epsilon_a}{2}, 1 - |1-\epsilon| CD\frac{1}{2\epsilon_a \alpha_0}\right) > 0.$$

□

Remark As shown in the proof for the coercivity, we can observe the boundedness of

$$\sum_{e \subset \Gamma_h \cup \Gamma_D} \int_e \{D\nabla v \cdot \underline{n}_e\}[v] de \; \forall v \in \mathcal{D}_k(\mathcal{E}_h)$$

if $\beta_0(d-1) \geq 1$. In a similar way, we have for any $v, w \in \mathcal{D}_k(\mathcal{E}_h)$

$$\left| \sum_{e \subset \Gamma_h \cup \Gamma_D} \int_e \{D\nabla v \cdot \underline{n}_e\}[w] de \right| \leq C \left(\epsilon_a \sum_{E \in \mathcal{E}_h} \left\| D^{1/2} \nabla v \right\|^2_{L_2(E)} + \frac{1}{\epsilon_a} \sum_{e \subset \Gamma_h \cup \Gamma_D} \frac{1}{|e|^{\beta_0}} \|[w]\|^2_{L_2(e)} \right)$$

for some positive constant C and for any positive ϵ_a. Hence, taking $\epsilon_a = 1/\sqrt{\alpha_0} > 0$ yields

$$\left| \sum_{e \subset \Gamma_h \cup \Gamma_D} \int_e \{D\nabla v \cdot \underline{n}_e\}[w] de \right| \leq C \frac{1}{\sqrt{\alpha_0}} \left(\sum_{E \in \mathcal{E}_h} \left\| D^{1/2} \nabla v \right\|^2_{L_2(E)} + \sum_{e \subset \Gamma_h \cup \Gamma_D} \frac{\alpha_0}{|e|^{\beta_0}} \|[w]\|^2_{L_2(e)} \right) \tag{3.2.1}$$

thus,

$$\begin{aligned} &\left| \sum_{e \subset \Gamma_h \cup \Gamma_D} \int_e \{D\nabla v \cdot \underline{n}_e\}[w] de \pm \sum_{e \subset \Gamma_h \cup \Gamma_D} \int_e \{D\nabla w \cdot \underline{n}_e\}[v] de \right| \\ &\leq C \frac{1}{\sqrt{\alpha_0}} \Bigg(\sum_{E \in \mathcal{E}_h} \left\| D^{1/2} \nabla v \right\|^2_{L_2(E)} + \sum_{e \subset \Gamma_h \cup \Gamma_D} \frac{\alpha_0}{|e|^{\beta_0}} \|[w]\|^2_{L_2(e)} \\ &\quad + \sum_{E \in \mathcal{E}_h} \left\| D^{1/2} \nabla w \right\|^2_{L_2(E)} + \sum_{e \subset \Gamma_h \cup \Gamma_D} \frac{\alpha_0}{|e|^{\beta_0}} \|[v]\|^2_{L_2(e)} \Bigg) \end{aligned}$$

$$\leq C\frac{1}{\sqrt{\alpha_0}}\left(\|v\|^2_{\mathcal{V}}+\|w\|^2_{\mathcal{V}}\right). \tag{3.2.2}$$

Note that C is a positive constant independent of any function in $\mathcal{D}_k(\mathcal{E}_h)$.

Theorem 3.4. *For $\alpha_0 > 0$ and $\beta_0(d-1) \geq 1$, $a_\epsilon(\cdot,\cdot)$ is continuous on $\mathcal{D}_k(\mathcal{E}_h)$ with respect to the norm $\|\cdot\|_{\mathcal{V}}$. Thus there exists a positive constant K such that for any $v, w \in \mathcal{D}_k(\mathcal{E}_h)$*

$$a_\epsilon(v,w) \leq K\|v\|_{\mathcal{V}}\|w\|_{\mathcal{V}}.$$

Proof. Let $v, w \in \mathcal{D}_k(\mathcal{E}_h)$. Then, since $|\epsilon| \leq 1$

$$\begin{aligned}|a_\epsilon(v,w)| \leq &\sum_{E\in\mathcal{E}_h}\left\|D^{1/2}\nabla v\right\|_{L_2(E)}\left\|D^{1/2}\nabla w\right\|_{L_2(E)}\\ &+\sum_{e\subset\Gamma_h\cup\Gamma_D}\|\{D\nabla v\cdot \underline{n}_e\}\|_{L_2(e)}\|[w]\|_{L_2(e)}\\ &+\sum_{e\subset\Gamma_h\cup\Gamma_D}\|\{D\nabla w\cdot \underline{n}_e\}\|_{L_2(e)}\|[v]\|_{L_2(e)}\\ &+\sum_{e\subset\Gamma_h\cup\Gamma_D}\frac{\alpha_0}{|e|^{\beta_0}}\|[v]\|_{L_2(e)}\|[w]\|_{L_2(e)}\end{aligned}$$

by Cauchy-Schwarz inequality. As shown in the proof of Theorem 3.3,

$$\|\{D\nabla w\cdot \underline{n}_e\}\|_{L_2(e)} \leq C\left(h_{E_1}^{-1/2}\|D^{1/2}\nabla v\|_{L_2(E_1)}+h_{E_2}^{-1/2}\|D^{1/2}\nabla v\|_{L_2(E_2)}\right)$$

for faced elements E_1 and E_2 with the shared edge e. Then, using discrete Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}|a_\epsilon(v,w)| \leq &\sum_{E\in\mathcal{E}_h}\left\|D^{1/2}\nabla v\right\|_{L_2(E)}\left\|D^{1/2}\nabla w\right\|_{L_2(E)}\\ &+\sum_{e\subset\Gamma_h\cup\Gamma_D}\|\{D\nabla v\cdot \underline{n}_e\}\|_{L_2(e)}\frac{|e|^{\beta_0/2}}{|e|^{\beta_0/2}}\|[w]\|_{L_2(e)}\\ &+\sum_{e\subset\Gamma_h\cup\Gamma_D}\|\{D\nabla w\cdot \underline{n}_e\}\|_{L_2(e)}\frac{|e|^{\beta_0/2}}{|e|^{\beta_0/2}}\|[v]\|_{L_2(e)}\\ &+\sum_{e\subset\Gamma_h\cup\Gamma_D}\frac{\alpha_0}{|e|^{\beta_0}}\|[v]\|_{L_2(e)}\|[w]\|_{L_2(e)}\\ \leq&\left(\sum_{E\in\mathcal{E}_h}\left\|D^{1/2}\nabla v\right\|^2_{L_2(E)}+\sum_{e\subset\Gamma_h\cup\Gamma_D}\|\{D\nabla v\cdot \underline{n}_e\}\|^2_{L_2(e)}|e|^{\beta_0}\right.\\ &\left.+\sum_{e\subset\Gamma_h\cup\Gamma_D}\frac{1}{|e|^{\beta_0}}\|[v]\|^2_{L_2(e)}+\sum_{e\subset\Gamma_h\cup\Gamma_D}\frac{\alpha_0}{|e|^{\beta_0}}\|[v]\|^2_{L_2(e)}\right)^{1/2}\\ &\times\left(\sum_{E\in\mathcal{E}_h}\left\|D^{1/2}\nabla w\right\|^2_{L_2(E)}+\sum_{e\subset\Gamma_h\cup\Gamma_D}\|\{D\nabla w\cdot \underline{n}_e\}\|^2_{L_2(e)}|e|^{\beta_0}\right.\end{aligned}$$

$$+ \sum_{e\subset\Gamma_h\cup\Gamma_D} \frac{1}{|e|^{\beta_0}} \|[w]\|^2_{L_2(e)} + \sum_{e\subset\Gamma_h\cup\Gamma_D} \frac{\alpha_0}{|e|^{\beta_0}} \|[w]\|^2_{L_2(e)} \Bigg)^{1/2}$$

($\because$ by discrete Cauchy-Schwarz inequality)

$$\leq \Bigg(\sum_{E\in\mathcal{E}_h} \left\| D^{1/2}\nabla v \right\|^2_{L_2(E)} + C \sum_{E\in\mathcal{E}_h} \left\| D^{1/2}\nabla v \right\|^2_{L_2(E)} \frac{|e|^{\beta_0}}{h_E}$$

$$+ \sum_{e\subset\Gamma_h\cup\Gamma_D} \frac{1}{|e|^{\beta_0}} \|[v]\|^2_{L_2(e)} + \sum_{e\subset\Gamma_h\cup\Gamma_D} \frac{\alpha_0}{|e|^{\beta_0}} \|[v]\|^2_{L_2(e)} \Bigg)^{1/2}$$

$$\times \Bigg(\sum_{E\in\mathcal{E}_h} \left\| D^{1/2}\nabla w \right\|^2_{L_2(E)} + C \sum_{E\in\mathcal{E}_h} \left\| D^{1/2}\nabla w \right\|^2_{L_2(E)} \frac{|e|^{\beta_0}}{h_E}$$

$$+ \sum_{e\subset\Gamma_h\cup\Gamma_D} \frac{1}{|e|^{\beta_0}} \|[w]\|^2_{L_2(e)} + \sum_{e\subset\Gamma_h\cup\Gamma_D} \frac{\alpha_0}{|e|^{\beta_0}} \|[w]\|^2_{L_2(e)} \Bigg)^{1/2}$$

($\because$ by inverse polynomial trace theorem for some positive C independent of h_E)

$$\leq \Bigg(\sum_{E\in\mathcal{E}_h} \left\| D^{1/2}\nabla v \right\|^2_{L_2(E)} + C \sum_{E\in\mathcal{E}_h} \left\| D^{1/2}\nabla v \right\|^2_{L_2(E)} \frac{h_E^{\beta_0(d-1)}}{h_E}$$

$$+ \sum_{e\subset\Gamma_h\cup\Gamma_D} \frac{1}{|e|^{\beta_0}} \|[v]\|^2_{L_2(e)} + \sum_{e\subset\Gamma_h\cup\Gamma_D} \frac{\alpha_0}{|e|^{\beta_0}} \|[v]\|^2_{L_2(e)} \Bigg)^{1/2}$$

$$\times \Bigg(\sum_{E\in\mathcal{E}_h} \left\| D^{1/2}\nabla w \right\|^2_{L_2(E)} + C \sum_{E\in\mathcal{E}_h} \left\| D^{1/2}\nabla w \right\|^2_{L_2(E)} \frac{h_E^{\beta_0(d-1)}}{h_E}$$

$$+ \sum_{e\subset\Gamma_h\cup\Gamma_D} \frac{1}{|e|^{\beta_0}} \|[w]\|^2_{L_2(e)} + \sum_{e\subset\Gamma_h\cup\Gamma_D} \frac{\alpha_0}{|e|^{\beta_0}} \|[w]\|^2_{L_2(e)} \Bigg)^{1/2}$$

($\because$ $|e| \leq h_E^{d-1}$ $\forall E \in \mathcal{E}_h$)

$$\leq \Bigg(\sum_{E\in\mathcal{E}_h} \left\| D^{1/2}\nabla v \right\|^2_{L_2(E)} + C \sum_{E\in\mathcal{E}_h} \left\| D^{1/2}\nabla v \right\|^2_{L_2(E)}$$

$$+ \sum_{e\subset\Gamma_h\cup\Gamma_D} \frac{1}{|e|^{\beta_0}} \|[v]\|^2_{L_2(e)} + \sum_{e\subset\Gamma_h\cup\Gamma_D} \frac{\alpha_0}{|e|^{\beta_0}} \|[v]\|^2_{L_2(e)} \Bigg)^{1/2}$$

$$\times \Bigg(\sum_{E\in\mathcal{E}_h} \left\| D^{1/2}\nabla w \right\|^2_{L_2(E)} + C \sum_{E\in\mathcal{E}_h} \left\| D^{1/2}\nabla w \right\|^2_{L_2(E)}$$

$$+ \sum_{e\subset\Gamma_h\cup\Gamma_D} \frac{1}{|e|^{\beta_0}} \|[w]\|^2_{L_2(e)} + \sum_{e\subset\Gamma_h\cup\Gamma_D} \frac{\alpha_0}{|e|^{\beta_0}} \|[w]\|^2_{L_2(e)} \Bigg)^{1/2}$$

($\because$ $\beta_0(d-1) \geq 1$ and $h_E \leq 1,\ \forall E \in \mathcal{E}_h$).

Therefore, there exists a positive constant K such that

$$\begin{aligned}|a_\epsilon(v,w)| \le & K\Bigg(\sum_{E\in\mathcal{E}_h}\left\|D^{1/2}\nabla v\right\|^2_{L_2(E)} + \sum_{e\subset\Gamma_h\cup\Gamma_D}\frac{\alpha_0}{|e|^{\beta_0}}\|[v]\|^2_{L_2(e)}\Bigg)^{1/2} \\ & \times\Bigg(\sum_{E\in\mathcal{E}_h}\left\|D^{1/2}\nabla w\right\|^2_{L_2(E)} + \sum_{e\subset\Gamma_h\cup\Gamma_D}\frac{\alpha_0}{|e|^{\beta_0}}\|[w]\|^2_{L_2(e)}\Bigg)^{1/2} \\ = & K\|v\|_{\mathcal{V}}\|w\|_{\mathcal{V}}\end{aligned}$$

where

$$K = \max\left(C, \frac{1}{\alpha_0}\right) + 1.$$

□

Theorem 3.5. *Under same conditions for continuity of the bilinear form $a_\epsilon(\cdot,\cdot)$, there exists a positive constant C such that*

$$\forall v, w \in \mathcal{D}_k(\mathcal{E}_h),\ a_\epsilon(v,w) \le Ch^{-1}\|v\|_{L_2(\Omega)}\|w\|_{\mathcal{V}}.$$

Proof. Let $v, w \in \mathcal{D}_k(\mathcal{E}_h)$. By the continuity of the bilinear form

$$\begin{aligned}|a_\epsilon(v,w)| \le & K\|v\|_{\mathcal{V}}\|w\|_{\mathcal{V}} \\ = & K\left(\sum_{E\in\mathcal{E}_h}\left\|D^{1/2}\nabla v\right\|^2_{L_2(E)} + \sum_{e\subset\Gamma_h\cup\Gamma_D}\frac{\alpha_0}{|e|^{\beta_0}}\|[v]\|^2_{L_2(e)}\right)^{1/2}\|w\|_{\mathcal{V}}.\end{aligned}$$

(1.4.11), inverse polynomial trace theorem and the quasi-uniform subdivision imply

$$\begin{aligned}|a_\epsilon(v,w)| \le & K\left(\sum_{E\in\mathcal{E}_h}Ch_E^{-2}\|v\|^2_{L_2(E)} + \sum_{e\subset\Gamma_h\cup\Gamma_D}\frac{\alpha_0}{|e|^{\beta_0}}Ch_E^{-1}\|[v]\|^2_{L_2(E)}\right)^{1/2}\|w\|_{\mathcal{V}} \\ \le & Ch^{-1}\left(\sum_{E\in\mathcal{E}_h}\|v\|^2_{L_2(E)} + \sum_{E\in\mathcal{E}_h}\|v\|^2_{L_2(E)}\right)^{1/2}\|w\|_{\mathcal{V}} \\ \le & Ch^{-1}\|v\|_{L_2(\Omega)}\|w\|_{\mathcal{V}}.\end{aligned}$$

□

Hereafter we suppose $\beta_0(d-1) \ge 1$ and we will introduce a skew symmetric bilinear form in NIPG such that

$$\begin{aligned}a_1(v,w) = & \sum_{E\in\mathcal{E}_h}\int_E D\nabla v\cdot\nabla w\, dE - \sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e\{D\nabla v\cdot\underline{n}_e\}[w]de \\ & + \sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e\{D\nabla w\cdot\underline{n}_e\}[v]de + J_0^{\alpha_0,\beta_0}(v,w)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{E\in\mathcal{E}_h} \int_E D\nabla w \cdot \nabla v \; dE - \sum_{e\subset\Gamma_h\cup\Gamma_D} \int_e \{D\nabla v \cdot \underline{n}_e\}[w] de \\
&\quad + \sum_{e\subset\Gamma_h\cup\Gamma_D} \int_e \{D\nabla w \cdot \underline{n}_e\}[v] de + J_0^{\alpha_0,\beta_0}(w,v) \\
&= \sum_{E\in\mathcal{E}_h} \int_E D\nabla w \cdot \nabla v \; dE - \sum_{e\subset\Gamma_h\cup\Gamma_D} \int_e \{D\nabla w \cdot \underline{n}_e\}[v] de \\
&\quad + \sum_{e\subset\Gamma_h\cup\Gamma_D} \int_e \{D\nabla v \cdot \underline{n}_e\}[w] de + J_0^{\alpha_0,\beta_0}(w,v) \\
&\quad - 2\left(\sum_{e\subset\Gamma_h\cup\Gamma_D} \int_e \{D\nabla v \cdot \underline{n}_e\}[w] de - \sum_{e\subset\Gamma_h\cup\Gamma_D} \int_e \{D\nabla w \cdot \underline{n}_e\}[v] de \right) \\
&= a_1(w,v) - 2\left(\sum_{e\subset\Gamma_h\cup\Gamma_D} \int_e \{D\nabla v \cdot \underline{n}_e\}[w] de - \sum_{e\subset\Gamma_h\cup\Gamma_D} \int_e \{D\nabla w \cdot \underline{n}_e\}[v] de \right) \\
&:= a_1(w,v) + B(v,w). \qquad (3.2.3)
\end{aligned}
$$

Furthermore, (3.2.2) allows us to obtain

$$
|B(v,w)| \le C \frac{1}{\sqrt{\alpha_0}} \left(\|v\|_{\mathcal{V}}^2 + \|w\|_{\mathcal{V}}^2 \right), \;\; \forall v, w \in V^h \qquad (3.2.4)
$$

for some positive C depending only on the domain Ω and $\mathcal{D}_k(\mathcal{E}_h)$. Note that by the definition of $B(v,w)$ we have

$$
a_1(v,w) = \sum_{E\in\mathcal{E}_h} \int_E D\nabla v \cdot \nabla w \; dE + J_0^{\alpha_0,\beta_0}(v,w) + \frac{1}{2} B(v,w).
$$

Thus,

$$
|a_1(v,w)| \le \|v\|_{\mathcal{V}} \|w\|_{\mathcal{V}} + \frac{C}{\sqrt{\alpha_0}} \left(\|v\|_{\mathcal{V}}^2 + \|w\|_{\mathcal{V}}^2 \right).
$$

In a similar way, (3.2.2) implies

$$
\begin{aligned}
|a_{-1}(v,w)| &\le \|v\|_{\mathcal{V}} \|w\|_{\mathcal{V}} + \frac{C}{\sqrt{\alpha_0}} \left(\|v\|_{\mathcal{V}}^2 + \|w\|_{\mathcal{V}}^2 \right), \\
|a_0(v,w)| &\le \|v\|_{\mathcal{V}} \|w\|_{\mathcal{V}} + \frac{C}{\sqrt{\alpha_0}} \left(\|v\|_{\mathcal{V}}^2 + \|w\|_{\mathcal{V}}^2 \right),
\end{aligned}
$$

hence for any $\epsilon = 1, 0, -1$,

$$
|a_\epsilon(v,w)| \le \|v\|_{\mathcal{V}} \|w\|_{\mathcal{V}} + \frac{C}{\sqrt{\alpha_0}} \left(\|v\|_{\mathcal{V}}^2 + \|w\|_{\mathcal{V}}^2 \right) \qquad (3.2.5)
$$

for some positive C.

Remark For any $v \in \mathcal{D}_k(\mathcal{E}_h)$,

$$
\|v\|_{\mathcal{V}}^2 = a_1(v,v) \le K \|v\|_{\mathcal{V}}^2, \text{ and } \kappa \|v\|_{\mathcal{V}}^2 \le a_{-1}(v,v) \le K \|v\|_{\mathcal{V}}^2
$$

with positive κ and K.

3.2.1 Displacement Form

Now, we can derive the semidiscrete formulation of **(Q1)** as
Find $u_h(t)$ and $\{\psi_{hq}(t)\}_{q=1}^{N_\varphi}$ in $\mathcal{D}_k(\mathcal{E}_h)$ such that for all $v \in \mathcal{D}_k(\mathcal{E}_h)$

$$(\rho \ddot{u}_h(t), v)_{L_2(\Omega)} + a_1(u_h(t), v) - \sum_{q=1}^{N_\varphi} a_{-1}(\psi_{hq}(t), v) + J_0^{\alpha_0,\beta_0}(\dot{u}_h, v) = F_d(t; v), \quad (3.2.6)$$

$$a_{-1}\left(\tau_q \dot{\psi}_{hq}(t) + \psi_{hq}(t), v\right) = a_{-1}(\varphi_q u(t), v), \ \forall q, \quad (3.2.7)$$

$$a_1(u_h(0), v) = a_1(u_0, v), \quad (3.2.8)$$

$$(\dot{u}_h(0), v)_{L_2(\Omega)} = (w_0, v)_{L_2(\Omega)}, \quad (3.2.9)$$

where $\psi_{hq}(0) = 0, \forall q \in \{1, \ldots, N_\varphi\}$, $\forall t$. Then our approximate solutions can be expressed by

$$u_h(\boldsymbol{x}, t) = \sum_{i=1}^{N_{V^h}} \mathtt{u}_i(t)\phi_i(\boldsymbol{x}),$$

$$\psi_{hq}(\boldsymbol{x}, t) = \sum_{i=1}^{N_{V^h}} \psi_{hq,i}(t)\phi_i(\boldsymbol{x}), \ \forall q.$$

By choosing $v = \phi_i$ in (3.2.6), we have a second order ODE system such that

$$\rho M \ddot{\underline{\mathtt{u}}}(t) + A\underline{\mathtt{u}}(t) - \sum_{q=1}^{N_\varphi} A^* \underline{\psi}_{hq}(t) + \mathcal{J}\dot{\underline{\mathtt{u}}}(t) = \underline{F}(t) \quad (3.2.10)$$

where for $i, j = 1, \ldots, N_{V^h}$

$$M_{ij} = (\phi_j, \phi_i)_{L_2(\Omega)}, \ A_{ij} = a_1(\phi_j, \phi_i), \ A^*_{ij} = a_{-1}(\phi_j, \phi_i), \ \mathcal{J}_{ij} = J_0^{\alpha_0,\beta_0}(\phi_j, \phi_i)$$

and

$$F_i(t) = F_d(t; \phi_i) \qquad \text{for } i = 1, \ldots, N_{V^h}$$

In a similar way, initial conditions yield the following system of ODEs

$$A\underline{\mathtt{u}}(0) = \underline{U}_0, \quad (3.2.11)$$

$$M\dot{\underline{\mathtt{u}}}(0) = \underline{W}_0, \quad (3.2.12)$$

where $(U_0)_i = a_1(u_0, \phi_i)$ and $(W_0)_i = (w_0, \phi_i)_{L_2(\Omega)}$ for $i = 1, \ldots, N_{V^h}$. To see the existence and uniqueness of solutions, we shall show that the matrices are invertible. As seen in Theorem 2.5, we can prove M, A^* and $\mathcal{J}$ are symmetric positive definite. Hence, (3.2.7) implies straight-forwardly

$$\tau_q \dot{\underline{\psi}}_{hq}(t) + \underline{\psi}_{hq}(t) = \varphi_q \dot{\underline{\mathtt{u}}}(t) \quad (3.2.13)$$

with $\underline{\psi}_{hq}(0) = \underline{0}, \forall q \in \{1,\dots,N_\varphi\}$. However, it is not obvious A is invertible but we already show that the bilinear form a_1 is coercive on $\mathcal{D}_k(\mathcal{E}_h)$. Then we can obtain the invertibility of A and the theory of ODE allows us to solve the system uniquely. However, as seen in CGFEM if stability bounds are given by data we would also have the existence and uniqueness.

Lemma 3.1. *Suppose $u_h \in H^2(0,T;L_2(\Omega)) \cap H^1(0,T;\mathcal{D}_k(\mathcal{E}_h))$ and $\psi_{hq} \in H^1(0,T;\mathcal{D}_k(\mathcal{E}_h))$, $\forall q \in \{1,\dots,N_\varphi\}$. Then we have for any $0 \le t \le T$*

$$-\int_0^t a_{-1}\left(\psi_{hq}(t'), \dot{u}_h(t')\right) dt' = \frac{\tau_q}{\varphi_q}\int_0^t a_{-1}\left(\dot{\psi}_{hq}(t'), \dot{\psi}_{hq}(t')\right) dt' + \frac{1}{2\varphi_q} a_{-1}\left(\psi_{hq}(t), \psi_{hq}(t)\right) - a_{-1}\left(u_h(t), \psi_{hq}(t)\right).$$

Proof. Put $v = \dot{\psi}_{hq}(t')$ for any $0 \le s \le t \le T$ into (3.2.7). Then we have

$$\begin{aligned}
&\tau_q a_{-1}\left(\dot{\psi}_{hq}(t'), \dot{\psi}_{hq}(t')\right) + a_{-1}\left(\dot{\psi}_{hq}(t'), \psi_{hq}(t')\right)\\
=&\tau_q a_{-1}\left(\dot{\psi}_{hq}(t'), \dot{\psi}_{hq}(t')\right) + \sum_{E\in\mathcal{E}_h}\int_E D\nabla\dot{\psi}_{hq}(t')\cdot\nabla\psi_{hq}(t')dE + J_0^{\alpha_0,\beta_0}(\dot{\psi}_{hq}(t'), \psi_{hq}(t'))\\
&- \sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e \{D\nabla\dot{\psi}_{hq}(t')\cdot\underline{n}_e\}[\psi_{hq}(t')]de - \sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e \{D\nabla\psi_{hq}(t')\cdot\underline{n}_e\}[\dot{\psi}_{hq}(t')]de\\
=&\tau_q a_{-1}\left(\dot{\psi}_{hq}(t'), \dot{\psi}_{hq}(t')\right) + \frac{1}{2}\frac{d}{dt'} a_{-1}\left(\psi_{hq}(t'), \psi_{hq}(t')\right)\\
=&\varphi_q a_{-1}\left(\dot{\psi}_{hq}(t'), u_h(t')\right)
\end{aligned}$$

by the Leibniz integral rule and the symmetric DG bilinear form. Then, integration over time gives

$$\tau_q\int_0^t a_{-1}\left(\dot{\psi}_{hq}(t'), \dot{\psi}_{hq}(t')\right) dt' + \frac{1}{2} a_{-1}\left(\psi_{hq}(t), \psi_{hq}(t)\right) = \varphi_q\int_0^t a_{-1}\left(\dot{\psi}_{hq}(t'), u_h(t')\right) dt',$$

since $\psi_{hq}(0) = 0,\ \forall q \in \{1,\dots,N_\varphi\}$. Integration by parts provides that

$$\int_0^t a_{-1}\left(\dot{\psi}_{hq}(t'), u_h(t')\right) dt' = a_{-1}\left(\psi_{hq}(t), u_h(t)\right) - \int_0^t a_{-1}\left(\psi_{hq}(t'), \dot{u}_h(t')\right) dt'.$$

This leads us to have

$$\frac{\tau_q}{\varphi_q}\int_0^t a_{-1}\left(\dot{\psi}_{hq}(t'), \dot{\psi}_{hq}(t')\right) dt' + \frac{1}{2\varphi_q} a_{-1}\left(\psi_{hq}(t), \psi_{hq}(t)\right) = a_{-1}\left(\psi_{hq}(t), u_h(t)\right) - \int_0^t a_{-1}\left(\psi_{hq}(t'), \dot{u}_h(t')\right) dt'$$

so that

$$-\int_0^t a_{-1}\left(\psi_{hq}(t'), \dot{u}_h(t')\right) dt' = \frac{\tau_q}{\varphi_q}\int_0^t a_{-1}\left(\dot{\psi}_{hq}(t'), \dot{\psi}_{hq}(t')\right) dt' + \frac{1}{2\varphi_q} a_{-1}\left(\psi_{hq}(t), \psi_{hq}(t)\right)$$

$$- a_{-1}\left(u_h(t), \psi_{hq}(t)\right).$$

□

Theorem 3.6. *Assume*

$$u_h \in H^2(0,T;L_2(\Omega)) \cap W^1_\infty(0,T;\mathcal{D}_k(\mathcal{E}_h)),$$
$$\psi_{hq} \in W^1_\infty(0,T;\mathcal{D}_k(\mathcal{E}_h)),\ \forall q \in \{1,\dots,N_\varphi\}.$$

If $\beta_0(d-1) \geq 1$ and α_0 is large enough, then there exists a positive constant C such that depends on T, Ω and $\mathcal{D}_k(\mathcal{E}_h)$ but is independent of the solutions and h_E, $\forall E \in \mathcal{E}_h$ with satisfying

$$\begin{aligned}
&\left\|\rho^{1/2}\dot{u}_h\right\|^2_{L_\infty(0,T;L_2(\Omega))} + \|u_h\|^2_{L_\infty(0,T;\mathcal{V})} + \sum_{q=1}^{N_\varphi}\|\psi_{hq}(t)\|^2_{\mathcal{V}} + \sum_{q=1}^{N_\varphi}\int_0^t \left\|\dot{\psi}_{hq}(t')\right\|^2_{\mathcal{V}} dt' \\
&+ \int_0^t J_0^{\alpha_0,\beta_0}(\dot{u}(t'),\dot{u}(t'))dt' \\
\leq& C\Bigg(\left\|\rho^{1/2}w_0\right\|^2_{L_2(\Omega)} + \|u_0\|^2_{\mathcal{V}} + \|f\|^2_{L_2(0,T;L_2(\Omega))} + h^{-1}\|g_N\|^2_{L_\infty(0,T;L_2(\Gamma_N))} \\
&+ h^{-1}\|\dot{g}_N\|^2_{L_2(0,T;L_2(\Gamma_N))}\Bigg),
\end{aligned}$$

and so

$$\begin{aligned}
&\left\|\rho^{1/2}\dot{u}_h(t)\right\|^2_{L_2(\Omega)} + \|u_h(t)\|^2_{\mathcal{V}} + \sum_{q=1}^{N_\varphi}\|\psi_{hq}(t)\|^2_{\mathcal{V}} + \sum_{q=1}^{N_\varphi}\int_0^t \left\|\dot{\psi}_{hq}(t')\right\|^2_{\mathcal{V}} dt' \\
&+ \int_0^t J_0^{\alpha_0,\beta_0}(\dot{u}(t'),\dot{u}(t'))dt' \\
\leq& C\Bigg(\left\|\rho^{1/2}w_0\right\|^2_{L_2(\Omega)} + \|u_0\|^2_{\mathcal{V}} + \|f\|^2_{L_2(0,T;L_2(\Omega))} + h^{-1}\|g_N\|^2_{L_\infty(0,T;L_2(\Gamma_N))} \\
&+ h^{-1}\|\dot{g}_N\|^2_{L_2(0,T;L_2(\Gamma_N))}\Bigg),
\end{aligned}$$

for $\forall t \in [0,T]$.

Proof. Let us consider (3.2.6) whence $v = \dot{u}_h(t')$ for $0 \leq s \leq t \leq T$.

$$\begin{aligned}
&\left(\rho\ddot{u}_h(t'), \dot{u}_h(t')\right)_{L_2(\Omega)} + a_1\left(u_h(t'), \dot{u}_h(t')\right) - \sum_{q=1}^{N_\varphi} a_{-1}\left(\psi_{hq}(t'), \dot{u}_h(t')\right) + J_0^{\alpha_0,\beta_0}(\dot{u}_h(t'), \dot{u}_h(t')) \\
&= F_d(t'; \dot{u}_h(t')).
\end{aligned}$$

With applying Leibniz's integral rule, we have

$$\left(\rho\ddot{u}_h(t'), \dot{u}_h(t')\right)_{L_2(\Omega)} = \frac{\rho}{2}\frac{d}{dt'}\left\|\dot{u}_h(t')\right\|^2_{L_2(\Omega)}$$

and

$$a_1\left(u_h(t'),\dot{u}_h(t')\right)=\sum_{E\in\mathcal{E}_h}\int_E D\nabla u_h(t')\cdot\nabla\dot{u}_h(t')dE-\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e\{D\nabla u_h(t')\cdot\underline{n}_e\}[\dot{u}_h(t')]de$$
$$+\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e\{D\nabla\dot{u}_h(t')\cdot\underline{n}_e\}[u_h(t')]de+J_0^{\alpha_0,\beta_0}(u_h(t'),\dot{u}_h(t'))$$
$$=\frac{1}{2}\frac{d}{dt'}\left\|u_h(t')\right\|_{\mathcal{V}}^2-\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e\{D\nabla u_h(t')\cdot\underline{n}_e\}[\dot{u}_h(t')]de$$
$$+\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e\{D\nabla\dot{u}_h(t')\cdot\underline{n}_e\}[u_h(t')]de.$$

Hence

$$\frac{\rho}{2}\frac{d}{dt'}\left\|\dot{u}_h(t')\right\|_{L_2(\Omega)}^2+\frac{1}{2}\frac{d}{dt'}\left\|u_h(t')\right\|_{\mathcal{V}}^2-\sum_{q=1}^{N_\varphi}a_{-1}\left(\psi_{hq}(t'),\dot{u}_h(t')\right)+J_0^{\alpha_0,\beta_0}(\dot{u}_h(t'),\dot{u}_h(t'))$$
$$=F_d(t';\dot{u}_h(t'))+\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e\{D\nabla u_h(t')\cdot\underline{n}_e\}[\dot{u}_h(t')]de$$
$$-\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e\{D\nabla\dot{u}_h(t')\cdot\underline{n}_e\}[u_h(t')]de,$$

then integrating over time gives

$$\frac{\rho}{2}\left\|\dot{u}_h(t)\right\|_{L_2(\Omega)}^2+\frac{1}{2}\left\|u_h(t)\right\|_{\mathcal{V}}^2-\sum_{q=1}^{N_\varphi}\int_0^t a_{-1}\left(\psi_{hq}(t'),\dot{u}_h(t')\right)dt'+\int_0^t J_0^{\alpha_0,\beta_0}(\dot{u}_h(t'),\dot{u}_h(t'))dt'$$
$$=\frac{\rho}{2}\left\|\dot{u}_h(0)\right\|_{L_2(\Omega)}^2+\frac{1}{2}\left\|u_h(0)\right\|_{\mathcal{V}}^2+\int_0^t F_d(t';\dot{u}_h(t'))dt'$$
$$+\int_0^t\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e\{D\nabla u_h(t')\cdot\underline{n}_e\}[\dot{u}_h(t')]dedt'$$
$$-\int_0^t\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e\{D\nabla\dot{u}_h(t')\cdot\underline{n}_e\}[u_h(t')]dedt'.$$

By Lemma 3.1, we have

$$\frac{\rho}{2}\left\|\dot{u}_h(t)\right\|_{L_2(\Omega)}^2+\frac{1}{2}\left\|u_h(t)\right\|_{\mathcal{V}}^2+\sum_{q=1}^{N_\varphi}\frac{1}{2\varphi_q}a_{-1}\left(\psi_{hq}(t),\psi_{hq}(t)\right)$$
$$+\sum_{q=1}^{N_\varphi}\frac{\tau_q}{\varphi_q}\int_0^t a_{-1}\left(\dot{\psi}_{hq}(t'),\dot{\psi}_{hq}(t')\right)dt'+\int_0^t J_0^{\alpha_0,\beta_0}(\dot{u}_h(t'),\dot{u}_h(t'))dt'$$
$$=\frac{\rho}{2}\left\|\dot{u}_h(0)\right\|_{L_2(\Omega)}^2+\frac{1}{2}\left\|u_h(0)\right\|_{\mathcal{V}}^2+\int_0^t F_d(t';\dot{u}_h(t'))dt'+\sum_{q=1}^{N_\varphi}a_{-1}\left(\psi_{hq}(t),u_h(t)\right)$$

$$
\begin{aligned}
&+\int_0^t \sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e \{D\nabla u_h(t')\cdot \underline{n}_e\}[\dot{u}_h(t')]dedt' \\
&-\int_0^t \sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e \{D\nabla \dot{u}_h(t')\cdot \underline{n}_e\}[u_h(t')]dedt',
\end{aligned}
$$

and note that by the definition of SIPG

$$
a_{-1}\left(\psi_{hq}(t),\psi_{hq}(t)\right)=\|\psi_{hq}(t)\|^2_{\mathcal{V}}-2\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e\{D\nabla\psi_{hq}(t)\cdot\underline{n}_e\}[\psi_{hq}(t)]de.
$$

Moreover, the coercivity of SIPG implies

$$
\begin{aligned}
&\frac{\rho}{2}\|\dot{u}_h(t)\|^2_{L_2(\Omega)}+\frac{1}{2}\|u_h(t)\|^2_{\mathcal{V}}+\sum_{q=1}^{N_\varphi}\frac{1}{2\varphi_q}\|\psi_{hq}(t)\|^2_{\mathcal{V}}+\sum_{q=1}^{N_\varphi}\frac{\kappa\tau_q}{\varphi_q}\int_0^t\left\|\dot{\psi}_{hq}(t')\right\|^2_{\mathcal{V}}dt' \\
&\quad+\int_0^t J_0^{\alpha_0,\beta_0}(\dot{u}_h(t'),\dot{u}_h(t'))dt' \\
&\leq\left|\frac{\rho}{2}\|\dot{u}_h(0)\|^2_{L_2(\Omega)}+\frac{1}{2}\|u_h(0)\|^2_{\mathcal{V}}+\int_0^t F_d(t';\dot{u}_h(t'))dt'+\sum_{q=1}^{N_\varphi}a_{-1}\left(\psi_{hq}(t),u_h(t)\right)\right. \\
&\quad+\int_0^t \sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e \{D\nabla u_h(t')\cdot \underline{n}_e\}[\dot{u}_h(t')]dedt' \\
&\quad-\int_0^t \sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e \{D\nabla \dot{u}_h(t')\cdot \underline{n}_e\}[u_h(t')]dedt' \\
&\quad\left.+\sum_{q=1}^{N_\varphi}\frac{1}{\varphi_q}\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e\{D\nabla\psi_{hq}(t)\cdot\underline{n}_e\}[\psi_{hq}(t)]de\right|.
\end{aligned}
$$

Consider $\int_0^t F_d(t';\dot{u}_h(t'))dt'$. By Cauchy-Schwarz inequality and integration by parts,

$$
\begin{aligned}
\left|\int_0^t F_d(t';\dot{u}_h(t'))dt'\right| &\leq\left|\int_0^t \left\|f(t')\right\|_{L_2(\Omega)}\left\|\dot{u}_h(t')\right\|_{L_2(\Omega)}dt'+\sum_{e\subset\Gamma_N}\int_e g_N(t)u_h(t)de\right. \\
&\quad\left.-\sum_{e\subset\Gamma_N}\int_e g_N(0)u_h(0)de-\sum_{e\subset\Gamma_N}\int_0^t\int_e \dot{g}_N(t')u_h(t')dedt'\right| \\
&\leq\int_0^t \left\|f(t')\right\|_{L_2(\Omega)}\left\|\dot{u}(t')\right\|_{L_2(\Omega)}dt'+\sum_{e\subset\Gamma_N}\|g_N(t)\|_{L_2(e)}\|u_h(t)\|_{L_2(e)} \\
&\quad+\sum_{e\subset\Gamma_N}\|g_N(0)\|_{L_2(e)}\|u_h(0)\|_{L_2(e)} \\
&\quad+\sum_{e\subset\Gamma_N}\int_0^t\left\|\dot{g}_N(t')\right\|_{L_2(e)}\left\|u_h(t')\right\|_{L_2(e)}dt'.
\end{aligned}
$$

Taking into account Young's inequality and L_∞ norm, for positive $\epsilon_a,\ \epsilon_b$ we have

$$\begin{aligned}\left|\int_0^t F_d(t';\dot{u}_h(t'))dt'\right| \leq & \frac{\epsilon_a}{2}\|\dot{u}_h\|^2_{L_\infty(0,T;L_2(\Omega))} + \frac{1}{2\epsilon_a}\left(\int_0^t \left\|f(t')\right\|_{L_2(\Omega)} dt'\right)^2 \\ &+ \frac{1}{2\epsilon_b}\|g_N\|^2_{L_\infty(0,T;L_2(\Gamma_N))} + \frac{\epsilon_b}{2}\sum_{e\subset\Gamma_N}\|u_h(t)\|^2_{L_2(e)} \\ &+ \frac{1}{2h}\|g_N\|^2_{L_\infty(0,T;L_2(\Gamma_N))} + \frac{h}{2}\sum_{e\subset\Gamma_N}\|u_h(0)\|^2_{L_2(e)} \\ &+ \sum_{e\subset\Gamma_N}\int_0^t \left\|\dot{g}_N(t')\right\|_{L_2(e)}\left\|u_h(t')\right\|_{L_2(e)} dt',\end{aligned}$$

where $Ch \leq h_E \leq h,\ \forall E \in \mathcal{E}_h$. Note that Theorem 1.11 provides

$$\left\|u_h(t')\right\|_{L_2(e)} \leq Ch_E^{-1/2}\left\|u_h(t')\right\|_{L_2(E)},\ \forall s \in [0,T]$$

hence clearly

$$\begin{aligned}\sum_{e\subset\Gamma_N}\left\|u_h(t')\right\|^2_{L_2(e)} \leq & \sum_{E\in\mathcal{E}_h} Ch_E^{-1}\left\|u_h(t')\right\|^2_{L_2(E)} \\ \leq & Ch^{-1}\left\|u_h(t')\right\|^2_{L_2(\Omega)},\end{aligned}$$

and (1.4.10) gives

$$\sum_{e\subset\Gamma_N}\left\|u_h(t')\right\|^2_{L_2(e)} \leq Ch^{-1}\left\|u_h(t')\right\|^2_{\mathcal{V}}.$$

Tidying up the results, for positive ϵ_c

$$\begin{aligned}\left|\int_0^t F_d(t';\dot{u}_h(t'))dt'\right| \leq & \frac{\epsilon_a}{2}\|\dot{u}_h\|^2_{L_\infty(0,T;L_2(\Omega))} + \frac{1}{2\epsilon_a}\left(\int_0^t \left\|f(t')\right\|_{L_2(\Omega)} dt'\right)^2 \\ &+ \frac{1}{2\epsilon_b}\|g_N\|^2_{L_\infty(0,T;L_2(\Gamma_N))} + \frac{\epsilon_b}{2}Ch^{-1}\|u_h\|^2_{L_\infty(0,T;\mathcal{V})} \\ &+ \frac{1}{2h}\|g_N\|^2_{L_\infty(0,T;L_2(\Gamma_N))} + \frac{C}{2}\|u_h(0)\|^2_{L_2(\Omega)} \\ &+ \frac{\epsilon_c}{2}Ch^{-1}\|u_h\|^2_{L_\infty(0,T;\mathcal{V})} + \frac{1}{2\epsilon_c}\left(\int_0^t \left\|\dot{g}_N(t')\right\|_{L_2(\Gamma_N)} dt'\right)^2.\end{aligned}$$

On the other hand, use of (3.2.5) yields

$$\begin{aligned}\left|a_{-1}\left(\psi_{hq}(t),u_h(t)\right)\right| \leq & \|\psi_{hq}(t)\|_{\mathcal{V}}\|u_h(t)\|_{\mathcal{V}} + \frac{C}{\sqrt{\alpha_0}}(\|\psi_{hq}(t)\|^2_{\mathcal{V}} + \|u_h(t)\|^2_{\mathcal{V}}) \\ \leq & (\frac{C}{\sqrt{\alpha_0}} + \frac{1}{2\epsilon_q})\|\psi_{hq}(t)\|^2_{\mathcal{V}} + (\frac{C}{\sqrt{\alpha_0}} + \frac{\epsilon_q}{2})\|u_h(t)\|^2_{\mathcal{V}}\end{aligned}$$

for any positive ϵ_q, $q \in \{1, \ldots, N_\varphi\}$ by Young's inequality. Additionally, integration by parts and (3.2.1) lead us to have

$$
\begin{aligned}
&-\int_0^t \sum_{e\subset\Gamma_h\cup\Gamma_D} \int_e \{D\nabla \dot{u}_h(t')\cdot \underline{n}_e\}[u_h(t')]dedt' \\
=&-\sum_{e\subset\Gamma_h\cup\Gamma_D} \int_e \{D\nabla u_h(t)\cdot \underline{n}_e\}[u_h(t)] + \sum_{e\subset\Gamma_h\cup\Gamma_D} \int_e \{D\nabla u_h(0)\cdot \underline{n}_e\}[u_h(0)] \\
&+\int_0^t \sum_{e\subset\Gamma_h\cup\Gamma_D} \int_e \{D\nabla u_h(t')\cdot \underline{n}_e\}[\dot{u}_h(t')]dedt' \\
\leq& \frac{C}{\sqrt{\alpha_0}} \|u_h(t)\|_{\mathcal{V}}^2 + \frac{C}{\sqrt{\alpha_0}} \|u_h(0)\|_{\mathcal{V}}^2 + \int_0^t \sum_{e\subset\Gamma_h\cup\Gamma_D} \int_e \{D\nabla u_h(t')\cdot \underline{n}_e\}[\dot{u}_h(t')]dedt'.
\end{aligned}
$$

In this sense, by (3.2.1)

$$
\begin{aligned}
&\int_0^t \sum_{e\subset\Gamma_h\cup\Gamma_D} \int_e \{D\nabla u_h(t')\cdot \underline{n}_e\}[\dot{u}_h(t')]dedt' \\
\leq& \frac{C}{\sqrt{\alpha_0}} \int_0^t \sum_{E\in\mathcal{E}_h} \left\| D^{1/2}\nabla u_h(t')\right\|_{L_2(E)}^2 dt' + \frac{C}{\sqrt{\alpha_0}} \int_0^t J_0^{\alpha_0,\beta_0}(\dot{u}(t'),\dot{u}(t'))dt' \\
\leq& \frac{C}{\sqrt{\alpha_0}} \int_0^t \left\| u_h(t')\right\|_{\mathcal{V}}^2 dt' + \frac{C}{\sqrt{\alpha_0}} \int_0^t J_0^{\alpha_0,\beta_0}(\dot{u}(t'),\dot{u}(t'))dt' \\
\leq& \frac{CT}{\sqrt{\alpha_0}} \|u_h\|_{L_\infty(0,T;\mathcal{V})}^2 + \frac{C}{\sqrt{\alpha_0}} \int_0^t J_0^{\alpha_0,\beta_0}(\dot{u}(t'),\dot{u}(t'))dt',
\end{aligned}
$$

and

$$
\left| \sum_{q=1}^{N_\varphi} \frac{1}{\varphi_q} \sum_{e\subset\Gamma_h\cup\Gamma_D} \int_e \{D\nabla\psi_{hq}(t)\cdot \underline{n}_e\}[\psi_{hq}(t)]de \right| \leq \sum_{q=1}^{N_\varphi} \frac{C}{\varphi_q\sqrt{\alpha_0}} \|\psi_{hq}(t)\|_{\mathcal{V}}^2 .
$$

Consequently, we have

$$
\begin{aligned}
&\frac{\rho}{2}\|\dot{u}_h(t)\|_{L_2(\Omega)}^2 + \left(\frac{1}{2} - \frac{C(N_\varphi+1)}{\sqrt{\alpha_0}} - \sum_{q=1}^{N_\varphi}\frac{\epsilon_q}{2}\right)\|u_h(t)\|_{\mathcal{V}}^2 \\
&+\sum_{q=1}^{N_\varphi}\left(\frac{1}{2\varphi_q} - \frac{1}{2\epsilon_q} - \frac{C(1+\varphi_q)}{\varphi_q\sqrt{\alpha_0}}\right)\|\psi_{hq}(t)\|_{\mathcal{V}}^2 + \sum_{q=1}^{N_\varphi}\frac{\kappa\tau_q}{\varphi_q}\int_0^t \left\|\dot{\psi}_{hq}(t')\right\|_{\mathcal{V}}^2 dt' \\
&+\left(1-\frac{C}{\sqrt{\alpha_0}}\right)\int_0^t J_0^{\alpha_0,\beta_0}(\dot{u}(t'),\dot{u}(t'))dt' \\
\leq& \frac{\rho}{2}\|\dot{u}_h(0)\|_{L_2(\Omega)}^2 + \frac{1}{2}\|u_h(0)\|_{\mathcal{V}}^2 + \frac{\epsilon_a}{2}\|\dot{u}_h\|_{L_\infty(0,T;L_2(\Omega))}^2 + \frac{1}{2\epsilon_a}\left(\int_0^t \left\|f(t')\right\|_{L_2(\Omega)} dt'\right)^2
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\frac{1}{2\epsilon_b}+\frac{1}{2h}\right)\|g_N\|^2_{L_\infty(0,T;L_2(\Gamma_N))}+\frac{\epsilon_b}{2}Ch^{-1}\|u_h\|^2_{L_\infty(0,T;\mathcal{V})}+\frac{C}{2}\|u_h(0)\|^2_{L_2(\Omega)}\\
&+\frac{\epsilon_c}{2}Ch^{-1}\|u_h\|^2_{L_\infty(0,T;\mathcal{V})}+\frac{1}{2\epsilon_c}\left(\int_0^t\left\|\dot{g}_N(t')\right\|_{L_2(\Gamma_N)}dt'\right)^2+\frac{C}{\sqrt{\alpha_0}}\|u_h(0)\|^2_{L_2(\Omega)}\\
&+\frac{CT}{\sqrt{\alpha_0}}\|u_h\|^2_{L_\infty(0,T;\mathcal{V})}\\
\leq&\frac{\rho}{2}\|\dot{u}_h(0)\|^2_{L_2(\Omega)}+\frac{1}{2}\|u_h(0)\|^2_{\mathcal{V}}+\frac{\epsilon_a}{2}\|\dot{u}_h\|^2_{L_\infty(0,T;L_2(\Omega))}+\frac{T}{2\epsilon_a}\|f\|^2_{L_2(0,T;L_2(\Omega))}\\
&+\left(\frac{1}{2\epsilon_b}+\frac{1}{2h}\right)\|g_N\|^2_{L_\infty(0,T;L_2(\Gamma_N))}+\frac{\epsilon_b}{2}Ch^{-1}\|u_h\|^2_{L_\infty(0,T;\mathcal{V})}+\frac{C}{2}\|u_h(0)\|^2_{L_2(\Omega)}\\
&+\frac{\epsilon_c}{2}Ch^{-1}\|u_h\|^2_{L_\infty(0,T;\mathcal{V})}+\frac{T}{2\epsilon_c}\|\dot{g}_N\|^2_{L_2(0,T;L_2(\Gamma_N))}+\frac{C}{\sqrt{\alpha_0}}\|u_h(0)\|^2_{L_2(\Omega)}\\
&+\frac{CT}{\sqrt{\alpha_0}}\|u_h\|^2_{L_\infty(0,T;\mathcal{V})}
\end{aligned}
$$

by Cauchy-Schwarz inequality. Here we suppose for our sake the coefficients of Young's inequalities,

$$
\begin{aligned}
\frac{1}{2}-\frac{C(N_\varphi+1)}{\sqrt{\alpha_0}}-\sum_{q=1}^{N_\varphi}\frac{\epsilon_q}{2}>&0,\\
\frac{1}{2\varphi_q}-\frac{1}{2\epsilon_q}-\frac{C(1+\varphi_q)}{\varphi_q\sqrt{\alpha_0}}>&0,\ \forall q\in\{1,\ldots,N_\varphi\}.
\end{aligned}
$$

Let us take $\epsilon_q=(\varphi_q+\frac{\varphi_0}{2N_\varphi})>0$ for each q then we have

$$
\begin{aligned}
\frac{1}{2}-\sum_{q=1}^{N_\varphi}\frac{\epsilon_q}{2}=&\frac{\varphi_0}{4}>0,\\
\frac{1}{2\varphi_q}-\frac{1}{2\epsilon_q}=&\frac{\varphi_0}{4N_\varphi\varphi_q^2+2\varphi_0\varphi_q}>0,\ \forall q\in\{1,\ldots,N_\varphi\}.
\end{aligned}
$$

Considering L_∞ norm on the left hand side gives

$$
\begin{aligned}
&\frac{\rho}{2}\|\dot{u}_h\|^2_{L_\infty(0,T;L_2(\Omega))}+\left(\frac{\varphi_0}{4}-\frac{C(N_\varphi+1)}{\sqrt{\alpha_0}}\right)\|u_h\|^2_{L_\infty(0,T;\mathcal{V})}\\
&\quad+\sum_{q=1}^{N_\varphi}\left(\frac{\varphi_0}{4N_\varphi\varphi_q^2+2\varphi_0\varphi_q}-\frac{C(1+\varphi_q)}{\varphi_q\sqrt{\alpha_0}}\right)\|\psi_{hq}(t)\|^2_{\mathcal{V}}+\sum_{q=1}^{N_\varphi}\frac{\kappa\tau_q}{\varphi_q}\int_0^t\left\|\dot{\psi}_{hq}(t')\right\|^2_{\mathcal{V}}dt'\\
&\quad+\left(1-\frac{C}{\sqrt{\alpha_0}}\right)\int_0^t J_0^{\alpha_0,\beta_0}(\dot{u}(t'),\dot{u}(t'))dt'\\
&\leq 3\Bigg(\frac{\rho}{2}\|\dot{u}_h(0)\|^2_{L_2(\Omega)}+\left(\frac{1+C}{2}+\frac{C}{\sqrt{\alpha_0}}\right)\|u_h(0)\|^2_{\mathcal{V}}+\frac{\epsilon_a}{2}\|\dot{u}_h\|^2_{L_\infty(0,T;L_2(\Omega))}
\end{aligned}
$$

$$+\frac{T}{2\epsilon_a}\|f\|^2_{L_2(0,T;L_2(\Omega))}+\left(\frac{1}{2\epsilon_b}+\frac{1}{2h}\right)\|g_N\|^2_{L_\infty(0,T;L_2(\Gamma_N))}$$
$$+\frac{\epsilon_b+\epsilon_c}{2}Ch^{-1}\|u_h\|^2_{L_\infty(0,T;\mathcal{V})}+\frac{T}{2\epsilon_c}\|\dot{g}_N\|^2_{L_2(0,T;L_2(\Gamma_N))}$$
$$\left.+\frac{CT}{\sqrt{\alpha_0}}\|u_h\|^2_{L_\infty(0,T;\mathcal{V})}\right).$$

If we take

$$\epsilon_a=\frac{\rho}{6}>0,\ \epsilon_b=\epsilon_c=\frac{\varphi_0 h}{24C}>0,$$

we can obtain

$$\frac{\rho}{4}\|\dot{u}_h\|^2_{L_\infty(0,T;L_2(\Omega))}+\left(\frac{\varphi_0}{8}-\frac{C(N_\varphi+1+3T)}{\sqrt{\alpha_0}}\right)\|u_h\|^2_{L_\infty(0,T;\mathcal{V})}$$
$$+\sum_{q=1}^{N_\varphi}\left(\frac{\varphi_0}{4N_\varphi\varphi_q^2+2\varphi_0\varphi_q}-\frac{C(1+\varphi_q)}{\varphi_q\sqrt{\alpha_0}}\right)\|\psi_{hq}(t)\|^2_{\mathcal{V}}+\sum_{q=1}^{N_\varphi}\frac{\kappa\tau_q}{\varphi_q}\int_0^t\left\|\dot{\psi}_{hq}(t')\right\|^2_{\mathcal{V}}dt'$$
$$+\left(1-\frac{C}{\sqrt{\alpha_0}}\right)\int_0^t J_0^{\alpha_0,\beta_0}(\dot{u}(t'),\dot{u}(t'))dt'$$
$$\leq\frac{3\rho}{2}\|\dot{u}_h(0)\|^2_{L_2(\Omega)}+\left(\frac{3+3C}{2}+\frac{3C}{\sqrt{\alpha_0}}\right)\|u_h(0)\|^2_{\mathcal{V}}+\frac{9T}{\rho}\|f\|^2_{L_2(0,T;L_2(\Omega))}$$
$$+\left(\frac{36C}{\varphi_0 h}+\frac{3}{2h}\right)\|g_N\|^2_{L_\infty(0,T;L_2(\Gamma_N))}+\frac{36CT}{\varphi_0 h}\|\dot{g}_N\|^2_{L_2(0,T;L_2(\Gamma_N))}.$$

Therefore a sufficiently large penalty coefficient α_0 leads us to have

$$\left\|\rho^{1/2}\dot{u}_h\right\|^2_{L_\infty(0,T;L_2(\Omega))}+\|u_h\|^2_{L_\infty(0,T;\mathcal{V})}+\sum_{q=1}^{N_\varphi}\|\psi_{hq}(t)\|^2_{\mathcal{V}}+\sum_{q=1}^{N_\varphi}\int_0^t\left\|\dot{\psi}_{hq}(t')\right\|^2_{\mathcal{V}}dt'$$
$$+\int_0^t J_0^{\alpha_0,\beta_0}(\dot{u}(t'),\dot{u}(t'))dt'$$
$$\leq C\Bigg(\left\|\rho^{1/2}\dot{u}_h(0)\right\|^2_{L_2(\Omega)}+\|u_h(0)\|^2_{\mathcal{V}}+\|f\|^2_{L_2(0,T;L_2(\Omega))}+h^{-1}\|g_N\|^2_{L_\infty(0,T;L_2(\Gamma_N))}$$
$$+h^{-1}\|\dot{g}_N\|^2_{L_2(0,T;L_2(\Gamma_N))}\Bigg),$$

for some positive constant C such that depends on final time T, the domain Ω and $\mathcal{D}_k(\mathcal{E}_h)$. As following (3.2.8) and (3.2.9) with Cauchy-Schwarz inequality, we can conclude that

$$\left\|\rho^{1/2}\dot{u}_h\right\|^2_{L_\infty(0,T;L_2(\Omega))}+\|u_h\|^2_{L_\infty(0,T;\mathcal{V})}+\sum_{q=1}^{N_\varphi}\|\psi_{hq}(t)\|^2_{\mathcal{V}}+\sum_{q=1}^{N_\varphi}\int_0^t\left\|\dot{\psi}_{hq}(t')\right\|^2_{\mathcal{V}}dt'$$
$$+\int_0^t J_0^{\alpha_0,\beta_0}(\dot{u}(t'),\dot{u}(t'))dt'$$

$$\leq C\Bigg(\left\|\rho^{1/2}w_0\right\|^2_{L_2(\Omega)} + \|u_0\|^2_{\mathcal{V}} + \|f\|^2_{L_2(0,T;L_2(\Omega))} + h^{-1}\|g_N\|^2_{L_\infty(0,T;L_2(\Gamma_N))}$$
$$+ h^{-1}\|\dot{g}_N\|^2_{L_2(0,T;L_2(\Gamma_N))}\Bigg),$$

and

$$\left\|\rho^{1/2}\dot{u}_h(t)\right\|^2_{L_2(\Omega)} + \|u_h(t)\|^2_{\mathcal{V}} + \sum_{q=1}^{N_\varphi}\|\psi_{hq}(t)\|^2_{\mathcal{V}} + \sum_{q=1}^{N_\varphi}\int_0^t \left\|\dot{\psi}_{hq}(t')\right\|^2_{\mathcal{V}} dt'$$
$$+ \int_0^t J_0^{\alpha_0,\beta_0}(\dot{u}(t'),\dot{u}(t'))dt'$$
$$\leq C\Bigg(\left\|\rho^{1/2}w_0\right\|^2_{L_2(\Omega)} + \|u_0\|^2_{\mathcal{V}} + \|f\|^2_{L_2(0,T;L_2(\Omega))} + h^{-1}\|g_N\|^2_{L_\infty(0,T;L_2(\Gamma_N))}$$
$$+ h^{-1}\|\dot{g}_N\|^2_{L_2(0,T;L_2(\Gamma_N))}\Bigg),$$

for any $0 \leq t \leq T$. □

This theorem implies indeed the existence and uniqueness of the solution for the finite dimensional ODEs system. However, as comparing with CGFEM, the stability bound for DGFEM has h^{-1} terms. This is not observed in a practical sense and has nothing to do. In fact, it implies only the boundary condition is imposed weakly[24, 26]. Hence we do not care about h^{-1} terms in detail.

Under the conditions satisfying the stability bounds for **(Q1)**, we shall consider the error estimates. Instead of using the elliptic operator defined for CGFEM, we would introduce another elliptic projection. [24] allows us to use the following approximation theory.

Theorem 3.7. Elliptic projection [24, 22]
Let $\epsilon \in \{-1, 0, 1\}$. Define a DG elliptic projector R_ϵ with respect to $a_\epsilon(\cdot,\cdot)$ by

$$\forall t \geq 0,\ \forall v \in \mathcal{D}_k(\mathcal{E}_h),\ a_\epsilon(u(t), v) = a_\epsilon(R_\epsilon u(t), v). \tag{3.2.14}$$

Then we have the property as Galerkin orthogonality such that

$$a_\epsilon(u - R_\epsilon(u), w) = 0,\ \forall v \in \mathcal{D}_k(\mathcal{E}_h) \text{ and } u \text{ is arbitrary.}$$

We will call this Galerkin orthogonality too. If $u \in L_2(0,T;H^s(\mathcal{E}_h))$ for $s \in \mathbb{N}$ such that $s > 3/2$, it satisfies

$$\forall t \geq 0,\ \|u(t) - R_\epsilon u(t)\|_{\mathcal{V}} \leq Ch^{\min(k+1,s)-1}|||u(t)|||_{H^s(\mathcal{E}_h)}, \tag{3.2.15}$$
$$\forall t \geq 0,\ \|u(t) - R_\epsilon u(t)\|_{L_2(\Omega)} \leq Ch^{\min(k+1,s)-1}|||u(t)|||_{H^s(\mathcal{E}_h)}. \tag{3.2.16}$$

Moreover, with the convex domain Ω and $\beta_0 \geq 3(d-1)^{-1}$

$$\forall t \geq 0,\ \|u(t) - R_\epsilon u(t)\|_{L_2(\Omega)} \leq Ch^{\min(k+1,s)}|||u(t)|||_{H^s(\mathcal{E}_h)}. \tag{3.2.17}$$

In terms of a penalty parameter β_0, a standard penalisation means that we assume $\beta_0(d-1) = 1$ for $d = 2, 3$. However, it is said to be super-penalised when $\beta_0(d-1) \geq 3$. Indeed, for SIPG (3.2.17) *also holds with the standard penalisation.*

Elliptic error estimates theorem can be proved by approximation properties e.g. see [22, 37, 21]. As seen in the above, the super-penalisation leads us to derive L_2 optimal error estimates of NIPG and IIPG, whereas SIPG requires only the standard penalisation. Meanwhile the super-penalisation implies the optimality of L_2 estimates, but the converse is not true. In 1D cases, NIPG and IIPG have optimal L_2 estimates for odd degrees of polynomials k with the standard penalisation, e.g. see details in [61, 62]. Furthermore, L_2 optimal error estimates without the super-penalisation in multi-dimensional spaces for rectangular meshes have been presented by [63].

As seen in the CGFEM cases, we consider the DG elliptic projection to derive error estimates theorems. Let us define

$$\begin{aligned}
\theta &= u - R_1 u, \\
\vartheta_q &= \psi_q - R_{-1}\psi_q, \ \forall q, \\
\chi &= u_h - R_1 u, \\
\varsigma_q &= \psi_{hq} - R_{-1}\psi_q, \ \forall q.
\end{aligned}$$

Lemma 3.2. *Suppose $u \in H^2(0,T;C^2(\Omega)) \cap W^1_\infty(0,T;H^s(\mathcal{E}_h))$ and $\beta_0(d-1) \geq 1$ for $s > 3/2$. There exists a positive constant C such that for any $t \in [0,T]$*

$$\|\dot{\chi}\|_{L_\infty(0,T;L_2(\Omega))} + \|\chi\|_{L_\infty(0,T;\mathcal{V})} \leq C h^{\min(k+1,s)-1}$$

when α_0 is large enough. Moreover, if Ω is convex and $\beta_0(d-1) \geq 3$

$$\|\dot{\chi}\|_{L_\infty(0,T;L_2(\Omega))} + \|\chi\|_{L_\infty(0,T;\mathcal{V})} \leq C h^{\min(k+1,s)}.$$

Proof. For any $v \in \mathcal{D}_k(\mathcal{E}_h)$, by subtraction from (3.1.1) to (3.2.6),

$$\begin{aligned}
&(\rho(\ddot{u}(t) - \ddot{u}_h(t)), v)_{L_2(\Omega)} + a_1(u(t) - u_h(t), v) - \sum_{q=1}^{N_\varphi} a_{-1}(\psi_q(t) - \psi_{hq}(t), v) \\
&+ J_0^{\alpha_0,\beta_0}(\dot{u}(t) - \dot{u}_h(t), v) = 0.
\end{aligned}$$

Hence adding zeros with using DG elliptic operators yields

$$\begin{aligned}
&(\rho(\ddot{u}(t) - R_1\ddot{u}(t) - (\ddot{u}_h(t) - R_1\ddot{u}(t))), v)_{L_2(\Omega)} + a_1(u(t) - R_1 u(t) - (u_h(t) - R_1 u(t)), v) \\
&- \sum_{q=1}^{N_\varphi} a_{-1}(\psi_q(t) - R_{-1}\psi_q(t) - (\psi_{hq}(t) - R_{-1}\psi_q(t)), v) \\
&+ J_0^{\alpha_0,\beta_0}(\dot{u}(t) - R_1\dot{u}(t) - (\dot{u}_h(t) - R_1\dot{u}(t)), v) = 0
\end{aligned}$$

so that

$$\left(\rho\ddot{\chi}(t),v\right)_{L_2(\Omega)}+a_1\left(\chi(t),v\right)-\sum_{q=1}^{N_\varphi}a_{-1}\left(\varsigma_q(t),v\right)+J_0^{\alpha_0,\beta_0}(\dot{\chi}(t),v)$$
$$=\Big(\rho\ddot{\theta}(t),v\Big)_{L_2(\Omega)}+a_1\left(\theta(t),v\right)-\sum_{q=1}^{N_\varphi}a_{-1}\left(\vartheta_q(t),v\right)+J_0^{\alpha_0,\beta_0}(\dot{\theta}(t),v).$$

When we apply Galerkin orthogonality,

$$\left(\rho\ddot{\chi}(t),v\right)_{L_2(\Omega)}+a_1\left(\chi(t),v\right)-\sum_{q=1}^{N_\varphi}a_{-1}\left(\varsigma_q(t),v\right)+J_0^{\alpha_0,\beta_0}(\dot{\chi}(t),v)$$
$$=\Big(\rho\ddot{\theta}(t),v\Big)_{L_2(\Omega)}+J_0^{\alpha_0,\beta_0}(\dot{\theta}(t),v).$$

If we put $v=\dot{\chi}(t)$, we have

$$\frac{\rho}{2}\frac{d}{dt}\left\|\dot{\chi}(t)\right\|^2_{L_2(\Omega)}+\frac{1}{2}\frac{d}{dt}\left\|\chi(t)\right\|^2_{\mathcal{V}}-\sum_{q=1}^{N_\varphi}a_{-1}\left(\varsigma_q(t),\dot{\chi}(t)\right)+J_0^{\alpha_0,\beta_0}(\dot{\chi}(t),\dot{\chi}(t))$$
$$=\Big(\rho\ddot{\theta}(t),\dot{\chi}(t)\Big)_{L_2(\Omega)}+\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e\{D\nabla\chi(t)\cdot\underline{n}_e\}[\dot{\chi}(t)]de$$
$$-\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e\{D\nabla\dot{\chi}(t)\cdot\underline{n}_e\}[\chi(t)]de+J_0^{\alpha_0,\beta_0}(\dot{\theta}(t),\dot{\chi}(t)). \quad (3.2.18)$$

Also subtracting (3.1.2) from (3.2.7)

$$a_{-1}\left(\tau_q\dot{\varsigma}_q(t)+\varsigma_q(t),v\right)-\varphi_q a_{-1}\left(\chi(t),v\right)=a_{-1}\Big(\tau_q\dot{\vartheta}_q(t)+\vartheta_q(t),v\Big)-\varphi_q a_{-1}\left(\theta(t),v\right)$$
$$=-\varphi_q a_{-1}\left(\theta(t),v\right).$$

by Galerkin orthogonality for any $v\in\mathcal{D}_k(\mathcal{E}_h)$, $\forall q$. Inserting $v=\dot{\varsigma}_q(t)$ gives

$$\tau_q\left\|\dot{\varsigma}_q(t)\right\|^2_{\mathcal{V}}+\frac{1}{2}\frac{d}{dt}a_{-1}\left(\varsigma_q(t),\varsigma_q(t)\right)-\varphi_q a_{-1}\left(\chi(t),\dot{\varsigma}_q(t)\right)=-\varphi_q a_{-1}\left(\theta(t),\dot{\varsigma}_q(t)\right).$$

Hence taking integration with respect to time and using integration by parts yield for each q

$$-\int_0^t a_{-1}\left(\varsigma_q(t'),\dot{\chi}(t')\right)dt'=\frac{\tau_q}{\varphi_q}\int_0^t\left\|\dot{\varsigma}_q(t')\right\|^2_{\mathcal{V}}dt'+\frac{1}{2\varphi_q}a_{-1}\left(\varsigma_q(t),\varsigma_q(t)\right)-a_{-1}\left(\varsigma_q(t),\chi(t)\right)$$
$$+\int_0^t a_{-1}\left(\theta(t'),\dot{\varsigma}_q(t')\right)dt'. \quad (3.2.19)$$

Turning back to (3.2.18), taking integration over time gives

$$\begin{aligned}
&\frac{\rho}{2}\left\|\dot{\chi}(t)\right\|_{L_2(\Omega)}^2+\frac{1}{2}\left\|\chi(t)\right\|_{\mathcal{V}}^2-\sum_{q=1}^{N_\varphi}\int_0^t a_{-1}\left(\varsigma_q(t'),\dot{\chi}(t')\right)dt'+\int_0^t J_0^{\alpha_0,\beta_0}(\dot{\chi}(t'),\dot{\chi}(t'))dt'\\
=&\frac{\rho}{2}\left\|\dot{\chi}(0)\right\|_{L_2(\Omega)}^2+\frac{1}{2}\left\|\chi(0)\right\|_{\mathcal{V}}^2+\int_0^t\left(\rho\ddot{\theta}(t'),\dot{\chi}(t')\right)_{L_2(\Omega)}dt'+\int_0^t J_0^{\alpha_0,\beta_0}(\dot{\theta}(t'),\dot{\chi}(t'))dt'\\
&+\int_0^t\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e\{D\nabla\chi(t')\cdot\underline{n}_e\}[\dot{\chi}(t')]de\,dt'-\int_0^t\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e\{D\nabla\dot{\chi}(t')\cdot\underline{n}_e\}[\chi(t')]de\,dt'.
\end{aligned}$$

By (3.2.19), we have

$$\begin{aligned}
&\frac{\rho}{2}\left\|\dot{\chi}(t)\right\|_{L_2(\Omega)}^2+\frac{1}{2}\left\|\chi(t)\right\|_{\mathcal{V}}^2+\sum_{q=1}^{N_\varphi}\frac{1}{2\varphi_q}a_{-1}\left(\varsigma_q(t),\varsigma_q(t)\right)+\sum_{q=1}^{N_\varphi}\int_0^t\frac{\tau_q}{\varphi_q}a_{-1}\left(\dot{\varsigma}_q(t'),\dot{\varsigma}_q(t')\right)dt'\\
&+\int_0^t J_0^{\alpha_0,\beta_0}(\dot{\chi}(t'),\dot{\chi}(t'))dt'\\
=&\frac{\rho}{2}\left\|\dot{\chi}(0)\right\|_{L_2(\Omega)}^2+\frac{1}{2}\left\|\chi(0)\right\|_{\mathcal{V}}^2+\int_0^t\left(\rho\ddot{\theta}(t'),\dot{\chi}(t')\right)_{L_2(\Omega)}dt'+\sum_{q=1}^{N_\varphi}a_{-1}\left(\varsigma_q(t),\chi(t)\right)\\
&+\int_0^t\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e\{D\nabla\chi(t')\cdot\underline{n}_e\}[\dot{\chi}(t')]de\,dt'-\int_0^t\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e\{D\nabla\dot{\chi}(t')\cdot\underline{n}_e\}[\chi(t')]de\,dt'\\
&-\sum_{q=1}^{N_\varphi}\int_0^t a_{-1}\left(\theta(t'),\dot{\varsigma}_q(t')\right)dt'+\int_0^t J_0^{\alpha_0,\beta_0}(\dot{\theta}(t'),\dot{\chi}(t'))dt',
\end{aligned}$$

and the coercivity and definition of SIPG imply

$$\begin{aligned}
&\frac{\rho}{2}\left\|\dot{\chi}(t)\right\|_{L_2(\Omega)}^2+\frac{1}{2}\left\|\chi(t)\right\|_{\mathcal{V}}^2+\sum_{q=1}^{N_\varphi}\frac{1}{2\varphi_q}\left\|\varsigma_q(t)\right\|_{\mathcal{V}}^2+\sum_{q=1}^{N_\varphi}\int_0^t\frac{\kappa\tau_q}{\varphi_q}\left\|\dot{\varsigma}_q(t')\right\|_{\mathcal{V}}^2dt'\\
&+\int_0^t J_0^{\alpha_0,\beta_0}(\dot{\chi}(t'),\dot{\chi}(t'))dt'\\
\leq&\left|\frac{\rho}{2}\left\|\dot{\chi}(0)\right\|_{L_2(\Omega)}^2+\frac{1}{2}\left\|\chi(0)\right\|_{\mathcal{V}}^2+\int_0^t\left(\rho\ddot{\theta}(t'),\dot{\chi}(t')\right)_{L_2(\Omega)}dt'+\sum_{q=1}^{N_\varphi}a_{-1}\left(\varsigma_q(t),\chi(t)\right)\right.\\
&+\int_0^t\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e\{D\nabla\chi(t')\cdot\underline{n}_e\}[\dot{\chi}(t')]de\,dt'-\int_0^t\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e\{D\nabla\dot{\chi}(t')\cdot\underline{n}_e\}[\chi(t')]de\,dt'\\
&-\sum_{q=1}^{N_\varphi}\int_0^t a_{-1}\left(\theta(t'),\dot{\varsigma}_q(t')\right)dt'+\int_0^t J_0^{\alpha_0,\beta_0}(\dot{\theta}(t'),\dot{\chi}(t'))dt'\\
&\left.+\sum_{q=1}^{N_\varphi}\frac{1}{\varphi_q}\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e\{D\nabla\varsigma_q(t)\cdot\underline{n}_e\}[\varsigma_q(t)]de\right|.
\end{aligned}$$

Note that for e such that is a shared edge between elements E_1 and E_2,

$$\begin{aligned}[\theta(t')]|_e =& u(t')|_{E_1^e} - R_1 u(t')|_{E_1^e} - (u(t')|_{E_2^e} - R_1 u(t')|_{E_2^e}) \\ =& u(t')|_{E_1^e} - u(t')|_{E_2^e} - R_1(u(t')|_{E_1^e} - u(t')|_{E_2^e}) \\ =& 0 - R_1(0) = 0\end{aligned}$$

for all $0 \leq s \leq t \leq T$ since $u(t')$ is continuous on Ω and hence

$$[\theta(t')] = 0 \text{ on } \Gamma_h \cup \Gamma_D,\ \forall s \tag{3.2.20}$$

by Dirichlet condition. In this manner, we can also have

$$[\dot{\theta}(t')] = 0 \text{ on } \Gamma_h \cup \Gamma_D,\ \forall s. \tag{3.2.21}$$

Use of Cauchy-Schwarz inequality, Young's inequality, (3.2.1), (3.2.2), (3.2.5), (3.2.20) and (3.2.21) makes some bounds as follows:

- $\left| \int_0^t \left(\rho\ddot{\theta}(t'), \dot{\chi}(t')\right)_{L_2(\Omega)} dt' \right|$

$$\begin{aligned}\left| \int_0^t \left(\rho\ddot{\theta}(t'), \dot{\chi}(t')\right)_{L_2(\Omega)} dt' \right| \leq& \int_0^t \left\|\rho\ddot{\theta}(t')\right\|_{L_2(\Omega)} \|\dot{\chi}(t')\|_{L_2(\Omega)} dt' \\ \leq& \|\dot{\chi}\|_{L_\infty(0,T;L_2(\Omega))} \int_0^t \left\|\rho\ddot{\theta}(t')\right\|_{L_2(\Omega)} dt' \\ \leq& \|\dot{\chi}\|_{L_\infty(0,T;L_2(\Omega))} \rho\sqrt{T} \left(\int_0^T \left\|\ddot{\theta}(t')\right\|^2_{L_2(\Omega)} dt'\right)^{1/2} \\ \leq& \frac{\epsilon_a}{2} \|\dot{\chi}\|^2_{L_\infty(0,T;L_2(\Omega))} + \frac{\rho^2 T}{2\epsilon_a} \left\|\ddot{\theta}\right\|^2_{L_2(0,T;L_2(\Omega))}\end{aligned}$$

  for any positive ϵ_a.

- $\left| \sum_{q=1}^{N_\varphi} a_{-1}(\varsigma_q(t), \chi(t)) \right|$

$$\begin{aligned}\left| \sum_{q=1}^{N_\varphi} a_{-1}(\varsigma_q(t), \chi(t)) \right| \leq& \sum_{q=1}^{N_\varphi} \|\varsigma_q(t)\|_{\mathcal{V}} \|\chi(t)\|_{\mathcal{V}} + \frac{C}{\sqrt{\alpha_0}} \sum_{q=1}^{N_\varphi} (\|\chi(t)\|^2_{\mathcal{V}} + \|\varsigma_q(t)\|^2_{\mathcal{V}}) \\ \leq& \sum_{q=1}^{N_\varphi} (\frac{C}{\sqrt{\alpha_0}} + \frac{\epsilon_q}{2}) \|\chi(t)\|^2_{\mathcal{V}} + \sum_{q=1}^{N_\varphi} (\frac{C}{\sqrt{\alpha_0}} + \frac{1}{2\epsilon_q}) \|\varsigma_q(t)\|^2_{\mathcal{V}}\end{aligned}$$

  for any positive ϵ_q for each q.

- $\left| \int_0^t \sum_{e\subset\Gamma_h\cup\Gamma_D} \int_e \{D\nabla\chi(t') \cdot \underline{n}_e\}[\dot{\chi}(t')] de dt' - \int_0^t \sum_{e\subset\Gamma_h\cup\Gamma_D} \int_e \{D\nabla\dot{\chi}(t') \cdot \underline{n}_e\}[\chi(t')] de dt' \right|$

$$\left| \int_0^t \sum_{e\subset\Gamma_h\cup\Gamma_D} \int_e \{D\nabla\chi(t') \cdot \underline{n}_e\}[\dot{\chi}(t')] de dt' - \int_0^t \sum_{e\subset\Gamma_h\cup\Gamma_D} \int_e \{D\nabla\dot{\chi}(t') \cdot \underline{n}_e\}[\chi(t')] de dt' \right|$$

$$
\begin{aligned}
=&\Bigg|2\int_0^t \sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e \{D\nabla\chi(t')\cdot\underline{n}_e\}[\dot{\chi}(t')]dedt' - \sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e \{D\nabla\chi(t)\cdot\underline{n}_e\}[\chi(t)]de\\
&+\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e \{D\nabla\chi(0)\cdot\underline{n}_e\}[\chi(0)]de\Bigg|\\
&(\because\ \text{by integration by parts})\\
\leq&\frac{2C}{\sqrt{\alpha_0}}\int_0^t \left\|\chi(t')\right\|_{\mathcal{V}}^2 + J_0^{\alpha_0,\beta_0}(\dot{\chi}(t'),\dot{\chi}(t'))dt' + \frac{C}{\sqrt{\alpha_0}}(\|\chi(t)\|_{\mathcal{V}}^2+\|\chi(0)\|_{\mathcal{V}}^2)\\
&(\because\ \text{use of (3.2.1) and (3.2.2)})\\
\leq&\frac{2CT}{\sqrt{\alpha_0}}\|\chi\|_{L_\infty(0,T;\mathcal{V})}^2 + \frac{2C}{\sqrt{\alpha_0}}\int_0^t J_0^{\alpha_0,\beta_0}(\dot{\chi}(t'),\dot{\chi}(t'))dt'\\
&+\frac{C}{\sqrt{\alpha_0}}(\|\chi(t)\|_{\mathcal{V}}^2+\|\chi(0)\|_{\mathcal{V}}^2).
\end{aligned}
$$

- $\left|\sum_{q=1}^{N_\varphi}\int_0^t a_{-1}\left(\theta(t'),\dot{\varsigma}_q(t')\right)dt'\right|$

  By the definition of the DG bilinear form,

  $$a_{-1}\left(\theta(t'),\dot{\varsigma}_q(t')\right) = a_1\left(\theta(t'),\dot{\varsigma}_q(t')\right) - 2\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e\{D\nabla\dot{\varsigma}_q(t')\cdot\underline{n}_e\}[\theta(t')]de$$

  and Galerkin orthogonality for NIPG yields

  $$a_{-1}\left(\theta(t'),\dot{\varsigma}_q(t')\right) = -2\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e\{D\nabla\dot{\varsigma}_q(t')\cdot\underline{n}_e\}[\theta(t')]de.$$

  Hence (3.2.20) implies

  $$a_{-1}\left(\theta(t'),\dot{\varsigma}_q(t')\right) = 0,$$

  and thus

  $$\left|\sum_{q=1}^{N_\varphi}\int_0^t a_{-1}\left(\theta(t'),\dot{\varsigma}_q(t')\right)dt'\right| = 0.$$

- $\left|\int_0^t J_0^{\alpha_0,\beta_0}(\dot{\theta}(t'),\dot{\chi}(t'))dt'\right|$

  (3.2.21) gives $J_0^{\alpha_0,\beta_0}(\dot{\theta}(t'),v)=0$ for any $v\in\mathcal{D}_k(\mathcal{E}_h)$ so that

  $$\left|\int_0^t J_0^{\alpha_0,\beta_0}(\dot{\theta}(t'),\dot{\chi}(t'))dt'\right| = 0.$$

- $\left|\sum_{q=1}^{N_\varphi}\frac{1}{\varphi_q}\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e\{D\nabla\varsigma_q(t)\cdot\underline{n}_e\}[\varsigma_q(t)]de\right|$

  By (3.2.1),

  $$\left|\sum_{q=1}^{N_\varphi}\frac{1}{\varphi_q}\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e\{D\nabla\varsigma_q(t)\cdot\underline{n}_e\}[\varsigma_q(t)]de\right| \leq \sum_{q=1}^{N_\varphi}\frac{C}{\varphi_q\sqrt{\alpha_0}}\left\|\varsigma_q(t)\right\|_{\mathcal{V}}^2.$$

Tidying up the results, we have

$$
\begin{aligned}
&\frac{\rho}{2}\left\|\dot{\chi}(t)\right\|^2_{L_2(\Omega)} + \left(\frac{1}{2} - \frac{C(1+N_\varphi)}{\sqrt{\alpha_0}} - \sum_{q=1}^{N_\varphi}\frac{\epsilon_q}{2}\right)\left\|\chi(t)\right\|^2_{\mathcal{V}} \\
&\quad + \sum_{q=1}^{N_\varphi}\left(\frac{1}{2\varphi_q} - \frac{1}{2\epsilon_q} - \frac{C(1+\varphi_q)}{\varphi_q\sqrt{\alpha_0}}\right)\left\|\varsigma_q(t)\right\|^2_{\mathcal{V}} + \sum_{q=1}^{N_\varphi}\frac{\kappa\tau_q}{\varphi_q}\int_0^t \left\|\dot{\varsigma}_q(t')\right\|^2_{\mathcal{V}} dt' \\
&\quad + \big(1 - \frac{2C}{\sqrt{\alpha_0}}\big)\int_0^t J_0^{\alpha_0,\beta_0}(\dot{\chi}(t'),\dot{\chi}(t'))dt' \\
\leq&\frac{\rho}{2}\left\|\dot{\chi}(0)\right\|^2_{L_2(\Omega)} + \big(\frac{1}{2} + \frac{C}{\sqrt{\alpha_0}}\big)\left\|\chi(0)\right\|^2_{\mathcal{V}} + \frac{\epsilon_a}{2}\left\|\dot{\chi}\right\|^2_{L_\infty(0,T;L_2(\Omega))} + \frac{\rho^2 T}{2\epsilon_a}\left\|\ddot{\theta}\right\|^2_{L_2(0,T;L_2(\Omega))} \\
&\quad + \frac{2CT}{\sqrt{\alpha_0}}\left\|\chi\right\|^2_{L_\infty(0,T;\mathcal{V})}.
\end{aligned}
$$

If we take $\epsilon_q = \varphi_q + \frac{\varphi_0}{2N_\varphi}$ for each q, then

$$
\begin{aligned}
&\frac{\rho}{2}\left\|\dot{\chi}(t)\right\|^2_{L_2(\Omega)} + \left(\frac{\varphi_0}{4} - \frac{C(1+N_\varphi)}{\sqrt{\alpha_0}}\right)\left\|\chi(t)\right\|^2_{\mathcal{V}} \\
&\quad + \sum_{q=1}^{N_\varphi}\left(\frac{\varphi_0}{4\varphi_q^2 + N_\varphi + \varphi_0} - \frac{C(1+\varphi_q)}{\varphi_q\sqrt{\alpha_0}}\right)\left\|\varsigma_q(t)\right\|^2_{\mathcal{V}} + \sum_{q=1}^{N_\varphi}\frac{\kappa\tau_q}{\varphi_q}\int_0^t \left\|\dot{\varsigma}_q(t')\right\|^2_{\mathcal{V}} dt' \\
&\quad + \big(1 - \frac{2C}{\sqrt{\alpha_0}}\big)\int_0^t J_0^{\alpha_0,\beta_0}(\dot{\chi}(t'),\dot{\chi}(t'))dt' \\
\leq&\frac{\rho}{2}\left\|\dot{\chi}(0)\right\|^2_{L_2(\Omega)} + \big(\frac{1}{2} + \frac{C}{\sqrt{\alpha_0}}\big)\left\|\chi(0)\right\|^2_{\mathcal{V}} + \frac{\epsilon_a}{2}\left\|\dot{\chi}\right\|^2_{L_\infty(0,T;L_2(\Omega))} + \frac{\rho^2 T}{2\epsilon_a}\left\|\ddot{\theta}\right\|^2_{L_2(0,T;L_2(\Omega))} \\
&\quad + \frac{2CT}{\sqrt{\alpha_0}}\left\|\chi\right\|^2_{L_\infty(0,T;\mathcal{V})}.
\end{aligned}
$$

Taking into account L_∞ norm on the left hand side, it becomes

$$
\begin{aligned}
&\frac{\rho}{2}\left\|\dot{\chi}\right\|^2_{L_\infty(0,T;L_2(\Omega))} + \left(\frac{\varphi_0}{4} - \frac{C(1+N_\varphi)}{\sqrt{\alpha_0}}\right)\left\|\chi\right\|^2_{L_\infty(0,T;\mathcal{V})} \\
&\quad + \sum_{q=1}^{N_\varphi}\left(\frac{\varphi_0}{4\varphi_q^2 + N_\varphi + \varphi_0} - \frac{C(1+\varphi_q)}{\varphi_q\sqrt{\alpha_0}}\right)\left\|\varsigma_q(t)\right\|^2_{\mathcal{V}} + \sum_{q=1}^{N_\varphi}\frac{\kappa\tau_q}{\varphi_q}\int_0^t \left\|\dot{\varsigma}_q(t')\right\|^2_{\mathcal{V}} dt' \\
&\quad + \big(1 - \frac{2C}{\sqrt{\alpha_0}}\big)\int_0^t J_0^{\alpha_0,\beta_0}(\dot{\chi}(t'),\dot{\chi}(t'))dt' \\
\leq&\frac{3\rho}{2}\left\|\dot{\chi}(0)\right\|^2_{L_2(\Omega)} + \frac{3}{2}\left\|\chi(0)\right\|^2_{\mathcal{V}} + \frac{3\epsilon_a}{2}\left\|\dot{\chi}\right\|^2_{L_\infty(0,T;L_2(\Omega))} + \frac{3\rho^2 T}{2\epsilon_a}\left\|\ddot{\theta}\right\|^2_{L_2(0,T;L_2(\Omega))} \\
&\quad + \frac{6CT}{\sqrt{\alpha_0}}\left\|\chi\right\|^2_{L_\infty(0,T;\mathcal{V})},
\end{aligned}
$$

so that

$$
\begin{aligned}
&\frac{\rho}{4}\|\dot{\chi}\|^2_{L_\infty(0,T;L_2(\Omega))} + \left(\frac{\varphi_0}{4} - \frac{C(1+N_\varphi+6T)}{\sqrt{\alpha_0}}\right)\|\chi\|^2_{L_\infty(0,T;\mathcal{V})} \\
&\quad + \sum_{q=1}^{N_\varphi}\left(\frac{\varphi_0}{4\varphi_q^2+N_\varphi+\varphi_0} - \frac{C(1+\varphi_q)}{\varphi_q\sqrt{\alpha_0}}\right)\|\varsigma_q(t)\|^2_{\mathcal{V}} + \sum_{q=1}^{N_\varphi}\frac{\kappa\tau_q}{\varphi_q}\int_0^t \left\|\dot{\varsigma}_q(t')\right\|^2_{\mathcal{V}}dt' \\
&\quad + \left(1 - \frac{2C}{\sqrt{\alpha_0}}\right)\int_0^t J_0^{\alpha_0,\beta_0}(\dot{\chi}(t'),\dot{\chi}(t'))dt' \\
&\leq \frac{3\rho}{2}\|\dot{\chi}(0)\|^2_{L_2(\Omega)} + \frac{3}{2}\|\chi(0)\|^2_{\mathcal{V}} + 9\rho T\left\|\ddot{\theta}\right\|^2_{L_2(0,T;L_2(\Omega))},
\end{aligned}
$$

when $\epsilon_a = \rho/6$. Furthermore, from (3.2.8) and elliptic projection,

$$
\begin{aligned}
\forall v \in \mathcal{D}_k(\mathcal{E}_h)),\ a_1(\chi(0),v) &= a_1(\chi(0),v) - 0 \\
&= a_1(\chi(0),v) - a_1(\theta(0),v) \\
&= a_1(\chi(0)-\theta(0),v) \\
&= a_1(u_h(0) - R_1u_0 - (u_0 - R_1u_0),v) \\
&= a_1(u_h(0) - u_0, v) \\
&= 0,
\end{aligned}
$$

so

$$
\|\chi(0)\|^2_{\mathcal{V}} = a_1(\chi(0),\chi(0)) = 0.
$$

Similarly, (3.2.9) implies

$$
\begin{aligned}
\|\dot{\chi}(0)\|^2_{L_2(\Omega)} &= (\dot{\chi}(0),\dot{\chi}(0))_{L_2(\Omega)} \\
&= (\dot{u}_h(0) - R_1w_0, \dot{u}_h(0) - R_1w_0)_{L_2(\Omega)} \\
&= (\dot{u}_h(0) - R_1w_0, w_0 - R_1w_0)_{L_2(\Omega)} \\
&\leq \|\dot{u}_h(0) - R_1w_0\|_{L_2(\Omega)}\|w_0 - R_1w_0\|_{L_2(\Omega)} \\
&= \|\dot{\chi}(0)\|_{L_2(\Omega)}\left\|\dot{\theta}(0)\right\|_{L_2(\Omega)},
\end{aligned}
$$

and so

$$
\|\dot{\chi}(0)\|_{L_2(\Omega)} \leq \left\|\dot{\theta}(0)\right\|_{L_2(\Omega)}.
$$

Note that Theorem 3.7 provides that

$$
\left\|\dot{\theta}(0)\right\|_{L_2(\Omega)}, \left\|\ddot{\theta}\right\|_{L_2(0,T;L_2(\Omega))} = O(h^{\min(k+1,s)-1})
$$

since $u \in H^2(0,T;C^2(\Omega))$. Therefore, there exists a positive constant C such that for any $t \in [0,T]$

$$
\|\dot{\chi}\|^2_{L_\infty(0,T;L_2(\Omega))} + \|\chi\|^2_{L_\infty(0,T;\mathcal{V})} + \sum_{q=1}^{N_\varphi}\|\varsigma_q(t)\|^2_{\mathcal{V}} + \sum_{q=1}^{N_\varphi}\int_0^t \left\|\dot{\varsigma}_q(t')\right\|^2_{\mathcal{V}}dt'
$$

$$
\begin{aligned}
&+\int_0^t J_0^{\alpha_0,\beta_0}(\dot\chi(t'),\dot\chi(t'))dt'\\
\le& Ch^{2(\min(k+1,s)-1)}
\end{aligned}
$$

when sufficiently large α_0 is given as

$$
\frac{\varphi_0}{4}-\frac{C(1+N_\varphi+6T)}{\sqrt{\alpha_0}}>0, \qquad 1-\frac{2C}{\sqrt{\alpha_0}},
$$

and

$$
\frac{\varphi_0}{4\varphi_q^2+N_\varphi+\varphi_0}-\frac{C(1+\varphi_q)}{\varphi_q\sqrt{\alpha_0}}>0, \qquad \forall q\in\{1,\dots,N_\varphi\}.
$$

Hence we can also prove

$$
\|\dot\chi\|_{L_\infty(0,T;L_2(\Omega))}+\|\chi\|_{L_\infty(0,T;\mathcal{V})}\le Ch^{\min(k+1,s)-1}.
$$

Furthermore, if our domain Ω is convex and $\beta_0(d-1)\ge 3$, Theorem 3.7 leads us to have

$$
\left\|\dot\theta(0)\right\|_{L_2(\Omega)},\left\|\ddot\theta\right\|_{L_2(0,T;L_2(\Omega))}=O(h^{\min(k+1,s)}).
$$

A sufficiently large α_0 enables us to conclude

$$
\|\dot\chi\|_{L_\infty(0,T;L_2(\Omega))}+\|\chi\|_{L_\infty(0,T;\mathcal{V})}\le Ch^{\min(k+1,s)}.
$$

□

Theorem 3.8. *Under the same conditions in* Lemma 3.2*, we have*

$$
\begin{aligned}
\|u-u_h\|_{L_\infty(0,T;\mathcal{V})}&\le Ch^{\min(k+1,s)-1},\\
\|\dot u-\dot u_h\|_{L_\infty(0,T;L_2(\Omega))}&\le Ch^{\min(k+1,s)-1}
\end{aligned}
$$

Moreover, if Ω is convex and $\beta_0(d-1)\ge 3$, then

$$
\|\dot u-\dot u_h\|_{L_\infty(0,T;L_2(\Omega))}\le Ch^{\min(k+1,s)}.
$$

Proof.

$$
\begin{aligned}
\|u-u_h\|_{L_\infty(0,T;\mathcal{V})}&=\|u-R_1u-(u_h-R_1u)\|_{L_\infty(0,T;\mathcal{V})}\\
&\le\|\theta\|_{L_\infty(0,T;\mathcal{V})}+\|\chi\|_{L_\infty(0,T;\mathcal{V})}
\end{aligned}
$$

by triangular inequality. (3.2.15) and Lemma 3.2 yield

$$
\|u-u_h\|_{L_\infty(0,T;\mathcal{V})}\le Ch^{\min(k+1,s)-1}.
$$

In this same sense, by Lemma 3.2 and (3.2.16)

$$
\|\dot u-\dot u_h\|_{L_\infty(0,T;L_2(\Omega))}=\|\dot u-R_1\dot u-(\dot u_h-R_1\dot u)\|_{L_\infty(0,T;L_2(\Omega))}
$$

$$
\begin{aligned}
&\leq \left\|\dot{\theta}\right\|_{L_\infty(0,T;L_2(\Omega))} + \|\dot{\chi}\|_{L_\infty(0,T;L_2(\Omega))} \\
&\leq Ch^{\min(k+1,s)-1},
\end{aligned}
$$

and if Ω is convex and $\beta_0(d-1) \geq 3$, (3.2.17) gives

$$
\|\dot{u} - \dot{u}_h\|_{L_\infty(0,T;L_2(\Omega))} \leq Ch^{\min(k+1,s)}.
$$

□

Corollary 3.1. *Under the same conditions for* Lemma 3.2*, we have*

$$
\|u - u_h\|_{L_\infty(0,T;L_2(\Omega))} \leq Ch^{\min(k+1,s)-1}.
$$

Moreover, if Ω *is convex and* $\beta_0(d-1) \geq 3$

$$
\|u - u_h\|_{L_\infty(0,T;L_2(\Omega))} \leq Ch^{\min(k+1,s)}.
$$

Proof. It is easy to show. By Theorem 1.12, (1.4.10) implies

$$
\|v\|_{L_2(\Omega)} \leq C\,\|v\|_{\mathcal{V}}\,,\ \forall v \in H^s(\mathcal{E}_h)
$$

for some positive C. Hence, by Lemma 3.2 and (3.2.16)

$$
\begin{aligned}
\|u - u_h\|_{L_\infty(0,T;L_2(\Omega))} &= \|u - R_1 u - (u_h - R_1 u)\|_{L_\infty(0,T;L_2(\Omega))} \\
&\leq \|\theta\|_{L_\infty(0,T;L_2(\Omega))} + \|\chi\|_{L_\infty(0,T;L_2(\Omega))} \\
&\leq \|\theta\|_{L_\infty(0,T;L_2(\Omega))} + C\,\|\chi\|_{L_\infty(0,T;\mathcal{V})} \\
&\leq Ch^{\min(k+1,s)-1}.
\end{aligned}
$$

In a similar way, if Ω is convex and $\beta_0(d-1) \geq 3$, Lemma 3.2 and (3.2.17) lead

$$
\|u - u_h\|_{L_\infty(0,T;L_2(\Omega))} \leq Ch^{\min(k+1,s)}.
$$

□

3.2.2 Velocity Form

As seen in the previous section, we can also have the semidiscrete formulation of **(Q2)**: Find $u_h(t)$ and $\{\zeta_{hq}(t)\}_{q=1}^{N_\varphi}$ such that for all $v \in \mathcal{D}_k(\mathcal{E}_h)$

$$
(\rho\ddot{u}_h(t), v)_{L_2(\Omega)} + \varphi_0 a_1\left(u_h(t), v\right) + \sum_{q=1}^{N_\varphi} a_{-1}\left(\zeta_{hq}(t), v\right) + J_0^{\alpha_0,\beta_0}(\dot{u}_h(t), v) = F_v(t; v), \tag{3.2.22}
$$

$$
a_{-1}\left(\tau_q\dot{\zeta}_{hq}(t) + \zeta_{hq}(t), v\right) = a_{-1}\left(\tau_q\varphi_q\dot{u}_h(t), v\right), \tag{3.2.23}
$$

$$a_1\left(u_h(0), v\right) = a_1\left(u_0, v\right), \tag{3.2.24}$$

$$\left(\dot{u}_h(0), v\right)_{L_2(\Omega)} = \left(w_0, v\right)_{L_2(\Omega)}, \tag{3.2.25}$$

where $\zeta_{hq}(0) = 0, \forall q \in \{1, \ldots, N_\varphi\}$, $\forall t$. Hence our approximate solutions can be expressed by

$$u_h(\boldsymbol{x}, t) = \sum_{i=1}^{N_{Vh}} \mathfrak{u}_i(t)\phi_i(\boldsymbol{x}),$$

$$\zeta_{hq}(\boldsymbol{x}, t) = \sum_{i=1}^{N_{Vh}} \zeta_{hq,i}(t)\phi_i(\boldsymbol{x}),\ \forall q \in \{1, \ldots, N_\varphi\}.$$

The semidiscrete problem for **(Q2)** and the strong form of internal variables present also a ODEs system and is solved uniquely if stability bounds given. The resulting ODE system is given as

$$\begin{gathered}
\rho M\ddot{\underline{\mathfrak{u}}}(t) + \varphi_0 A\underline{\mathfrak{u}}(t) + \sum_{q=1}^{N_\varphi} A^*\underline{\zeta}_{hq}(t) + \mathcal{J}\dot{\underline{\mathfrak{u}}}(t) = \underline{\tilde{F}}(t),\\
\tau_q\dot{\underline{\zeta}}_{hq}(t) + \underline{\zeta}_{hq}(t) = \tau_q\varphi_q\dot{\underline{\mathfrak{u}}}(t),\ \text{for each } q,\\
A\underline{\mathfrak{u}}(0) = \underline{U}_0,\\
M\dot{\underline{\mathfrak{u}}}(0) = \underline{W}_0,\\
\underline{\zeta}_{hq}(0) = \underline{0},\ \text{for each } q,
\end{gathered}$$

where $(\underline{\tilde{F}}(t))_i = F_v(t; \phi_i)$ for $i = 1, \ldots, N_{V^h}$, M is the mass matrix, A and A^* are the stiffness matrix governed by the DG bilinear forms, and $\mathcal{J}$ is the jump matrix from the jump operator $J_0^{\alpha_0,\beta_0}$. In a similar way with the proof of Theorem 3.6, we will show the stability bounds for the existence and uniqueness of the semidiscrete solution of **(Q2)**.

Lemma 3.3.
Suppose $u_h \in H^2(0, T; L_2(\Omega)) \cap H^1(0, T; \mathcal{D}_k(\mathcal{E}_h))$ and $\zeta_{hq} \in H^1(0, T; \mathcal{D}_k(\mathcal{E}_h))$, for each $q \in \{1, \ldots, N_\varphi\}$. Then we have for any $0 \le t \le T$,

$$\begin{aligned}
\int_0^t a_{-1}\left(\zeta_{hq}(t'), \dot{u}_h(t')\right) dt' =& \frac{1}{2\varphi_q}\int_0^t a_{-1}\left(\zeta_{hq}(t'), \zeta_{hq}(t')\right) dt'\\
&+ \frac{1}{\tau_q\varphi_q}\int_0^t a_{-1}\left(\zeta_{hq}(t'), \zeta_{hq}(t')\right) dt'.
\end{aligned}$$

Proof. Put $v = \zeta_{hq}(t')$ into (3.2.23). Then Leibniz's integral rule gives

$$\begin{aligned}
&\tau_q a_{-1}\left(\dot{\zeta}_{hq}(t'), \zeta_{hq}(t')\right) + a_{-1}\left(\zeta_{hq}(t'), \zeta_{hq}(t')\right)\\
=&\frac{\tau_q}{2}\frac{d}{dt'}a_{-1}\left(\zeta_{hq}(t'), \zeta_{hq}(t')\right) + a_{-1}\left(\zeta_{hq}(t'), \zeta_{hq}(t')\right)
\end{aligned}$$

$$=\tau_q\varphi_q a_{-1}\left(\dot{u}_h(t'),\zeta_{hq}(t')\right).$$

While taking into account integration from $t'=0$ to $t'=t$ for $0\le t\le T$,

$$\begin{aligned}
&\frac{\tau_q}{2}\left(a_{-1}\left(\zeta_{hq}(t),\zeta_{hq}(t)\right)-a_{-1}\left(\zeta_{hq}(0),\zeta_{hq}(0)\right)\right)+\int_0^t a_{-1}\left(\zeta_{hq}(t'),\zeta_{hq}(t')\right)dt'\\
=&\tau_q\varphi_q\int_0^t a_{-1}\left(\dot{u}_h(t'),\zeta_{hq}(t')\right)dt'\\
=&\tau_q\varphi_q\int_0^t a_{-1}\left(\zeta_{hq}(t'),\dot{u}_h(t')\right)dt'.
\end{aligned}$$

Since $\zeta_{hq}(0)=0,\ \forall q\in\{1,\dots,N_\varphi\}$,

$$\int_0^t a_{-1}\left(\zeta_{hq}(t'),\dot{u}_h(t')\right)dt'=\frac{1}{2\varphi_q}a_{-1}\left(\zeta_{hq}(t),\zeta_{hq}(t)\right)+\frac{1}{\tau_q\varphi_q}\int_0^t a_{-1}\left(\zeta_{hq}(t'),\zeta_{hq}(t')\right)dt'.$$

□

Theorem 3.9. *Suppose*

$$\begin{aligned}
u_h\in&H^2(0,T;L_2(\Omega))\cap W^1_\infty(0,T;\mathcal{D}_k(\mathcal{E}_h)),\\
\zeta_{hq}\in&W^1_\infty(0,T;\mathcal{D}_k(\mathcal{E}_h)),\ \forall q\in\{1,\dots,N_\varphi\}.
\end{aligned}$$

If $\beta_0(d-1)\ge 1$, then there exists a positive constant C such that depends on T and Ω but is independent of h_E, for any E with satisfying

$$\begin{aligned}
&\left\|\rho^{1/2}\dot{u}_h\right\|^2_{L_\infty(0,T;L_2(\Omega))}+\|u_h\|^2_{L_\infty(0,T;\mathcal{V})}+\sum_{q=1}^{N_\varphi}\|\zeta_{hq}(t)\|^2_{\mathcal{V}}+\sum_{q=1}^{N_\varphi}\int_0^t\left\|\zeta_{hq}(t')\right\|^2_{\mathcal{V}}dt'\\
&+\int_0^t J_0^{\alpha_0,\beta_0}(\dot{u}_h(t'),\dot{u}_h(t'))dt'\\
\le&C\Bigg(\left\|\rho^{1/2}w_0\right\|^2_{L_2(\Omega)}+\|u_0\|^2_{\mathcal{V}}+\|f\|^2_{L_2(0,T;L_2(\Omega))}+h^{-1}\|g_N\|^2_{L_\infty(0,T;L_2(\Gamma_N))}\\
&+h^{-1}\|\dot{g}_N\|_{L_2(0,T;L_2(\Gamma_N))}\Bigg)
\end{aligned}$$

for any $t\in[0,T]$ with sufficiently large α_0.

Proof. For any $0\le s\le t\le T$, let $v=\dot{u}_h(t')$. (3.2.22) becomes

$$\begin{aligned}
&\left(\rho\ddot{u}_h(t'),\dot{u}_h(t')\right)_{L_2(\Omega)}+\varphi_0 a_1\left(u_h(t'),\dot{u}_h(t')\right)+\sum_{q=1}^{N_\varphi}a_{-1}\left(\zeta_{hq}(t'),\dot{u}_h(t')\right)\\
&+J_0^{\alpha_0,\beta_0}(\dot{u}_h(t'),\dot{u}_h(t'))\\
&=F_v(t';\dot{u}_h(t')).
\end{aligned}$$

Integrating from $t' = 0$ to $t' = t$ yields

$$\begin{aligned}
&\frac{\rho}{2}\left\|\dot{u}_h(t)\right\|^2_{L_2(\Omega)} + \frac{\varphi_0}{2}\left\|u_h(t)\right\|^2_{\mathcal{V}} + \sum_{q=1}^{N_\varphi}\int_0^t a_{-1}\left(\zeta_{hq}(t'), \dot{u}_h(t')\right)dt' \\
&+ \int_0^t J_0^{\alpha_0,\beta_0}(\dot{u}_h(t'), \dot{u}_h(t'))dt' \\
=&\frac{\rho}{2}\left\|\dot{u}_h(0)\right\|^2_{L_2(\Omega)} + \frac{\varphi_0}{2}\left\|u_h(0)\right\|^2_{\mathcal{V}} + \int_0^t F_v(t';\dot{u}_h(t'))dt' \\
&+ \int_0^t \sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e \{D\nabla u_h(t')\cdot \underline{n}_e\}[\dot{u}_h(t')]dedt' \\
&- \int_0^t \sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e \{D\nabla \dot{u}_h(t')\cdot \underline{n}_e\}[u_h(t')]dedt'.
\end{aligned}$$

By Lemma 3.3, we have

$$\begin{aligned}
&\frac{\rho}{2}\left\|\dot{u}_h(t)\right\|^2_{L_2(\Omega)} + \frac{\varphi_0}{2}\left\|u_h(t)\right\|^2_{\mathcal{V}} + \sum_{q=1}^{N_\varphi}\frac{1}{2\varphi_q}a_{-1}\left(\zeta_{hq}(t), \zeta_{hq}(t)\right) \\
&+ \sum_{q=1}^{N_\varphi}\frac{1}{\tau_q\varphi_q}\int_0^t a_{-1}\left(\zeta_{hq}(t'), \zeta_{hq}(t')\right)dt' + \int_0^t J_0^{\alpha_0,\beta_0}(\dot{u}_h(t'), \dot{u}_h(t'))dt' \\
=&\frac{\rho}{2}\left\|\dot{u}_h(0)\right\|^2_{L_2(\Omega)} + \frac{\varphi_0}{2}\left\|u_h(0)\right\|^2_{\mathcal{V}} + \int_0^t F_v(t';\dot{u}_h(t'))dt' \\
&+ \int_0^t \sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e \{D\nabla u_h(t')\cdot \underline{n}_e\}[\dot{u}_h(t')]dedt' \\
&- \int_0^t \sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e \{D\nabla \dot{u}_h(t')\cdot \underline{n}_e\}[u_h(t')]dedt'
\end{aligned}$$

and since SIPG is coercive,

$$\begin{aligned}
&\frac{\rho}{2}\left\|\dot{u}_h(t)\right\|^2_{L_2(\Omega)} + \frac{\varphi_0}{2}\left\|u_h(t)\right\|^2_{\mathcal{V}} + \sum_{q=1}^{N_\varphi}\frac{\kappa}{2\varphi_q}\left\|\zeta_{hq}(t)\right\|^2_{\mathcal{V}} + \sum_{q=1}^{N_\varphi}\frac{\kappa}{\tau_q\varphi_q}\int_0^t \left\|\zeta_{hq}(t')\right\|^2_{\mathcal{V}}dt' \\
&+ \int_0^t J_0^{\alpha_0,\beta_0}(\dot{u}_h(t'), \dot{u}_h(t'))dt' \\
\leq&\left|\frac{\rho}{2}\left\|\dot{u}_h(0)\right\|^2_{L_2(\Omega)} + \frac{\varphi_0}{2}\left\|u_h(0)\right\|^2_{\mathcal{V}} + \int_0^t F_v(t';\dot{u}_h(t'))dt'\right. \\
&+ \int_0^t \sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e \{D\nabla u_h(t')\cdot \underline{n}_e\}[\dot{u}_h(t')]dedt' \\
&\left.- \int_0^t \sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e \{D\nabla \dot{u}_h(t')\cdot \underline{n}_e\}[u_h(t')]dedt'\right|.
\end{aligned}$$

Consider $|\int_0^t F_v(t';\dot{u}_h(t'))dt'|$. By the definition of F_v, integration by parts and Cauchy-Schwarz inequality yield

$$\begin{aligned}
\left|\int_0^t F_v(t';\dot{u}_h(t'))dt'\right| &= \Bigg|\int_0^t \left(f(t'),\dot{u}_h(t')\right)_{L_2(\Omega)} dt' + \sum_{e\subset\Gamma_N}\int_0^t\int_e g_N(t')\dot{u}_h(t')dedt' \\
&\quad -\sum_{q=1}^{N_\varphi}\int_0^t \varphi_q e^{-t'/\tau_q} a_1\left(u_0,\dot{u}(t')\right)dt'\Bigg| \\
&= \Bigg|\int_0^t \left(f(t'),\dot{u}_h(t')\right)_{L_2(\Omega)} dt' + \sum_{e\subset\Gamma_N}\int_e g_N(t)u_h(t)de \\
&\quad -\sum_{e\subset\Gamma_N}\int_e g_N(0)u_h(0)de - \sum_{e\subset\Gamma_N}\int_0^t\int_e \dot{g}_N(t')u_h(t')dedt' \\
&\quad -\sum_{q=1}^{N_\varphi}\varphi_q e^{-t/\tau_q}a_1\left(u_0,u_h(t)\right) + \sum_{q=1}^{N_\varphi}\varphi_q a_1\left(u_0,u_h(0)\right) \\
&\quad -\sum_{q=1}^{N_\varphi}\int_0^t \frac{\varphi_q}{\tau_q}e^{-t'/\tau_q}a_1\left(u_0,u_h(t')\right)dt'\Bigg| \\
&\leq \int_0^t \left\|f(t')\right\|_{L_2(\Omega)}\left\|\dot{u}(t')\right\|_{L_2(\Omega)} dt' + \sum_{e\subset\Gamma_N}\|g_N(t)\|_{L_2(e)}\|u_h(t)\|_{L_2(e)} \\
&+\sum_{e\subset\Gamma_N}\|g_N(0)\|_{L_2(e)}\|u_h(0)\|_{L_2(e)} \\
&\quad +\sum_{e\subset\Gamma_N}\int_0^t \left\|\dot{g}_N(t')\right\|_{L_2(e)}\left\|u_h(t')\right\|_{L_2(e)}dt' \\
&\quad +\sum_{q=1}^{N_\varphi}\varphi_q K\|u_0\|_{\mathcal{V}}\|u_h(t)\|_{\mathcal{V}} + \sum_{q=1}^{N_\varphi}\varphi_q K\|u_0\|_{\mathcal{V}}\|u_h(0)\|_{\mathcal{V}} \\
&\quad +\sum_{q=1}^{N_\varphi}\int_0^t \frac{\varphi_q}{\tau_q}K\|u_0\|_{\mathcal{V}}\left\|u_h(t')\right\|_{\mathcal{V}}dt' \\
&\leq \int_0^t \left\|f(t')\right\|_{L_2(\Omega)}\left\|\dot{u}(t')\right\|_{L_2(\Omega)} dt' + \sum_{e\subset\Gamma_N}\|g_N(t)\|_{L_2(e)}\|u_h(t)\|_{L_2(e)} \\
&+\sum_{e\subset\Gamma_N}\|g_N(0)\|_{L_2(e)}\|u_h(0)\|_{L_2(e)} \\
&\quad +\sum_{e\subset\Gamma_N}\int_0^t \left\|\dot{g}_N(t')\right\|_{L_2(e)}\left\|u_h(t')\right\|_{L_2(e)}dt' \\
&\quad +K\|u_0\|_{\mathcal{V}}\|u_h(t)\|_{\mathcal{V}} + K\|u_0\|_{\mathcal{V}}\|u_h(0)\|_{\mathcal{V}}
\end{aligned}$$

$$+\sum_{q=1}^{N_\varphi}\int_0^t \frac{\varphi_q}{\tau_q} K \|u_0\|_{\mathcal{V}} \left\|u_h(t')\right\|_{\mathcal{V}} dt'$$

since $a_1(\cdot,\cdot)$ is continuous, $0 < e^{-t/\tau_q} \le 1,\ \forall t \ge 0,\ \forall q \in \{1,\dots,N_\varphi\}$ and $\sum_{q=1}^{N_\varphi}\varphi_q < 1$. Let us consider L_∞ norm then trace inequalities, inverse inequalities and Young's inequalities imply

$$\begin{aligned}
\left|\int_0^t F_v(t';\dot{u}_h(t'))dt'\right| \le& \frac{\epsilon_a}{2}\|\dot{u}\|^2_{L_\infty(0,T;L_2(\Omega))} + \frac{1}{2\epsilon_a}\|f\|^2_{L_2(0,T;L_2(\Omega))}\\
&+ \frac{1}{2\epsilon_b}\|g_N\|^2_{L_\infty(0,T;L_2(\Gamma_N))} + Ch^{-1}\frac{\epsilon_b}{2}\|u_h\|^2_{L_\infty(0,T;\mathcal{V})}\\
&+ \frac{h^{-1}}{2}\|g_N\|^2_{L_\infty(0,T;L_2(\Gamma_N))} + \frac{C}{2}\|u_h(0)\|^2_{\mathcal{V}}\\
&+ \frac{1}{2\epsilon_c}\|\dot{g}_N\|_{L_2(0,T;L_2(\Gamma_N))} + Ch^{-1}\frac{\epsilon_c}{2}\|u_h\|^2_{L_\infty(0,T;\mathcal{V})}\\
&+ \frac{K^2}{2\epsilon_d}\|u_0\|^2_{\mathcal{V}} + \frac{\epsilon_d}{2}\|u_h\|^2_{L_\infty(0,T;\mathcal{V})} + \frac{K}{2}\|u_0\|^2_{\mathcal{V}} + \frac{K}{2}\|u_h(0)\|^2_{\mathcal{V}}\\
&+\sum_{q=1}^{N_\varphi}\frac{T^2K^2\varphi_q^2}{\tau_q^2}\frac{1}{2\epsilon_q}\|u_0\|^2_{\mathcal{V}} + \sum_{q=1}^{N_\varphi}\frac{\epsilon_q}{2}\|u_h\|^2_{L_\infty(0,T;\mathcal{V})},
\end{aligned}$$

for positive $\epsilon_a, \epsilon_b, \epsilon_c, \epsilon_d$ and $\{\epsilon_q\}$.

On the other hand, observe

$$\int_0^t \sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e \{D\nabla u_h(t')\cdot \underline{n}_e\}[\dot{u}_h(t')]dedt'.$$

By (3.2.1),

$$\begin{aligned}
&\left|\int_0^t \sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e \{D\nabla u_h(t')\cdot \underline{n}_e\}[\dot{u}_h(t')]dedt'\right|\\
\le& \frac{C}{\sqrt{\alpha_0}}\int_0^t \left\|u_h(t')\right\|^2_{\mathcal{V}} + J_0^{\alpha_0,\beta_0}(\dot{u}_h(t'),\dot{u}_h(t'))dt'\\
\le& \frac{CT}{\sqrt{\alpha_0}}\|u_h\|^2_{L_\infty(0,T;\mathcal{V})} + \frac{C}{\sqrt{\alpha_0}}\int_0^t J_0^{\alpha_0,\beta_0}(\dot{u}_h(t'),\dot{u}_h(t'))dt'.
\end{aligned}$$

In this manner, integration by parts implies

$$\begin{aligned}
&\left| -\int_0^t \sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e \{D\nabla \dot{u}_h(t')\cdot \underline{n}_e\}[u_h(t')]dedt'\right|\\
&\quad=\left| -\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e \{D\nabla u_h(t)\cdot \underline{n}_e\}[u_h(t)]de + \sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e \{D\nabla u_h(0)\cdot \underline{n}_e\}[u_h(0)]de\right.
\end{aligned}$$

$$
\begin{aligned}
&\quad + \int_0^t \sum_{e\subset\Gamma_h\cup\Gamma_D} \int_e \{D\nabla u_h(t')\cdot \underline{n}_e\}[\dot{u}_h(t')]dedt'\Bigg| \\
&\leq \frac{C}{\sqrt{\alpha_0}}\|u_h(t)\|_{\mathcal{V}}^2 + \frac{C}{\sqrt{\alpha_0}}\|u_h(0)\|_{\mathcal{V}}^2 + \frac{CT}{\sqrt{\alpha_0}}\|u_h\|_{L_\infty(0,T;\mathcal{V})}^2 \\
&\quad + \frac{C}{\sqrt{\alpha_0}}\int_0^t J_0^{\alpha_0,\beta_0}(\dot{u}_h(t'),\dot{u}_h(t'))dt'.
\end{aligned}
$$

Taking these results, we have the following inequality such that

$$
\begin{aligned}
&\frac{\rho}{2}\|\dot{u}_h(t)\|_{L_2(\Omega)}^2 + \big(\frac{\varphi_0}{2} - \frac{C}{\sqrt{\alpha_0}}\big)\|u_h(t)\|_{\mathcal{V}}^2 + \sum_{q=1}^{N_\varphi}\frac{\kappa}{2\varphi_q}\|\zeta_{hq}(t)\|_{\mathcal{V}}^2 \\
&\quad + \sum_{q=1}^{N_\varphi}\frac{\kappa}{\tau_q\varphi_q}\int_0^t \|\zeta_{hq}(t')\|_{\mathcal{V}}^2 dt' + \big(1 - \frac{2C}{\sqrt{\alpha_0}}\big)\int_0^t J_0^{\alpha_0,\beta_0}(\dot{u}_h(t'),\dot{u}_h(t'))dt' \\
&\leq \frac{\rho}{2}\|\dot{u}_h(0)\|_{L_2(\Omega)}^2 + \left(\frac{\varphi_0}{2} + \frac{K}{2} + \frac{C}{2} + \frac{C}{\sqrt{\alpha_0}}\right)\|u_h(0)\|_{\mathcal{V}}^2 \\
&\quad + \left(\frac{K^2}{2\epsilon_d} + \frac{K}{2} + \sum_{q=1}^{N_\varphi}\frac{T^2K^2\varphi_q^2}{\tau_q^2}\frac{1}{2\epsilon_q}\right)\|u_0\|_{\mathcal{V}}^2 + \frac{1}{2\epsilon_a}\|f\|_{L_2(0,T;L_2(\Omega))}^2 \\
&\quad + \left(\frac{1}{2\epsilon_b} + \frac{h^{-1}}{2}\right)\|g_N\|_{L_\infty(0,T;L_2(\Gamma_N))}^2 + \frac{1}{2\epsilon_c}\|\dot{g}_N\|_{L_2(0,T;L_2(\Gamma_N))} + \frac{\epsilon_a}{2}\|\dot{u}_h\|_{L_\infty(0,T;L_2(\Omega))}^2 \\
&\quad + \left(Ch^{-1}\frac{\epsilon_b}{2} + Ch^{-1}\frac{\epsilon_c}{2} + \frac{\epsilon_d}{2} + \sum_{q=1}^{N_\varphi}\frac{\epsilon_q}{2} + \frac{2CT}{\sqrt{\alpha_0}}\right)\|u_h\|_{L_\infty(0,T;\mathcal{V})}^2.
\end{aligned}
$$

Considering L_∞ norm in time for $\dot{u}_h$ and u_h on the left hand side yields

$$
\begin{aligned}
&\frac{\rho}{2}\|\dot{u}_h\|_{L_\infty(0,T;L_2(\Omega))}^2 + \frac{\varphi_0}{2}\|u_h\|_{L_\infty(0,T;\mathcal{V})}^2 + \sum_{q=1}^{N_\varphi}\frac{\kappa}{2\varphi_q}\|\zeta_{hq}(t)\|_{\mathcal{V}}^2 \\
&\quad + \sum_{q=1}^{N_\varphi}\frac{\kappa}{\tau_q\varphi_q}\int_0^t \|\zeta_{hq}(t')\|_{\mathcal{V}}^2 dt' + \big(1 - \frac{2C}{\sqrt{\alpha_0}}\big)\int_0^t J_0^{\alpha_0,\beta_0}(\dot{u}_h(t'),\dot{u}_h(t'))dt' \\
&\leq \frac{3}{2}\Bigg(\rho\|\dot{u}_h(0)\|_{L_2(\Omega)}^2 + \left(\varphi_0 + K + C + \frac{2C}{\sqrt{\alpha_0}}\right)\|u_h(0)\|_{\mathcal{V}}^2 \\
&\quad + \left(\frac{K^2}{\epsilon_d} + K + \sum_{q=1}^{N_\varphi}\frac{T^2K^2\varphi_q^2}{\tau_q^2}\frac{1}{\epsilon_q}\right)\|u_0\|_{\mathcal{V}}^2 + \frac{1}{\epsilon_a}\|f\|_{L_2(0,T;L_2(\Omega))}^2 \\
&\quad + \left(\frac{1}{\epsilon_b} + \frac{1}{h}\right)\|g_N\|_{L_\infty(0,T;L_2(\Gamma_N))}^2 + \frac{1}{\epsilon_c}\|\dot{g}_N\|_{L_2(0,T;L_2(\Gamma_N))} \\
&\quad + \epsilon_a\|\dot{u}_h\|_{L_\infty(0,T;L_2(\Omega))}^2 + \left(C\frac{\epsilon_b}{h} + C\frac{\epsilon_c}{h} + \epsilon_d + \sum_{q=1}^{N_\varphi}\epsilon_q + \frac{4CT}{\sqrt{\alpha_0}}\right)\|u_h\|_{L_\infty(0,T;\mathcal{V})}^2\Bigg).
\end{aligned}
$$

Hence if we take

$$\begin{aligned}
\epsilon_a =&\frac{\rho}{6},\\
\epsilon_b =&\frac{\varphi_0 h}{24C},\\
\epsilon_c =&\frac{\varphi_0 h}{24C},\\
\epsilon_d =&\frac{\varphi_0}{24},\\
\epsilon_q =&\frac{\varphi_0}{24N_\varphi},\ \forall q\in\{1,\dots,N_\varphi\},
\end{aligned}$$

then we can obtain

$$\begin{aligned}
&\frac{\rho}{4}\left\|\dot{u}_h\right\|^2_{L_\infty(0,T;L_2(\Omega))}+\left(\frac{\varphi_0}{4}-\frac{4CT}{\sqrt{\alpha_0}}\right)\left\|u_h\right\|^2_{L_\infty(0,T;\mathcal{V})}+\sum_{q=1}^{N_\varphi}\frac{\kappa}{2\varphi_q}\left\|\zeta_{hq}(t)\right\|^2_{\mathcal{V}}\\
&+\sum_{q=1}^{N_\varphi}\frac{\kappa}{\tau_q\varphi_q}\int_0^t\left\|\zeta_{hq}(t')\right\|^2_{\mathcal{V}}dt'+\big(1-\frac{2C}{\sqrt{\alpha_0}}\big)\int_0^t J_0^{\alpha_0,\beta_0}(\dot{u}_h(t'),\dot{u}_h(t'))dt'\\
\le& C\Bigg(\left\|\rho^{1/2}\dot{u}_h(0)\right\|^2_{L_2(\Omega)}+\left\|u_h(0)\right\|^2_{\mathcal{V}}+\left\|u_0\right\|^2_{\mathcal{V}}+\left\|f\right\|^2_{L_2(0,T;L_2(\Omega))}\\
&+h^{-1}\left\|g_N\right\|^2_{L_\infty(0,T;L_2(\Gamma_N))}+h^{-1}\left\|\dot{g}_N\right\|_{L_2(0,T;L_2(\Gamma_N))}\Bigg),
\end{aligned}$$

for some positive C. Therefore, with sufficiently large α_0 by

$$\frac{\varphi_0}{4}-\frac{6CT}{\sqrt{\alpha_0}}>0,\qquad 1-\frac{2C}{\sqrt{\alpha_0}}>0,$$

we can conclude that

$$\begin{aligned}
&\left\|\rho^{1/2}\dot{u}_h\right\|^2_{L_\infty(0,T;L_2(\Omega))}+\left\|u_h\right\|^2_{L_\infty(0,T;\mathcal{V})}+\sum_{q=1}^{N_\varphi}\left\|\zeta_{hq}(t)\right\|^2_{\mathcal{V}}+\sum_{q=1}^{N_\varphi}\int_0^t\left\|\zeta_{hq}(t')\right\|^2_{\mathcal{V}}dt'\\
&+\int_0^t J_0^{\alpha_0,\beta_0}(\dot{u}_h(t'),\dot{u}_h(t'))dt'\\
\le& C\Bigg(\left\|\rho^{1/2}\dot{u}_h(0)\right\|^2_{L_2(\Omega)}+\left\|u_h(0)\right\|^2_{\mathcal{V}}+\left\|u_0\right\|^2_{\mathcal{V}}+\left\|f\right\|^2_{L_2(0,T;L_2(\Omega))}+h^{-1}\left\|g_N\right\|^2_{L_\infty(0,T;L_2(\Gamma_N))}\\
&+h^{-1}\left\|\dot{g}_N\right\|_{L_2(0,T;L_2(\Gamma_N))}\Bigg)\\
\le& C\Bigg(\left\|\rho^{1/2}w_0\right\|^2_{L_2(\Omega)}+\left\|u_0\right\|^2_{\mathcal{V}}+\left\|f\right\|^2_{L_2(0,T;L_2(\Omega))}+h^{-1}\left\|g_N\right\|^2_{L_\infty(0,T;L_2(\Gamma_N))}\\
&+h^{-1}\left\|\dot{g}_N\right\|_{L_2(0,T;L_2(\Gamma_N))}\Bigg)
\end{aligned}$$

for some positive C. $\square$

According to Theorem 3.9, if the given initial conditions, boundary conditions and source terms are zero, our semidiscrete solution becomes also zero. Since our problem is equivalent to solving a linear system, it implies the existence and uniqueness of the semidiscrete solution.

Next, we will consider error bounds for the semidiscrete formulation **(Q2)**. Let us define

$$\begin{aligned}\theta &= u - R_1 u,\\ \nu_q &= \zeta_q - R_{-1}\zeta_q,\ \forall q \in \{1,\ldots,N_\varphi\},\\ \chi &= u_h - R_1 u,\\ \Upsilon_q &= \zeta_{hq} - R_{-1}\zeta_q,\ \forall q \in \{1,\ldots,N_\varphi\}.\end{aligned}$$

Recall (3.2.21) and Theorem 3.7, for error estimates as seen in Lemma 3.2.

Lemma 3.4. *Suppose $u \in H^2(0,T;C^2(\Omega)) \cap W^1_\infty(0,T;H^s(\mathcal{E}_h))$ and $\beta_0(d-1) \geq 1$ for $s > 3/2$. There exists a positive constant C such that for any $t \in [0,T]$*

$$\|\dot{\chi}\|_{L_\infty(0,T;L_2(\Omega))} + \|\chi\|_{L_\infty(0,T;\mathcal{V})} \leq Ch^{\min(k+1,s)-1},$$

and if Ω is convex and $\beta_0(d-1) \geq 3$

$$\|\dot{\chi}\|_{L_\infty(0,T;L_2(\Omega))} + \|\chi\|_{L_\infty(0,T;\mathcal{V})} \leq Ch^{\min(k+1,s)}$$

with large enough α_0.

Proof. In a similar way with the proof of Lemma 3.2, subtracting (3.1.5) from (3.2.22) gives

$$\begin{aligned}&(\rho\ddot{\chi}(t),v)_{L_2(\Omega)} + \varphi_0 a_1(\chi(t),v) + \sum_{q=1}^{N_\varphi} a_{-1}(\Upsilon_q(t),v) + J_0^{\alpha_0,\beta_0}(\dot{\chi}(t),v)\\ &= \left(\rho\ddot{\theta}(t),v\right)_{L_2(\Omega)} + \varphi_0 a_1(\theta(t),v) + \sum_{q=1}^{N_\varphi} a_{-1}(\nu_q(t),v) + J_0^{\alpha_0,\beta_0}(\dot{\theta}(t),v),\end{aligned}$$

for any $v \in \mathcal{D}_k(\mathcal{E}_h)$. By (3.2.21) and the Galerkin orthogonality,

$$(\rho\ddot{\chi}(t),v)_{L_2(\Omega)} + \varphi_0 a_1(\chi(t),v) + \sum_{q=1}^{N_\varphi} a_{-1}(\Upsilon_q(t),v) + J_0^{\alpha_0,\beta_0}(\dot{\chi}(t),v) = \left(\rho\ddot{\theta}(t),v\right)_{L_2(\Omega)}. \tag{3.2.26}$$

Also, the difference between (3.1.6) and (3.2.23) shows

$$\tau_q a_{-1}\left(\dot{\Upsilon}_q(t),v\right) + a_{-1}(\Upsilon_q(t),v) - \tau_q\varphi_q a_{-1}(\dot{\chi}(t),v)$$

$$=\tau_q a_{-1}\left(\dot{\nu}_q(t),v\right)+a_{-1}\left(\nu_q(t),v\right)-\tau_q\varphi_q a_{-1}\left(\dot{\theta}(t),v\right)$$
$$=-\tau_q\varphi_q a_{-1}\left(\dot{\theta}(t),v\right)$$

for any $v\in\mathcal{D}_k(\mathcal{E}_h)$, $\forall q\in\{1,\dots,N_\varphi\}$, by using Galerkin orthogonality. Since

$$a_{-1}\left(\dot{\theta}(t),v\right)=a_1\left(\dot{\theta}(t),v\right)-2\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e\{D\nabla v\cdot\underline{n}_e\}[\dot{\theta}(t)]de=0$$

by Galerkin orthogonality and (3.2.21)

$$\tau_q a_{-1}\left(\dot{\Upsilon}_q(t),v\right)+a_{-1}\left(\Upsilon_q(t),v\right)=\tau_q\varphi_q a_{-1}\left(\dot{\chi}(t),v\right).$$

Put $v=\Upsilon_q(t)$ here. Then for each q

$$a_{-1}\left(\Upsilon_q(t),\dot{\chi}(t)\right)=\frac{1}{2\varphi_q}\frac{d}{dt}a_{-1}\left(\Upsilon_q(t),\Upsilon_q(t)\right)+\frac{1}{\tau_q\varphi_q}a_{-1}\left(\Upsilon_q(t),\Upsilon_q(t)\right).\qquad(3.2.27)$$

On the other hand, by substitution $v=\dot{\chi}(t)$ into (3.2.26), we have

$$\frac{\rho}{2}\frac{d}{dt}\|\dot{\chi}(t)\|^2_{L_2(\Omega)}+\frac{\varphi_0}{2}\frac{d}{dt}\|\chi(t)\|^2_{\mathcal{V}}+\sum_{q=1}^{N_\varphi}a_{-1}\left(\Upsilon_q(t),\dot{\chi}(t)\right)+J_0^{\alpha_0,\beta_0}(\dot{\chi}(t),\dot{\chi}(t))$$
$$=\left(\rho\ddot{\theta}(t),\dot{\chi}(t)\right)_{L_2(\Omega)}+\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e\{D\nabla\chi(t)\cdot\underline{n}_e\}[\dot{\chi}(t)]de$$
$$-\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e\{D\nabla\dot{\chi}(t)\cdot\underline{n}_e\}[\chi(t)]de.$$

Inserting (3.2.27) into this implies

$$\frac{\rho}{2}\frac{d}{dt}\|\dot{\chi}(t)\|^2_{L_2(\Omega)}+\frac{\varphi_0}{2}\frac{d}{dt}\|\chi(t)\|^2_{\mathcal{V}}+\sum_{q=1}^{N_\varphi}\frac{1}{2\varphi_q}\frac{d}{dt}a_{-1}\left(\Upsilon_q(t),\Upsilon_q(t)\right)$$
$$+\sum_{q=1}^{N_\varphi}\frac{1}{\tau_q\varphi_q}a_{-1}\left(\Upsilon_q(t),\Upsilon_q(t)\right)+J_0^{\alpha_0,\beta_0}(\dot{\chi}(t),\dot{\chi}(t))$$
$$=\left(\rho\ddot{\theta}(t),\dot{\chi}(t)\right)_{L_2(\Omega)}+\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e\{D\nabla\chi(t)\cdot\underline{n}_e\}[\dot{\chi}(t)]de$$
$$-\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e\{D\nabla\dot{\chi}(t)\cdot\underline{n}_e\}[\chi(t)]de.$$

With applying integration over time and using the coercive constant κ, it yields

$$\frac{\rho}{2}\|\dot{\chi}(t)\|^2_{L_2(\Omega)}+\frac{\varphi_0}{2}\|\chi(t)\|^2_{\mathcal{V}}+\sum_{q=1}^{N_\varphi}\frac{\kappa}{2\varphi_q}\|\Upsilon_q(t)\|^2_{\mathcal{V}}+\sum_{q=1}^{N_\varphi}\frac{\kappa}{\tau_q\varphi_q}\int_0^t\left\|\Upsilon_q(t')\right\|^2_{\mathcal{V}}dt'$$

$$
\begin{aligned}
&+\int_0^t J_0^{\alpha_0,\beta_0}(\dot{\chi}(t'),\dot{\chi}(t'))dt' \\
\leq&\left|\frac{\rho}{2}\|\dot{\chi}(0)\|^2_{L_2(\Omega)}+\frac{\varphi_0}{2}\|\chi(0)\|^2_{\mathcal{V}}+\int_0^t\left(\rho\ddot{\theta}(t'),\dot{\chi}(t')\right)_{L_2(\Omega)}dt'\right. \\
&\left.+\int_0^t\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e\{D\nabla\chi(t')\cdot\underline{n}_e\}[\dot{\chi}(t')]dedt'-\int_0^t\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e\{D\nabla\dot{\chi}(t')\cdot\underline{n}_e\}[\chi(t')]dedt'\right| \\
\leq&\frac{\rho}{2}\left\|\dot{\theta}(0)\right\|^2_{L_2(\Omega)}+\left|\int_0^t\left(\rho\ddot{\theta}(t'),\dot{\chi}(t')\right)_{L_2(\Omega)}dt'+\int_0^t\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e\{D\nabla\chi(t')\cdot\underline{n}_e\}[\dot{\chi}(t')]dedt'\right. \\
&\left.-\int_0^t\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e\{D\nabla\dot{\chi}(t')\cdot\underline{n}_e\}[\chi(t')]dedt'\right|
\end{aligned}
$$

since

$$
\|\dot{\chi}(0)\|_{L_2(\Omega)}\leq\left\|\dot{\theta}(0)\right\|_{L_2(\Omega)},\ \|\chi(0)\|_{\mathcal{V}}=0,\ \|\Upsilon_q(0)\|_{\mathcal{V}}=0,\ \forall q\in\{1,\ldots,N_\varphi\},
$$

as shown in the proof of Lemma 3.2. Now we shall use Cauchy-Schwarz inequality, Young's inequality and the boundedness of skew symmetric part $B(\cdot,\cdot)$. Hence, we can observe the followings

- $|\int_0^t\left(\rho\ddot{\theta}(t'),\dot{\chi}(t')\right)_{L_2(\Omega)}dt'|$

$$
\begin{aligned}
\left|\int_0^t\left(\rho\ddot{\theta}(t'),\dot{\chi}(t')\right)_{L_2(\Omega)}dt'\right|\leq&\int_0^t\rho\left\|\ddot{\theta}(t')\right\|_{L_2(\Omega)}\|\dot{\chi}(t')\|_{L_2(\Omega)}dt' \\
\leq&\|\dot{\chi}\|_{L_\infty(0,T;L_2(\Omega))}\int_0^t\rho\left\|\ddot{\theta}(t')\right\|_{L_2(\Omega)}dt' \\
\leq&\|\dot{\chi}\|_{L_\infty(0,T;L_2(\Omega))}\int_0^T\rho\left\|\ddot{\theta}(t')\right\|_{L_2(\Omega)}dt' \\
\leq&\rho\|\dot{\chi}\|_{L_\infty(0,T;L_2(\Omega))}\sqrt{T}\left\|\ddot{\theta}\right\|_{L_2(0,T;L_2(\Omega))} \\
\leq&\frac{\rho\epsilon_a}{2}\|\dot{\chi}\|^2_{L_\infty(0,T;L_2(\Omega))}+\frac{\rho T}{2\epsilon_a}\left\|\ddot{\theta}\right\|^2_{L_2(0,T;L_2(\Omega))}
\end{aligned}
$$

  for any positive ϵ_a.

- $|\int_0^t\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e\{D\nabla\chi(t')\cdot\underline{n}_e\}[\dot{\chi}(t')]dedt'-\int_0^t\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e\{D\nabla\dot{\chi}(t')\cdot\underline{n}_e\}[\chi(t')]dedt'|$

$$
\begin{aligned}
&\left|\int_0^t\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e\{D\nabla\chi(t')\cdot\underline{n}_e\}[\dot{\chi}(t')]dedt'-\int_0^t\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e\{D\nabla\dot{\chi}(t')\cdot\underline{n}_e\}[\chi(t')]dedt'\right| \\
&=\left|2\int_0^t\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e\{D\nabla\chi(t')\cdot\underline{n}_e\}[\dot{\chi}(t')]dedt'-\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e\{D\nabla\chi(t)\cdot\underline{n}_e\}[\chi(t)]de\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e\{D\nabla\chi(0)\cdot\underline{n}_e\}[\chi(0)]de\Bigg|\\
&\quad(\because\text{ by integration by parts})\\
\le&\frac{2C}{\sqrt{\alpha_0}}\int_0^t\left\|\chi(t')\right\|_{\mathcal{V}}^2+J_0^{\alpha_0,\beta_0}(\dot\chi(t'),\dot\chi(t'))dt'+\frac{C}{\sqrt{\alpha_0}}(\|\chi(t)\|_{\mathcal{V}}^2+\|\chi(0)\|_{\mathcal{V}}^2)\\
&\quad(\because\text{ by (3.2.1) and (3.2.2)})\\
\le&\frac{2CT}{\sqrt{\alpha_0}}\|\chi\|_{L_\infty(0,T;\mathcal{V})}^2+\frac{2C}{\sqrt{\alpha_0}}\int_0^t J_0^{\alpha_0,\beta_0}(\dot\chi(t'),\dot\chi(t'))dt'+\frac{C}{\sqrt{\alpha_0}}(\|\chi(t)\|_{\mathcal{V}}^2+\|\chi(0)\|_{\mathcal{V}}^2)\\
=&\frac{2CT}{\sqrt{\alpha_0}}\|\chi\|_{L_\infty(0,T;\mathcal{V})}^2+\frac{2C}{\sqrt{\alpha_0}}\int_0^t J_0^{\alpha_0,\beta_0}(\dot\chi(t'),\dot\chi(t'))dt'+\frac{C}{\sqrt{\alpha_0}}\|\chi(t)\|_{\mathcal{V}}^2.
\end{aligned}
$$

Turning to the main proof, the above bounds give

$$
\begin{aligned}
&\frac{\rho}{2}\|\dot\chi(t)\|_{L_2(\Omega)}^2+\big(\frac{\varphi_0}{2}-\frac{C}{\sqrt{\alpha_0}}\big)\|\chi(t)\|_{\mathcal{V}}^2+\sum_{q=1}^{N_\varphi}\frac{\kappa}{2\varphi_q}\|\Upsilon_q(t)\|_{\mathcal{V}}^2\\
&\quad+\sum_{q=1}^{N_\varphi}\frac{\kappa}{\tau_q\varphi_q}\int_0^t\left\|\Upsilon_q(t')\right\|_{\mathcal{V}}^2dt'+\big(1-\frac{2C}{\sqrt{\alpha_0}}\big)\int_0^t J_0^{\alpha_0,\beta_0}(\dot\chi(t'),\dot\chi(t'))dt'\\
\le&\frac{\rho}{2}\left\|\dot\theta(0)\right\|_{L_2(\Omega)}^2+\frac{\rho T}{2\epsilon_a}\left\|\ddot\theta\right\|_{L_2(0,T;L_2(\Omega))}^2+\frac{\rho\epsilon_a}{2}\|\dot\chi\|_{L_\infty(0,T;L_2(\Omega))}^2+\frac{2CT}{\sqrt{\alpha_0}}\|\chi\|_{L_\infty(0,T;\mathcal{V})}^2.
\end{aligned}
$$

Additionally, if we consider L_∞ norm, we get

$$
\begin{aligned}
&\frac{\rho}{2}\|\dot\chi\|_{L_\infty(0,T;L_2(\Omega))}^2+\big(\frac{\varphi_0}{2}-\frac{C}{\sqrt{\alpha_0}}\big)\|\chi\|_{L_\infty(0,T;\mathcal{V})}^2+\sum_{q=1}^{N_\varphi}\frac{\kappa}{2\varphi_q}\|\Upsilon_q(t)\|_{\mathcal{V}}^2\\
&\quad+\sum_{q=1}^{N_\varphi}\frac{\kappa}{\tau_q\varphi_q}\int_0^t\left\|\Upsilon_q(t')\right\|_{\mathcal{V}}^2dt'+\big(1-\frac{2C}{\sqrt{\alpha_0}}\big)\int_0^t J_0^{\alpha_0,\beta_0}(\dot\chi(t'),\dot\chi(t'))dt'\\
\le&\frac{3\rho}{2}\left\|\dot\theta(0)\right\|_{L_2(\Omega)}^2+\frac{3\rho T}{2\epsilon_a}\left\|\ddot\theta\right\|_{L_2(0,T;L_2(\Omega))}^2+\frac{3\rho\epsilon_a}{2}\|\dot\chi\|_{L_\infty(0,T;L_2(\Omega))}^2+\frac{6CT}{\sqrt{\alpha_0}}\|\chi\|_{L_\infty(0,T;\mathcal{V})}^2.
\end{aligned}
$$

Let us take ϵ_a such that

$$\epsilon_a=\frac{1}{6}>0,$$

then

$$
\begin{aligned}
&\frac{\rho}{4}\|\dot\chi\|_{L_\infty(0,T;L_2(\Omega))}^2+\big(\frac{\varphi_0}{2}-\frac{C(1+6T)}{\sqrt{\alpha_0}}\big)\|\chi\|_{L_\infty(0,T;\mathcal{V})}^2+\sum_{q=1}^{N_\varphi}\frac{\kappa}{2\varphi_q}\|\Upsilon_q(t)\|_{\mathcal{V}}^2\\
&\quad+\sum_{q=1}^{N_\varphi}\frac{\kappa}{2\tau_q\varphi_q}\int_0^t\left\|\Upsilon_q(t')\right\|_{\mathcal{V}}^2dt'+\big(1-\frac{2C}{\sqrt{\alpha_0}}\big)\int_0^t J_0^{\alpha_0,\beta_0}(\dot\chi(t'),\dot\chi(t'))dt'
\end{aligned}
$$

$$\leq \frac{3\rho}{2} \left\|\dot{\theta}(0)\right\|^2_{L_2(\Omega)} + 9\rho T \left\|\ddot{\theta}\right\|^2_{L_2(0,T;L_2(\Omega))}.$$

Suppose α_0 is sufficiently large as

$$\frac{\varphi_0}{2} - \frac{C(6T+1)}{\sqrt{\alpha_0}} > 0, \qquad 1 - \frac{2C}{\sqrt{\alpha_0}} > 0.$$

Thus, we have

$$\|\dot{\chi}\|^2_{L_\infty(0,T;L_2(\Omega))} + \|\chi\|^2_{L_\infty(0,T;\mathcal{V})} + \sum_{q=1}^{N_\varphi} \|\Upsilon_q(t)\|^2_{\mathcal{V}} + \sum_{q=1}^{N_\varphi} \int_0^t \|\Upsilon_q(t')\|^2_{\mathcal{V}}\, dt'$$
$$+ \int_0^t J_0^{\alpha_0,\beta_0}(\dot{\chi}(t'), \dot{\chi}(t'))dt' \leq C\Big(\left\|\dot{\theta}(0)\right\|^2_{L_2(\Omega)} + \left\|\ddot{\theta}\right\|^2_{L_2(0,T;L_2(\Omega))} \Big),$$

for some positive C. Therefore, by Theorem 3.7, we can conclude that

$$\|\dot{\chi}\|_{L_\infty(0,T;L_2(\Omega))} + \|\chi\|_{L_\infty(0,T;\mathcal{V})} \leq C h^{\min(k+1,s)-1},$$

and if Ω is convex and $\beta_0(d-1) \geq 3$

$$\|\dot{\chi}\|_{L_\infty(0,T;L_2(\Omega))} + \|\chi\|_{L_\infty(0,T;\mathcal{V})} \leq C h^{\min(k+1,s)}.$$

□

Theorem 3.10. *Under the same conditions for* Lemma 3.4*, we have*

$$\|u - u_h\|_{L_\infty(0,T;\mathcal{V})} \leq C h^{\min(k+1,s)-1},$$
$$\|\dot{u} - \dot{u}_h\|_{L_\infty(0,T;L_2(\Omega))} \leq C h^{\min(k+1,s)-1},$$
$$\|\dot{u} - \dot{u}_h\|_{L_\infty(0,T;L_2(\Omega))} \leq C h^{\min(k+1,s)} \text{ if elliptic regularity estimates available.}$$

Proof. It can be proved in the same sense with the proof of Theorem 3.8. By Theorem 3.7 and Lemma 3.4,

$$\|u - u_h\|_{L_\infty(0,T;\mathcal{V})} \leq \|\theta\|_{L_\infty(0,T;\mathcal{V})} + \|\chi\|_{L_\infty(0,T;\mathcal{V})}$$
$$\leq C h^{\min(k+1,s)-1},$$

and

$$\|\dot{u} - \dot{u}_h\|_{L_\infty(0,T;L_2(\Omega))} \leq \left\|\dot{\theta}\right\|_{L_\infty(0,T;L_2(\Omega))} + \|\dot{\chi}\|_{L_\infty(0,T;L_2(\Omega))}$$
$$\leq C h^{\min(k+1,s)-1},$$

by (3.2.16) and if Ω is convex and $\beta_0(d-1) \geq 3$, we can use (3.2.17) so that

$$\|\dot{u} - \dot{u}_h\|_{L_\infty(0,T;L_2(\Omega))} \leq C h^{\min(k+1,s)}.$$

□

Corollary 3.2. *Under the same conditions for* Lemma 3.4, *we have*

$$\|u - u_h\|_{L_\infty(0,T;L_2(\Omega))} \leq Ch^{\min(k+1,s)-1},$$
$$\|u - u_h\|_{L_\infty(0,T;L_2(\Omega))} \leq Ch^{\min(k+1,s)} \text{ if } \Omega \text{ is convex and } \beta_0(d-1) \geq 3.$$

Proof. Theorems 1.12, 3.7 and Lemma 3.4 allow us to have

$$\begin{aligned}\|u - u_h\|_{L_\infty(0,T;L_2(\Omega))} &\leq \|\theta\|_{L_\infty(0,T;L_2(\Omega))} + \|\chi\|_{L_\infty(0,T;L_2(\Omega))} \\ &\leq \|\theta\|_{L_\infty(0,T;L_2(\Omega))} + C\,\|\chi\|_{L_\infty(0,T;\mathcal{V})} \\ &\leq Ch^{\min(k+1,s)-1}.\end{aligned}$$

Moreover, if Ω is convex and $\beta_0(d-1) \geq 3$

$$\|u - u_h\|_{L_\infty(0,T;L_2(\Omega))} \leq Ch^{\min(k+1,s)}.$$

□

3.3 Fully Discrete Formulation for DGFEM

In the previous section, we took into account the semidiscrete formulations for the both displacement form and velocity form, that is, we considered continuous problems in time. However, by applying Crank-Nicolson method with respect to time and spatially DGFEM, fully discrete formulations would be derived. Then we will observe stability bounds and error bounds for both formulations **(Q1)** and **(Q2)**, respectively. At last, numerical experiments would have been carried out to verify the error convergence rates.

As following the same argument to introduce Crank-Nicolson method in Section 2.3, we have the time step $\Delta t > 0$ such that $T = N\Delta t$ for $N \in \mathbb{N}$. Then our numerical solution U_h can be expressed as

$$U_h(\boldsymbol{x}, t_n) = U_h^n = \sum_{i=1}^{N_{Vh}} \mathrm{u}_i^n \phi_i(\boldsymbol{x}),$$

for $t_n = n\Delta t$ with the relation

$$\frac{W_h^{n+1} + W_h^n}{2} = \frac{U_h^{n+1} - U_h^n}{\Delta t}, \tag{3.3.1}$$

where W_h^n is a numerical approximation of $\dot{u}(t_n)$. Moreover, we recall and use time average notation.

3.3.1 Displacement Form

A fully discrete formulation of **(Q1)** is given by
Find $U_h^n, W_h^n, \Psi_{hq}^n \in \mathcal{D}_k(\mathcal{E}_h)$ for $n = 0, \ldots, N$, and $q = 1, \ldots, N_\varphi$ such that satisfy for

any $v \in \mathcal{D}_k(\mathcal{E}_h)$, $\forall 0 \leq n \leq N-1$

$$\left(\rho \frac{W_h^{n+1} - W_h^n}{\Delta t}, v\right)_{L_2(\Omega)} + a_1\left(\frac{U_h^{n+1} + U_h^n}{2}, v\right) - \sum_{q=1}^{N_\varphi} a_{-1}\left(\frac{\Psi_{hq}^{n+1} + \Psi_{hq}^n}{2}, v\right)$$
$$+ J_0^{\alpha_0,\beta_0}\left(\frac{W_h^{n+1} + W_h^n}{2}, v\right) = \bar{F}_d^n(v), \tag{3.3.2}$$

$$a_{-1}\left(\tau_q \frac{\Psi_{hq}^{n+1} - \Psi_{hq}^n}{\Delta t} + \frac{\Psi_{hq}^{n+1} + \Psi_{hq}^n}{2}, v\right) = a_{-1}\left(\varphi_q \frac{U_h^{n+1} + U_h^n}{2}, v\right), \tag{3.3.3}$$

$$a_1\left(U_h^0, v\right) = a_1\left(u_0, v\right), \tag{3.3.4}$$
$$\left(W_h^0, v\right)_{L_2(\Omega)} = \left(w_0, v\right)_{L_2(\Omega)}, \tag{3.3.5}$$

where $\Psi_{hq}^0 = 0$, $\forall q \in \{1, \ldots, N_\varphi\}$. From the fully discrete formulation, the equivalent linear system can be obtained by for $1 \leq n \leq N-1$

$$\begin{aligned}\left(\frac{2\rho}{\Delta t^2} M + \mathcal{A} + \frac{1}{\Delta t}\mathcal{J}\right)\underline{\mathfrak{u}}^{n+1} =& \Bigg[\frac{2\rho}{\Delta t} M \underline{\mathfrak{w}}^n + \left(\frac{2\rho}{\Delta t^2} M - \mathcal{A} + \frac{1}{\Delta t}\mathcal{J}\right)\underline{\mathfrak{u}}^n \\ &+ \sum_{q=1}^{N_\varphi} \frac{2\tau_q}{2\tau_q + \Delta t} A^* \underline{\Psi}_{hq}^n + \frac{1}{2}(\underline{F}^{n+1} + \underline{F}^n)\Bigg] \qquad (3.3.6) \\ \underline{\mathfrak{w}}^{n+1} =& \frac{2}{\Delta t}(\underline{\mathfrak{u}}^{n+1} - \underline{\mathfrak{u}}^n) - \underline{\mathfrak{w}}^n, \qquad (3.3.7) \\ \underline{\Psi}_{hq}^{n+1} =& \frac{2\Delta t}{2\tau_q + \Delta t}\left(\frac{2\tau_q - \Delta t}{2\Delta t}\underline{\Psi}_{hq}^n + \frac{\varphi_q}{2}(\underline{\mathfrak{u}}^{n+1} + \underline{\mathfrak{u}}^n)\right), \qquad (3.3.8)\end{aligned}$$

$\forall q \in \{1, \ldots, N_\varphi\}$ where

$$\mathcal{A} := \frac{1}{2}\left(A - \sum_{q=1}^{N_\varphi} \frac{\varphi_q}{2}\left(\frac{\tau_q}{\Delta t} + \frac{1}{2}\right)^{-1} A^*\right)$$

and $\underline{\mathfrak{u}}^0 = A^{-1}\underline{U}_0$, $\underline{\mathfrak{w}}^0 = M^{-1}\underline{W}_0$. The existence and uniqueness of the solution will be given by stability bounds.

<u>Remark</u> By (3.3.4) and (3.3.5), Cauchy-Schwarz inequalities and continuity of NIPG give

$$\left\|U_h^0\right\|_{\mathcal{V}}^2 = a_1\left(U_h^0, U_h^0\right) = a_1\left(u_0, U_h^0\right) \leq K\left\|u_0\right\|_{\mathcal{V}}\left\|U_h^0\right\|_{\mathcal{V}},$$
$$\left\|W_h^0\right\|_{L_2(\Omega)}^2 = \left(W_h^0, W_h^0\right)_{L_2(\Omega)} = \left(w_0, W_h^0\right)_{L_2(\Omega)} \leq \left\|w_0\right\|_{L_2(\Omega)}\left\|W_h^0\right\|_{L_2(\Omega)},$$

thus

$$\left\|U_h^0\right\|_{\mathcal{V}} \leq K\left\|u_0\right\|_{\mathcal{V}}, \qquad \left\|W_h^0\right\|_{L_2(\Omega)} \leq \left\|w_0\right\|_{L_2(\Omega)}.$$

Lemma 3.5. *For any $n = 0, \ldots, N-1$, it holds for any $q \in \{1, \ldots, N_\varphi\}$*

$$\begin{aligned} a_{-1}\left(\Psi_{hq}^{n+1} + \Psi_{hq}^{n}, U_h^{n+1} - U_h^{n}\right) =& 2a_{-1}\left(\Psi_{hq}^{n+1}, U_h^{n+1}\right) - 2a_{-1}\left(\Psi_{hq}^{n}, U_h^{n}\right) \\ &- \frac{2\tau_q}{\Delta t \varphi_q} a_{-1}\left(\Psi_{hq}^{n+1} - \Psi_{hq}^{n}, \Psi_{hq}^{n+1} - \Psi_{hq}^{n}\right) \\ &- \frac{1}{\varphi_q}\left(a_{-1}\left(\Psi_{hq}^{n+1}, \Psi_{hq}^{n+1}\right) - a_{-1}\left(\Psi_{hq}^{n}, \Psi_{hq}^{n}\right)\right). \end{aligned}$$

Proof. Let $v = (\Psi_{hq}^{n+1} - \Psi_{hq}^{n})/\Delta t$. By substitution into (3.3.3),

$$\begin{aligned} &\frac{\tau_q}{\Delta t^2} a_{-1}\left(\Psi_{hq}^{n+1} - \Psi_{hq}^{n}, \Psi_{hq}^{n+1} - \Psi_{hq}^{n}\right) + \frac{1}{2\Delta t}\left(a_{-1}\left(\Psi_{hq}^{n+1}, \Psi_{hq}^{n+1}\right) - a_{-1}\left(\Psi_{hq}^{n}, \Psi_{hq}^{n}\right)\right) \\ &= \frac{\varphi_q}{2\Delta t} a_{-1}\left(U_h^{n+1} + U_h^{n}, \Psi_{hq}^{n+1} - \Psi_{hq}^{n}\right) \\ &= \frac{\varphi_q}{2\Delta t} a_{-1}\left(\Psi_{hq}^{n+1} - \Psi_{hq}^{n}, U_h^{n+1} + U_h^{n}\right) \end{aligned}$$

Since

$$\begin{aligned} a_{-1}\left(\Psi_{hq}^{n+1} - \Psi_{hq}^{n}, U_h^{n+1}\right) =& a_{-1}\left(\Psi_{hq}^{n+1}, U_h^{n+1}\right) - a_{-1}\left(\Psi_{hq}^{n}, U_h^{n}\right) + a_{-1}\left(\Psi_{hq}^{n}, U_h^{n}\right) \\ &- a_{-1}\left(\Psi_{hq}^{n}, U_h^{n+1}\right) \\ =& a_{-1}\left(\Psi_{hq}^{n+1}, U_h^{n+1}\right) - a_{-1}\left(\Psi_{hq}^{n}, U_h^{n}\right) - a_{-1}\left(\Psi_{hq}^{n}, U_h^{n+1} - U_h^{n}\right), \end{aligned}$$

and

$$\begin{aligned} a_{-1}\left(\Psi_{hq}^{n+1} - \Psi_{hq}^{n}, U_h^{n}\right) =& a_{-1}\left(\Psi_{hq}^{n+1}, U_h^{n+1}\right) - a_{-1}\left(\Psi_{hq}^{n+1}, U_h^{n+1}\right) - a_{-1}\left(\Psi_{hq}^{n}, U_h^{n}\right) \\ &+ a_{-1}\left(\Psi_{hq}^{n+1}, U_h^{n}\right) \\ =& a_{-1}\left(\Psi_{hq}^{n+1}, U_h^{n+1}\right) - a_{-1}\left(\Psi_{hq}^{n}, U_h^{n}\right) - a_{-1}\left(\Psi_{hq}^{n+1}, U_h^{n+1} - U_h^{n}\right), \end{aligned}$$

we have

$$\begin{aligned} a_{-1}\left(\Psi_{hq}^{n+1} - \Psi_{hq}^{n}, U_h^{n+1} + U_h^{n}\right) =& 2a_{-1}\left(\Psi_{hq}^{n+1}, U_h^{n+1}\right) - 2a_{-1}\left(\Psi_{hq}^{n}, U_h^{n}\right) \\ &- a_{-1}\left(\Psi_{hq}^{n+1} + \Psi_{hq}^{n}, U_h^{n+1} - U_h^{n}\right). \end{aligned}$$

This implies

$$\begin{aligned} a_{-1}\left(\Psi_{hq}^{n+1} + \Psi_{hq}^{n}, U_h^{n+1} - U_h^{n}\right) =& 2a_{-1}\left(\Psi_{hq}^{n+1}, U_h^{n+1}\right) - 2a_{-1}\left(\Psi_{hq}^{n}, U_h^{n}\right) \\ &- \frac{2\tau_q}{\Delta t \varphi_q} a_{-1}\left(\Psi_{hq}^{n+1} - \Psi_{hq}^{n}, \Psi_{hq}^{n+1} - \Psi_{hq}^{n}\right) \\ &- \frac{1}{\varphi_q}\left(a_{-1}\left(\Psi_{hq}^{n+1}, \Psi_{hq}^{n+1}\right) - a_{-1}\left(\Psi_{hq}^{n}, \Psi_{hq}^{n}\right)\right). \end{aligned}$$

□

Theorem 3.11. *For any $m = 1, \ldots, N$, there exists a positive constant C such that*

$$\max_{0\le n\le N}\left\|\rho^{1/2}W_h^n\right\|_{L_2(\Omega)}^2+\max_{0\le n\le N}\|U_h^n\|_{\mathcal{V}}^2+\sum_{q=1}^{N_\varphi}\left\|\Psi_{hq}^m\right\|_{\mathcal{V}}^2+\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi}\frac{1}{\Delta t}\left\|\Psi_{hq}^{n+1}-\Psi_{hq}^n\right\|_{\mathcal{V}}^2$$
$$+\Delta t\sum_{n=0}^{m-1}J_0^{\alpha_0,\beta_0}(W_h^{n+1}+W_h^n,W_h^{n+1}+W_h^n)$$
$$\le C\bigg(\left\|\rho^{1/2}w_0\right\|_{L_2(\Omega)}^2+\|u_0\|_{\mathcal{V}}^2+\|f\|_{L_\infty(0,T;L_2(\Omega))}^2+h^{-1}\|g_N\|_{L_\infty(0,T;L_2(\Gamma_N))}^2$$
$$+h^{-1}\|\dot{g}_N\|_{L_2(0,T;L_2(\Gamma_N))}^2\bigg)$$

if $\beta_0(d-1)\ge 1$ and α_0 is large enough.

Proof. Let $0 < m \le N$. Choosing $v = W_h^{n+1} + W_h^n$ into (3.3.2) gives

$$\frac{\rho}{\Delta t}\left(\|W_h^{n+1}\|_{L_2(\Omega)}^2-\|W_h^n\|_{L_2(\Omega)}^2\right)+\frac{1}{\Delta t}a_1\left(U_h^{n+1}+U_h^n,U_h^{n+1}-U_h^n\right)$$
$$-\frac{1}{\Delta t}\sum_{q=1}^{N_\varphi}a_{-1}\left(\Psi_{hq}^{n+1}+\Psi_{hq}^n,U_h^{n+1}-U_h^n\right)+\frac{1}{2}J_0^{\alpha_0,\beta_0}(W_h^{n+1}+W_h^n,W_h^{n+1}+W_h^n)$$
$$=\bar{F}_d^n(W_h^{n+1}+W_h^n)$$

with the relation (3.3.1). Expanding $a_1\left(U_h^{n+1}+U_h^n,U_h^{n+1}-U_h^n\right)$ yields

$$\frac{\rho}{\Delta t}\left(\|W_h^{n+1}\|_{L_2(\Omega)}^2-\|W_h^n\|_{L_2(\Omega)}^2\right)+\frac{1}{\Delta t}\left(\|U_h^{n+1}\|_{\mathcal{V}}^2-\|U_h^n\|_{\mathcal{V}}^2\right)$$
$$-\frac{1}{\Delta t}\left(a_1\left(U_h^{n+1},U_h^n\right)-a_1\left(U_h^n,U_h^{n+1}\right)\right)-\frac{1}{\Delta t}\sum_{q=1}^{N_\varphi}a_{-1}\left(\Psi_{hq}^{n+1}+\Psi_{hq}^n,U_h^{n+1}-U_h^n\right)$$
$$+\frac{1}{2}J_0^{\alpha_0,\beta_0}(W_h^{n+1}+W_h^n,W_h^{n+1}+W_h^n)$$
$$=\frac{\rho}{\Delta t}\left(\|W_h^{n+1}\|_{L_2(\Omega)}^2-\|W_h^n\|_{L_2(\Omega)}^2\right)+\frac{1}{\Delta t}\left(\|U_h^{n+1}\|_{\mathcal{V}}^2-\|U_h^n\|_{\mathcal{V}}^2\right)$$
$$-\frac{1}{\Delta t}B\left(U_h^{n+1},U_h^n\right)-\frac{1}{\Delta t}\sum_{q=1}^{N_\varphi}a_{-1}\left(\Psi_{hq}^{n+1}+\Psi_{hq}^n,U_h^{n+1}-U_h^n\right)$$
$$+\frac{1}{2}J_0^{\alpha_0,\beta_0}(W_h^{n+1}+W_h^n,W_h^{n+1}+W_h^n)$$
$$=\bar{F}_d^n(W_h^{n+1}+W_h^n).$$

By Lemma 3.5 and multiplying Δt on both sides, the equation becomes

$$\rho\left(\|W_h^{n+1}\|_{L_2(\Omega)}^2-\|W_h^n\|_{L_2(\Omega)}^2\right)+\left(\|U_h^{n+1}\|_{\mathcal{V}}^2-\|U_h^n\|_{\mathcal{V}}^2\right)$$

$$
\begin{aligned}
&+\sum_{q=1}^{N_\varphi}\frac{1}{\varphi_q}\left(a_{-1}\left(\Psi_{hq}^{n+1},\Psi_{hq}^{n+1}\right)-a_{-1}\left(\Psi_{hq}^{n},\Psi_{hq}^{n}\right)\right)\\
&+\sum_{q=1}^{N_\varphi}\frac{2\tau_q}{\Delta t\varphi_q}a_{-1}\left(\Psi_{hq}^{n+1}-\Psi_{hq}^{n},\Psi_{hq}^{n+1}-\Psi_{hq}^{n}\right)\\
&+\frac{\Delta t}{2}J_0^{\alpha_0,\beta_0}(W_h^{n+1}+W_h^n,W_h^{n+1}+W_h^n)\\
=&\Delta t\bar{F}_d^n(W_h^{n+1}+W_h^n)+2\sum_{q=1}^{N_\varphi}\left(a_{-1}\left(\Psi_{hq}^{n+1},U_h^{n+1}\right)-a_{-1}\left(\Psi_{hq}^{n},U_h^{n}\right)\right)\\
&+B\left(U_h^{n+1},U_h^n\right).
\end{aligned}
$$

Consider the summation from $n=0$ to $n=m-1$. Then,

$$
\begin{aligned}
&\rho\left\|W_h^m\right\|_{L_2(\Omega)}^2+\left\|U_h^m\right\|_{\mathcal{V}}^2+\sum_{q=1}^{N_\varphi}\frac{1}{\varphi_q}a_{-1}\left(\Psi_{hq}^{m},\Psi_{hq}^{m}\right)\\
&+\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi}\frac{2\tau_q}{\Delta t\varphi_q}a_{-1}\left(\Psi_{hq}^{n+1}-\Psi_{hq}^{n},\Psi_{hq}^{n+1}-\Psi_{hq}^{n}\right)\\
&+\frac{\Delta t}{2}\sum_{n=0}^{m-1}J_0^{\alpha_0,\beta_0}(W_h^{n+1}+W_h^n,W_h^{n+1}+W_h^n)\\
=&\rho\left\|W_h^0\right\|_{L_2(\Omega)}^2+\left\|U_h^0\right\|_{\mathcal{V}}^2+\sum_{q=1}^{N_\varphi}\frac{1}{\varphi_q}a_{-1}\left(\Psi_{q}^{0},\Psi_{q}^{0}\right)+\sum_{n=0}^{m-1}\Delta t\bar{F}_d^n(W_h^{n+1}+W_h^n)\\
&+2\sum_{q=1}^{N_\varphi}a_{-1}\left(\Psi_{hq}^{m},U_h^{m}\right)-2\sum_{q=1}^{N_\varphi}a_{-1}\left(\Psi_{hq}^{0},U_h^{0}\right)+\sum_{n=0}^{m-1}B\left(U_h^{n+1},U_h^n\right)\\
\leq&\rho\left\|w_0\right\|_{L_2(\Omega)}^2+K\left\|u_0\right\|_{\mathcal{V}}^2+\sum_{n=0}^{m-1}\Delta t\bar{F}_d^n(W_h^{n+1}+W_h^n)+2\sum_{q=1}^{N_\varphi}a_{-1}\left(\Psi_{hq}^{m},U_h^{m}\right)\\
&+\sum_{n=0}^{m-1}B\left(U_h^{n+1},U_h^n\right)
\end{aligned}
\tag{3.3.9}
$$

since $\Psi_{hq}^0=0,\ \forall q\in\{1,\ldots,N_\varphi\}$,

$$
\left\|W_h^0\right\|_{L_2(\Omega)}\leq\left\|w_0\right\|_{L_2(\Omega)}\ \text{and}\ \left\|U_h^0\right\|_{\mathcal{V}}\leq K\left\|u_0\right\|_{\mathcal{V}}.
$$

Using coercivity on the left hand side of (3.3.9) and the definition of SIPG, we can have

$$
\rho\left\|W_h^m\right\|_{L_2(\Omega)}^2+\left\|U_h^m\right\|_{\mathcal{V}}^2+\sum_{q=1}^{N_\varphi}\frac{1}{\varphi_q}\left\|\Psi_{hq}^{m}\right\|_{\mathcal{V}}^2+\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi}\frac{2\kappa\tau_q}{\Delta t\varphi_q}\left\|\Psi_{hq}^{n+1}-\Psi_{hq}^{n}\right\|_{\mathcal{V}}^2
$$

$$
\begin{aligned}
&+\frac{\Delta t}{2}\sum_{n=0}^{m-1} J_0^{\alpha_0,\beta_0}(W_h^{n+1}+W_h^n, W_h^{n+1}+W_h^n) \\
\leq &\Bigg|\rho\left\|w_0\right\|_{L_2(\Omega)}^2 + K\left\|u_0\right\|_{\mathcal{V}}^2 + \sum_{n=0}^{m-1}\Delta t \bar{F}_d^n(W_h^{n+1}+W_h^n) + 2\sum_{q=1}^{N_\varphi} a_{-1}\left(\Psi_{hq}^m, U_h^m\right) \\
&+\sum_{n=0}^{m-1} B\left(U_h^{n+1}, U_h^n\right) + \sum_{q=1}^{N_\varphi}\frac{2}{\varphi_q}\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e \{D\nabla\Psi_{hq}^m\cdot \underline{n}_e\}[\Psi_{hq}^m]de\Bigg| \qquad (3.3.10)
\end{aligned}
$$

We will observe each upper bound for the right hand side of (3.3.10).

- $|\sum_{n=0}^{m-1}\Delta t\bar{F}_d^n(W_h^{n+1}+W_h^n)|$

$$
\begin{aligned}
\Delta t\sum_{n=0}^{m-1}\bar{F}_d^n\left(W_h^{n+1}+W_h^n\right) = &\Delta t\sum_{n=0}^{m-1}\left(\bar{f}^n, W_h^{n+1}+W_h^n\right)_{L_2(\Omega)} \\
&+2\sum_{n=0}^{m-1}\left(\bar{g}_N^n, U_h^{n+1}-U_h^n\right)_{L_2(\Gamma_N)}
\end{aligned}
$$

with (3.3.1). Recall summation by parts and whence it is applied,

$$
\begin{aligned}
\sum_{n=0}^{m-1}\left(\bar{g}_N^n, U_h^{n+1}-U_h^n\right)_{L_2(\Gamma_N)} = &\left(\bar{g}_N^{m-1}, U_h^m\right)_{L_2(\Gamma_N)} - \left(\bar{g}_N^0, U_h^0\right)_{L_2(\Gamma_N)} \\
&-\sum_{n=1}^{m-1}\left(\bar{g}_N^n-\bar{g}_N^{n-1}, U_h^n\right)_{L_2(\Gamma_N)}.
\end{aligned}
$$

Moreover, since g_N is continuous and differentiable in time,

$$
\sum_{n=1}^{m-1}\left(\bar{g}_N^n-\bar{g}_N^{n-1}, U_h^n\right)_{L_2(\Gamma_N)} = \frac{1}{2}\sum_{n=1}^{m-1}\int_{t_{n-1}}^{t_{n+1}}\left(\dot{g}_N(t'), U_h^n\right)_{L_2(\Gamma_N)}dt'.
$$

So we have

$$
\begin{aligned}
&\Delta t\sum_{n=0}^{m-1}\bar{F}_d^n\left(W_h^{n+1}+W_h^n\right) \\
=&\Delta t\sum_{n=0}^{m-1}\left(\bar{f}^n, W_h^{n+1}+W_h^n\right)_{L_2(\Omega)} + 2\left(\bar{g}_N^{m-1}, U_h^m\right)_{L_2(\Gamma_N)} \\
&-2\left(\bar{g}_N^0, U_h^0\right)_{L_2(\Gamma_N)} - \sum_{n=1}^{m-1}\int_{t_{n-1}}^{t_{n+1}}\left(\dot{g}_N(t'), U_h^n\right)_{L_2(\Gamma_N)}dt'.
\end{aligned}
$$

Using Cauchy-Schwarz inequality,

$$
\begin{aligned}
&\left|\Delta t \sum_{n=0}^{m-1} \bar{F}_d^n \left(W_h^{n+1} + W_h^n\right)\right| \\
&\leq \Delta t \sum_{n=0}^{m-1} \left\|\bar{f}^n\right\|_{L_2(\Omega)} \left\|W_h^{n+1} + W_h^n\right\|_{L_2(\Omega)} \\
&\quad + 2 \sum_{e\subset\Gamma_N} \left\|\bar{g}_N^{m-1}\right\|_{L_2(e)} \left\|U_h^m\right\|_{L_2(e)} + 2 \sum_{e\subset\Gamma_N} \left\|\bar{g}_N^0\right\|_{L_2(e)} \left\|U_h^0\right\|_{L_2(e)} \\
&\quad + \sum_{n=1}^{m-1} \int_{t_{n-1}}^{t_{n+1}} \sum_{e\subset\Gamma_N} \left\|\dot{g}_N(t')\right\|_{L_2(e)} \left\|U_h^n\right\|_{L_2(e)} dt'.
\end{aligned}
$$

By applying triangular inequality, Young's inequality, inverse polynomial trace inequality and inverse inequality, it implies

$$
\begin{aligned}
&\left|\Delta t \sum_{n=0}^{m-1} \bar{F}_d^n \left(W_h^{n+1} + W_h^n\right)\right| \\
&\leq \frac{\Delta t}{\epsilon_a} \sum_{n=0}^{m-1} \left\|\bar{f}^n\right\|_{L_2(\Omega)}^2 + \frac{\Delta t \epsilon_a}{2} \sum_{n=0}^{m-1} (\left\|W_h^{n+1}\right\|_{L_2(\Omega)}^2 + \|W_h^n\|_{L_2(\Omega)}^2) \\
&\quad + \frac{1}{\epsilon_b} \left\|\bar{g}_N^{m-1}\right\|_{L_2(\Gamma_N)}^2 + C\frac{\epsilon_b}{h} \|U_h^m\|_{\mathcal{V}} + \frac{1}{h} \left\|\bar{g}_N^0\right\|_{L_2(\Gamma_N)}^2 + C\left\|U_h^0\right\|_{\mathcal{V}}^{\ 2} \\
&\quad + \sum_{n=1}^{m-1} \int_{t_{n-1}}^{t_{n+1}} \frac{1}{2} \left(\frac{1}{\epsilon_b} \left\|\dot{g}_N(t')\right\|_{L_2(\Gamma_N)}^2 + C\frac{\epsilon_b}{h} \|U_h^n\|_{\mathcal{V}}^2\right) dt' \\
&\leq \frac{\Delta t}{\epsilon_a} \sum_{n=0}^{m-1} \left\|\bar{f}^n\right\|_{L_2(\Omega)}^2 + \frac{\Delta t \epsilon_a}{2} \sum_{n=0}^{m-1} (\left\|W_h^{n+1}\right\|_{L_2(\Omega)}^2 + \|W_h^n\|_{L_2(\Omega)}^2) \\
&\quad + \frac{1}{\epsilon_b} \left\|\bar{g}_N^{m-1}\right\|_{L_2(\Gamma_N)}^2 + \frac{C\epsilon_b}{h} \|U_h^m\|_{\mathcal{V}}^2 + \frac{1}{h} \left\|\bar{g}_N^0\right\|_{L_2(\Gamma_N)}^2 + C\left\|U_h^0\right\|_{\mathcal{V}}^2 \\
&\quad + \frac{1}{\epsilon_b} \int_0^{t_m} \left\|\dot{g}_N(t')\right\|_{L_2(\Gamma_N)}^2 dt' + \frac{C\Delta t \epsilon_b}{h} \sum_{n=1}^{m-1} \|U_h^n\|_{\mathcal{V}}^2 .
\end{aligned}
$$

When we consider maxima of the right hand side, since $m \leq N$, it is observed that

$$
\begin{aligned}
&\left|\Delta t \sum_{n=0}^{m-1} \bar{F}_d^n \left(W_h^{n+1} + W_h^n\right)\right| \\
&\leq \frac{\Delta t}{\epsilon_a} \sum_{n=0}^{N-1} \left\|\bar{f}^n\right\|_{L_2(\Omega)}^2 + \Delta t \epsilon_a N \max_{0\leq n\leq N} \|W_h^n\|_{L_2(\Omega)}^2 \\
&\quad + \frac{1}{\epsilon_b} \max_{0\leq n\leq N-1} \|\bar{g}_N^n\|_{L_2(\Gamma_N)}^2 + \frac{C\epsilon_b}{h} \max_{0\leq n\leq N} \|U_h^m\|_{\mathcal{V}}^2
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{h}\max_{0\le n\le N-1}\|\bar{g}_N^n\|_{L_2(\Gamma_N)}^2+C\big\|U_h^0\big\|_{\mathcal{V}}^2\\
&+\frac{1}{\epsilon_b}\int_0^T\big\|\dot{g}_N(t')\big\|_{L_2(\Gamma_N)}^2\,dt'+\frac{C\Delta t\epsilon_b}{h}N\max_{0\le n\le N}\|U_h^n\|_{\mathcal{V}}^2\\
\le&\frac{\Delta t}{\epsilon_a}\sum_{n=0}^{N-1}\big\|\bar{f}^n\big\|_{L_2(\Omega)}^2+T\epsilon_a\max_{0\le n\le N}\|W_h^n\|_{L_2(\Omega)}^2\\
&+\left(\frac{1}{\epsilon_b}+\frac{1}{h}\right)\max_{0\le n\le N-1}\|\bar{g}_N^n\|_{L_2(\Gamma_N)}^2+\frac{1}{\epsilon_b}\|\dot{g}_N\|_{L_2(0,T;L_2(\Gamma_N))}^2\\
&+C\big\|U_h^0\big\|_{\mathcal{V}}^2+\frac{C(T+1)\epsilon_b}{h}\max_{0\le n\le N}\|U_h^n\|_{\mathcal{V}}^2\\
\le&\frac{\Delta t}{\epsilon_a}\sum_{n=0}^{N-1}\big\|\bar{f}^n\big\|_{L_2(\Omega)}^2+T\epsilon_a\max_{0\le n\le N}\|W_h^n\|_{L_2(\Omega)}^2\\
&+\left(\frac{1}{\epsilon_b}+\frac{1}{h}\right)\max_{0\le n\le N-1}\|\bar{g}_N^n\|_{L_2(\Gamma_N)}^2+\frac{1}{\epsilon_b}\|\dot{g}_N\|_{L_2(0,T;L_2(\Gamma_N))}^2\\
&+CK\|u_0\|_{\mathcal{V}}^2+\frac{C(T+1)\epsilon_b}{h}\max_{0\le n\le N}\|U_h^n\|_{\mathcal{V}}^2
\end{aligned}
$$

since $T=N\Delta t$ and $\big\|U_h^0\big\|_{\mathcal{V}}\le K\,\|u_0\|_{\mathcal{V}}$.

- $|2\sum_{q=1}^{N_\varphi}a_{-1}\left(\Psi_{hq}^m,U_h^m\right)|$

  With (3.2.5), Cauchy-Schwarz inequality and Young's inequality, it can be rewritten as

$$
\begin{aligned}
\left|2\sum_{q=1}^{N_\varphi}a_{-1}\left(\Psi_{hq}^m,U_h^m\right)\right|\le&2\sum_{q=1}^{N_\varphi}\|U_h^m\|_{\mathcal{V}}\left\|\Psi_{hq}^m\right\|_{\mathcal{V}}+\frac{C}{\sqrt{\alpha_0}}\sum_{q=1}^{N_\varphi}(\|U_h^m\|_{\mathcal{V}}^2+\left\|\Psi_{hq}^m\right\|_{\mathcal{V}}^2)\\
\le&\sum_{q=1}^{N_\varphi}(\epsilon_q+\frac{C}{\sqrt{\alpha_0}})\|U_h^m\|_{\mathcal{V}}^2+\sum_{q=1}^{N_\varphi}(\frac{1}{\epsilon_q}+\frac{C}{\sqrt{\alpha_0}})\left\|\Psi_{hq}^m\right\|_{\mathcal{V}}^2,
\end{aligned}
$$

  for any positive ϵ_q, $\forall q\in\{1,\dots,N_\varphi\}$.

- $|\sum_{n=0}^{m-1}B\left(U_h^{n+1},U_h^n\right)|$

  Note that

$$
B\left(v,v\right)=0,\ \forall v\in\mathcal{D}_k(\mathcal{E}_h),
$$

  so adding this zero gives

$$
\sum_{n=0}^{m-1}B\left(U_h^{n+1},U_h^n\right)=\sum_{n=0}^{m-1}B\left(U_h^{n+1}-U_h^n,U_h^n\right)
$$

$$=\frac{\Delta t}{2}\sum_{n=0}^{m-1} B\left(W_h^{n+1}+W_h^n, U_h^n\right)$$

with (3.3.1). By the definition and summation by parts,

$$\begin{aligned}
&\frac{\Delta t}{2}\sum_{n=0}^{m-1} B\left(W_h^{n+1}+W_h^n, U_h^n\right)\\
=&\Delta t\sum_{n=0}^{m-1}\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e \{D\nabla U_h^n\cdot \underline{n}_e\}[W_h^{n+1}+W_h^n]de\\
&-\Delta t\sum_{n=0}^{m-1}\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e \{D\nabla (W_h^{n+1}+W_h^n)\cdot \underline{n}_e\}[U_h^n]de\\
=&\Delta t\sum_{n=0}^{m-1}\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e \{D\nabla U_h^n\cdot \underline{n}_e\}[W_h^{n+1}+W_h^n]de\\
&-2\sum_{n=0}^{m-1}\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e \{D\nabla (U_h^{n+1}-U_h^n)\cdot \underline{n}_e\}[U_h^n]de\\
=&\Delta t\sum_{n=0}^{m-1}\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e \{D\nabla U_h^n\cdot \underline{n}_e\}[W_h^{n+1}+W_h^n]de\\
&-2\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e \{D\nabla U_h^m\cdot \underline{n}_e\}[U_h^{m-1}]de\\
&+2\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e \{D\nabla U_h^0\cdot \underline{n}_e\}[U_h^0]de\\
&+2\sum_{n=0}^{m-2}\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e \{D\nabla U_h^{n+1}\cdot \underline{n}_e\}[U_h^{n+1}-U_h^n]de\\
=&\Delta t\sum_{n=0}^{m-1}\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e \{D\nabla U_h^n\cdot \underline{n}_e\}[W_h^{n+1}+W_h^n]de\\
&-2\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e \{D\nabla U_h^m\cdot \underline{n}_e\}[U_h^{m-1}]de\\
&+2\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e \{D\nabla U_h^0\cdot \underline{n}_e\}[U_h^0]de\\
&+\Delta t\sum_{n=0}^{m-2}\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e \{D\nabla U_h^{n+1}\cdot \underline{n}_e\}[W_h^{n+1}+W_h^n]de.
\end{aligned}$$

Use of (3.2.1) yields

$$\begin{aligned}
\Big|\sum_{n=0}^{m-1} B\left(U_h^{n+1}, U_h^n\right)\Big| =& \Big|\frac{\Delta t}{2}\sum_{n=0}^{m-1} B\left(W_h^{n+1}+W_h^n, U_h^n\right)\Big| \\
\leq & \frac{C\Delta t}{\sqrt{\alpha_0}}\sum_{n=0}^{m-1}\|U_h^n\|_{\mathcal{V}}^2 \\
& + \frac{C\Delta t}{\sqrt{\alpha_0}}\sum_{n=0}^{m-1} J_0^{\alpha_0,\beta_0}(W_h^{n+1}+W_h^n, W_h^{n+1}+W_h^n) \\
& + \frac{C}{\sqrt{\alpha_0}}\|U_h^m\|_{\mathcal{V}}^2 + \frac{C}{\sqrt{\alpha_0}}\left\|U_h^{m-1}\right\|_{\mathcal{V}}^2 + \frac{C}{\sqrt{\alpha_0}}\left\|U_h^0\right\|_{\mathcal{V}}^2 \\
& + \frac{C\Delta t}{\sqrt{\alpha_0}}\sum_{n=0}^{m-2}\|U_h^n\|_{\mathcal{V}}^2 \\
& + \frac{C\Delta t}{\sqrt{\alpha_0}}\sum_{n=0}^{m-2} J_0^{\alpha_0,\beta_0}(W_h^{n+1}+W_h^n, W_h^{n+1}+W_h^n) \\
\leq & \frac{2C\Delta t}{\sqrt{\alpha_0}}\sum_{n=0}^{m-1}\|U_h^n\|_{\mathcal{V}}^2 \\
& + \frac{2C\Delta t}{\sqrt{\alpha_0}}\sum_{n=0}^{m-1} J_0^{\alpha_0,\beta_0}(W_h^{n+1}+W_h^n, W_h^{n+1}+W_h^n) \\
& + \frac{C}{\sqrt{\alpha_0}}\|U_h^m\|_{\mathcal{V}}^2 + \frac{C}{\sqrt{\alpha_0}}\left\|U_h^{m-1}\right\|_{\mathcal{V}}^2 + \frac{CK}{\sqrt{\alpha_0}}\|u_0\|_{\mathcal{V}}^2
\end{aligned}$$

by initial condition. Taking account into maximum with respect to U_h^n, we have

$$\begin{aligned}
\Big|\sum_{n=0}^{m-1} B\left(U_h^{n+1}, U_h^n\right)\Big| \leq & \frac{CK}{\sqrt{\alpha_0}}\|u_0\|_{\mathcal{V}}^2 + \frac{2C(T+1)}{\sqrt{\alpha_0}}\max_{0\leq n\leq N}\|U_h^n\|_{\mathcal{V}}^2 \\
& + \frac{2C\Delta t}{\sqrt{\alpha_0}}\sum_{n=0}^{m-1} J_0^{\alpha_0,\beta_0}(W_h^{n+1}+W_h^n, W_h^{n+1}+W_h^n)
\end{aligned}$$

- $\Big|\sum_{q=1}^{N_\varphi}\frac{2}{\varphi_q}\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e\{D\nabla\Psi_{hq}^m\cdot \underline{n}_e\}[\Psi_{hq}^m]de\Big|$

It is easy to see

$$\Big|\sum_{q=1}^{N_\varphi}\frac{2}{\varphi_q}\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e\{D\nabla\Psi_{hq}^m\cdot \underline{n}_e\}[\Psi_{hq}^m]de\Big| \leq \sum_{q=1}^{N_\varphi}\frac{2C}{\varphi_q\sqrt{\alpha_0}}\left\|\Psi_{hq}^m\right\|_{\mathcal{V}}^2$$

by (3.2.1).

Turning to the main proof, as a result, tidying up into (3.3.10) implies

$$
\begin{aligned}
&\rho\left\|W_h^m\right\|_{L_2(\Omega)}^2+\left\|U_h^m\right\|_{\mathcal{V}}^2+\sum_{q=1}^{N_\varphi} \frac{1}{\varphi_q}\left\|\Psi_{hq}^m\right\|_{\mathcal{V}}^2+\sum_{n=0}^{m-1} \sum_{q=1}^{N_\varphi} \frac{2\kappa\tau_q}{\Delta t\varphi_q}\left\|\Psi_{hq}^{n+1}-\Psi_{hq}^n\right\|_{\mathcal{V}}^2 \\
&+\frac{\Delta t}{2} \sum_{n=0}^{m-1} J_0^{\alpha_0,\beta_0}(W_h^{n+1}+W_h^n, W_h^{n+1}+W_h^n) \\
\leq&\rho\left\|w_0\right\|_{L_2(\Omega)}^2+K\big(1+C+\frac{C}{\sqrt{\alpha_0}}\big)\left\|u_0\right\|_{\mathcal{V}}^2+\frac{\Delta t}{\epsilon_a} \sum_{n=0}^{N-1}\left\|\bar{f}^n\right\|_{L_2(\Omega)}^2 \\
&+\left(\frac{1}{\epsilon_b}+\frac{1}{h}\right) \max_{0 \leq n \leq N-1}\left\|\bar{g}_N^n\right\|_{L_2(\Gamma_N)}^2+\frac{1}{\epsilon_b}\left\|\dot{g}_N\right\|_{L_2(0,T;L_2(\Gamma_N))}^2 \\
&+T\epsilon_a \max_{0 \leq n \leq N}\left\|W_h^n\right\|_{L_2(\Omega)}^2+\frac{C(T+1)\epsilon_b}{h} \max_{0 \leq n \leq N}\left\|U_h^n\right\|_{\mathcal{V}}^2+\sum_{q=1}^{N_\varphi}(\epsilon_q+\frac{C}{\sqrt{\alpha_0}})\left\|U_h^m\right\|_{\mathcal{V}}^2 \\
&+\sum_{q=1}^{N_\varphi}(\frac{1}{\epsilon_q}+\frac{C}{\sqrt{\alpha_0}})\left\|\Psi_{hq}^m\right\|_{\mathcal{V}}^2+\frac{2C(T+1)}{\sqrt{\alpha_0}} \max_{0 \leq n \leq N}\left\|U_h^n\right\|_{\mathcal{V}}^2 \\
&+\frac{2C\Delta t}{\sqrt{\alpha_0}} \sum_{n=0}^{m-1} J_0^{\alpha_0,\beta_0}(W_h^{n+1}+W_h^n, W_h^{n+1}+W_h^n)+\sum_{q=1}^{N_\varphi} \frac{2C}{\varphi_q\sqrt{\alpha_0}}\left\|\Psi_{hq}^m\right\|_{\mathcal{V}}^2 .
\end{aligned}
$$

Set $\epsilon_q=\varphi_q+\varphi_0/(2N_\varphi)>0$ for each q. Then we have

$$
\begin{aligned}
&\rho\left\|W_h^m\right\|_{L_2(\Omega)}^2+(\frac{\varphi_0}{2}-\frac{CN_\varphi}{\sqrt{\alpha_0}})\left\|U_h^m\right\|_{\mathcal{V}}^2+\sum_{q=1}^{N_\varphi}\left(\frac{\varphi_0}{2\varphi_q^2N_\varphi+\varphi_0\varphi_q}-\frac{C(2+\varphi_q)}{\varphi_q\sqrt{\alpha_0}}\right)\left\|\Psi_{hq}^m\right\|_{\mathcal{V}}^2 \\
&+\sum_{n=0}^{m-1} \sum_{q=1}^{N_\varphi} \frac{2\kappa\tau_q}{\Delta t\varphi_q}\left\|\Psi_{hq}^{n+1}-\Psi_{hq}^n\right\|_{\mathcal{V}}^2 \\
&+\Delta t\left(\frac{1}{2}-\frac{2C}{\sqrt{\alpha_0}}\right) \sum_{n=0}^{m-1} J_0^{\alpha_0,\beta_0}(W_h^{n+1}+W_h^n, W_h^{n+1}+W_h^n) \\
\leq&\rho\left\|w_0\right\|_{L_2(\Omega)}^2+K(1+C+\frac{C}{\sqrt{\alpha_0}})\left\|u_0\right\|_{\mathcal{V}}^2+\frac{\Delta t}{\epsilon_a} \sum_{n=0}^{N-1}\left\|\bar{f}^n\right\|_{L_2(\Omega)}^2 \\
&+\left(\frac{1}{\epsilon_b}+\frac{1}{h}\right) \max_{0 \leq n \leq N-1}\left\|\bar{g}_N^n\right\|_{L_2(\Gamma_N)}^2+\frac{1}{\epsilon_b}\left\|\dot{g}_N\right\|_{L_2(0,T;L_2(\Gamma_N))}^2 \\
&+T\epsilon_a \max_{0 \leq n \leq N}\left\|W_h^n\right\|_{L_2(\Omega)}^2+\left(\frac{C(T+1)\epsilon_b}{h}+\frac{2C(T+1)}{\sqrt{\alpha_0}}\right) \max_{0 \leq n \leq N}\left\|U_h^n\right\|_{\mathcal{V}}^2 .
\end{aligned}
$$

Taking into account the property of maximum, we can obtain

$$
\rho \max_{0 \leq n \leq N}\left\|W_h^n\right\|_{L_2(\Omega)}^2+(\frac{\varphi_0}{2}-\frac{CN_\varphi}{\sqrt{\alpha_0}}) \max_{0 \leq n \leq N}\left\|U_h^n\right\|_{\mathcal{V}}^2
$$

$$
\begin{aligned}
&+\sum_{q=1}^{N_\varphi}\left(\frac{\varphi_0}{2\varphi_q^2N_\varphi+\varphi_0\varphi_q}-\frac{C(2+\varphi_q)}{\varphi_q\sqrt{\alpha_0}}\right)\left\|\Psi_{hq}^m\right\|_{\mathcal{V}}^2+\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi}\frac{2\kappa\tau_q}{\Delta t\varphi_q}\left\|\Psi_{hq}^{n+1}-\Psi_{hq}^n\right\|_{\mathcal{V}}^2\\
&+\Delta t\left(\frac{1}{2}-\frac{2C}{\sqrt{\alpha_0}}\right)\sum_{n=0}^{m-1}J_0^{\alpha_0,\beta_0}(W_h^{n+1}+W_h^n,W_h^{n+1}+W_h^n)\\
\le&3\biggl(\rho\left\|w_0\right\|_{L_2(\Omega)}^2+K(1+C+\frac{C}{\sqrt{\alpha_0}})\left\|u_0\right\|_{\mathcal{V}}^2+\frac{\Delta t}{\epsilon_a}\sum_{n=0}^{N-1}\left\|\bar{f}^n\right\|_{L_2(\Omega)}^2\\
&+\left(\frac{1}{\epsilon_b}+\frac{1}{h}\right)\max_{0\le n\le N-1}\left\|\bar{g}_N^n\right\|_{L_2(\Gamma_N)}^2+\frac{1}{\epsilon_b}\left\|\dot{g}_N\right\|_{L_2(0,T;L_2(\Gamma_N))}^2\\
&+T\epsilon_a\max_{0\le n\le N}\left\|W_h^n\right\|_{L_2(\Omega)}^2+\left(\frac{C(T+1)\epsilon_b}{h}+\frac{2C(T+1)}{\sqrt{\alpha_0}}\right)\max_{0\le n\le N}\left\|U_h^n\right\|_{\mathcal{V}}^2.
\end{aligned}
$$

By setting

$$
\epsilon_a=\frac{\rho}{6T}>0,\;\epsilon_b=\frac{\varphi_0 h}{6C(T+1)}>0,
$$

it yields

$$
\begin{aligned}
&\frac{\rho}{2}\max_{0\le n\le N}\left\|W_h^n\right\|_{L_2(\Omega)}^2+\left(\frac{\varphi_0}{4}-\frac{6C(T+1)+CN_\varphi}{\sqrt{\alpha_0}}\right)\max_{0\le n\le N}\left\|U_h^n\right\|_{\mathcal{V}}^2\\
&+\sum_{q=1}^{N_\varphi}\left(\frac{\varphi_0}{2\varphi_q^2N_\varphi+\varphi_0\varphi_q}-\frac{C(2+\varphi_q)}{\varphi_q\sqrt{\alpha_0}}\right)\left\|\Psi_{hq}^m\right\|_{\mathcal{V}}^2+\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi}\frac{2\kappa\tau_q}{\Delta t\varphi_q}\left\|\Psi_{hq}^{n+1}-\Psi_{hq}^n\right\|_{\mathcal{V}}^2\\
&+\Delta t\left(\frac{1}{2}-\frac{2C}{\sqrt{\alpha_0}}\right)\sum_{n=0}^{m-1}J_0^{\alpha_0,\beta_0}(W_h^{n+1}+W_h^n,W_h^{n+1}+W_h^n)\\
\le&3\biggl(\rho\left\|w_0\right\|_{L_2(\Omega)}^2+K(1+C+\frac{C}{\sqrt{\alpha_0}})\left\|u_0\right\|_{\mathcal{V}}^2+\Delta t\frac{6T}{\rho}\sum_{n=0}^{N-1}\left\|\bar{f}^n\right\|_{L_2(\Omega)}^2\\
&+\left(\frac{6C(T+1)}{\varphi_0 h}+\frac{1}{h}\right)\max_{0\le n\le N-1}\left\|\bar{g}_N^n\right\|_{L_2(\Gamma_N)}^2\\
&+\frac{6C(T+1)}{\varphi_0 h}\left\|\dot{g}_N\right\|_{L_2(0,T;L_2(\Gamma_N))}^2\biggr).
\end{aligned}
$$

If we take a large α_0 such that

$$
\begin{aligned}
\frac{\varphi_0}{4}-\frac{6C(T+1)+CN_\varphi}{\sqrt{\alpha_0}}&>0,\\
\frac{\varphi_0\kappa}{2\varphi_q^2N_\varphi+2\varphi_0\varphi_q}-\frac{C}{\sqrt{\alpha_0}}&>0,\;\forall q\in\{1,\dots,N_\varphi\},\\
\frac{1}{2}-\frac{2C}{\sqrt{\alpha_0}}&>0,
\end{aligned}
$$

it is concluded that

$$
\begin{aligned}
&\max_{0\le n\le N}\left\|\rho^{1/2}W_h^n\right\|_{L_2(\Omega)}^2+\max_{0\le n\le N}\left\|U_h^n\right\|_{\mathcal{V}}^2+\sum_{q=1}^{N_\varphi}\left\|\Psi_{hq}^m\right\|_{\mathcal{V}}^2+\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi}\frac{1}{\Delta t}\left\|\Psi_{hq}^{n+1}-\Psi_{hq}^n\right\|_{\mathcal{V}}^2\\
&+\Delta t\sum_{n=0}^{m-1}J_0^{\alpha_0,\beta_0}(W_h^{n+1}+W_h^n,W_h^{n+1}+W_h^n)\\
\le&C\Bigg(\left\|\rho^{1/2}w_0\right\|_{L_2(\Omega)}^2+\left\|u_0\right\|_{\mathcal{V}}^2+\Delta t\sum_{n=0}^{N-1}\left\|\bar{f}^n\right\|_{L_2(\Omega)}^2\\
&+h^{-1}\max_{0\le n\le N-1}\left\|\bar{g}_N^n\right\|_{L_2(\Gamma_N)}^2+h^{-1}\left\|\dot{g}_N\right\|_{L_2(0,T;L_2(\Gamma_N))}^2\Bigg),
\end{aligned}
$$

and, since

$$
\Delta t\sum_{n=0}^{N-1}\left\|\bar{f}^n\right\|_{L_2(\Omega)}^2\le 2T\left\|f\right\|_{L_\infty(0,T;L_2(\Omega))}^2,
$$

and

$$
\max_{0\le n\le N-1}\left\|\bar{g}_N^n\right\|_{L_2(\Gamma_N)}^2\le 2\left\|g_N\right\|_{L_\infty(0,T;L_2(\Gamma_N))}^2,
$$

we can also obtain

$$
\begin{aligned}
&\max_{0\le n\le N}\left\|\rho^{1/2}W_h^n\right\|_{L_2(\Omega)}^2+\max_{0\le n\le N}\left\|U_h^n\right\|_{\mathcal{V}}^2+\sum_{q=1}^{N_\varphi}\left\|\Psi_{hq}^m\right\|_{\mathcal{V}}^2+\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi}\frac{1}{\Delta t}\left\|\Psi_{hq}^{n+1}-\Psi_{hq}^n\right\|_{\mathcal{V}}^2\\
&+\Delta t\sum_{n=0}^{m-1}J_0^{\alpha_0,\beta_0}(W_h^{n+1}+W_h^n,W_h^{n+1}+W_h^n)\\
\le&C\Bigg(\left\|\rho^{1/2}w_0\right\|_{L_2(\Omega)}^2+\left\|u_0\right\|_{\mathcal{V}}^2+\left\|f\right\|_{L_\infty(0,T;L_2(\Omega))}^2+h^{-1}\left\|g_N\right\|_{L_\infty(0,T;L_2(\Gamma_N))}^2\\
&+h^{-1}\left\|\dot{g}_N\right\|_{L_2(0,T;L_2(\Gamma_N))}^2\Bigg)
\end{aligned}
$$

for some positive C. □

We proved the stability bound for the fully discrete formulation of **(Q1)** without using Grönwall's inequality. For the stability, it requires sufficiently large α_0. From this stability bound, we can also show the uniqueness and existence of the fully discrete solution of **(Q1)**. Then we will consider error bounds by introducing DG elliptic projection.

Let us define

$$
\begin{aligned}
\theta &:= u-R_1u,\\
\chi^n &:= U_h^n-R_1u^n,\\
\varpi^n &:= W_h^n-R_1\dot{u}^n,
\end{aligned}
$$

$$\vartheta_q := \psi_q - R_{-1}\psi_q \ \forall q \in \{1,\dots,N_\varphi\},$$
$$\varsigma_q^n := \Psi_{hq}^n - R_{-1}\psi_q^n \ \forall q \in \{1,\dots,N_\varphi\},$$

where $u^n = u(t_n)$ for $0 \le n \le N$. By using DG elliptic projection (3.2.14), we shall show the error bounds for the fully discrete formulation of **(Q1)**.

<u>Remark</u> For any $v \in \mathcal{D}_k(\mathcal{E}_h)$, $\forall t$, Galerkin orthogonality gives

$$a_1\left(\theta(t), v\right) = 0, \tag{3.3.11}$$
$$a_1\left(\dot{\theta}(t), v\right) = 0, \tag{3.3.12}$$
$$a_{-1}\left(\vartheta_q(t), v\right) = 0, \ \forall q \in \{1,\dots,N_\varphi\}, \tag{3.3.13}$$
$$a_{-1}\left(\dot{\vartheta}_q(t), v\right) = 0, \ \forall q \in \{1,\dots,N_\varphi\}. \tag{3.3.14}$$

In addition, the continuity of the strong solution and homogeneous Dirichlet boundary condition impose

$$[\theta(t)] = 0, \ [\dot{\theta}(t)] = 0, \quad [\vartheta_q(t)] = 0, \ [\dot{\vartheta}_q(t)] = 0, \tag{3.3.15}$$

for any t, $\forall q \in \{1,\dots,N_\varphi\}$ on $\Gamma_h \cup \Gamma_D$. Moreover, for any $v \in \mathcal{D}_k(\mathcal{E}_h)$, SIPG can be written as

$$a_{-1}\left(\theta(t), v\right) = a_1\left(\theta(t), v\right) - 2\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e \{D\nabla v\cdot \underline{n}_e\}[\theta(t)]de$$

so that (3.3.11) and (3.3.15) imply

$$a_{-1}\left(\theta(t), v\right) = 0. \tag{3.3.16}$$

So is $\dot{\theta}(t)$ by (3.3.12).

Lemma 3.6. *Suppose $u \in H^4(0,T;C^2(\Omega)) \cap W^1_\infty(0,T;H^s(\mathcal{E}_h))$ and $\beta_0(d-1) \ge 1$ for $s > 3/2$. For large enough α_0, there exists a positive constant C such that*

$$\begin{aligned}
&\max_{0\le j\le N}\left\|\varpi^j\right\|_{L_2(\Omega)} + \max_{0\le j\le N}\left\|\chi^j\right\|_{\mathcal{V}} + \sum_{q=1}^{N_\varphi}\max_{0\le j\le N}\left\|\varsigma_q^j\right\|_{\mathcal{V}} + \Delta t\sum_{q=1}^{N_\varphi}\sum_{n=0}^{N-1}\left\|\frac{\varsigma_q^{n+1}-\varsigma_q^n}{\Delta t}\right\|_{\mathcal{V}} \\
&+\left(\Delta t\sum_{n=0}^{N-1} J_0^{\alpha_0,\beta_0}\left(\varpi^{n+1}+\varpi^n, \varpi^{n+1}+\varpi^n\right)\right)^{1/2} \\
&\le C(h^{\min(k+1,s)-1} + \Delta t^2).
\end{aligned}$$

Furthermore, if Ω is convex and $\beta_0(d-1) \ge 3$ is given, we have

$$\max_{0\le j\le N}\left\|\varpi^j\right\|_{L_2(\Omega)} + \max_{0\le j\le N}\left\|\chi^j\right\|_{\mathcal{V}} + \sum_{q=1}^{N_\varphi}\max_{0\le j\le N}\left\|\varsigma_q^j\right\|_{\mathcal{V}} + \Delta t\sum_{q=1}^{N_\varphi}\sum_{n=0}^{N-1}\left\|\frac{\varsigma_q^{n+1}-\varsigma_q^n}{\Delta t}\right\|_{\mathcal{V}}$$

$$
+\left(\Delta t\sum_{n=0}^{N-1} J_0^{\alpha_0,\beta_0}\left(\varpi^{n+1}+\varpi^n,\varpi^{n+1}+\varpi^n\right)\right)^{1/2}
\leq C(h^{\min(k+1,s)}+\Delta t^2).
$$

Proof. Consider subtracting (3.3.2) from (3.1.1)

$$
\begin{aligned}
&\left(\frac{\rho}{2}(\ddot{u}^{n+1}+\ddot{u}^n)-\frac{\rho}{\Delta t}(W_h^{n+1}-W_h^n),v\right)_{L_2(\Omega)}+\frac{1}{2}a_1\left((u^{n+1}+u^n)-(U_h^{n+1}+U_h^n),v\right)\\
&-\frac{1}{2}\sum_{q=1}^{N_\varphi}a_{-1}\left((\psi_q^{n+1}+\psi_q^n)-(\Psi_{hq}^{n+1}+\Psi_{hq}^n),v\right)\\
&+\frac{1}{2}J_0^{\alpha_0,\beta_0}\left((\dot{u}^{n+1}+\dot{u}^n)-(W_h^{n+1}+W_h^n),v\right)=0,
\end{aligned}
$$

for any $v\in\mathcal{D}_k(\mathcal{E}_h)$. Hence it holds

$$
\begin{aligned}
&\frac{\rho}{\Delta t}\left(\varpi^{n+1}-\varpi^n,v\right)_{L_2(\Omega)}+\frac{1}{2}a_1\left(\chi^{n+1}+\chi^n,v\right)-\frac{1}{2}\sum_{q=1}^{N_\varphi}a_{-1}\left(\varsigma_q^{n+1}+\varsigma_q^n,v\right)\\
&\quad+\frac{1}{2}J_0^{\alpha_0,\beta_0}(\varpi^{n+1}+\varpi^n,v)\\
&=\frac{\rho}{\Delta t}\left(\dot{\theta}^{n+1}-\dot{\theta}^n,v\right)_{L_2(\Omega)}+\frac{1}{2}a_1\left(\theta^{n+1}+\theta^n,v\right)-\frac{1}{2}\sum_{q=1}^{N_\varphi}a_{-1}\left(\vartheta_q^{n+1}+\vartheta_q^n,v\right)\\
&\quad+\rho\left(\frac{\ddot{u}^{n+1}+\ddot{u}^n}{2}-\frac{\dot{u}^{n+1}-\dot{u}^n}{\Delta t},v\right)_{L_2(\Omega)}+\frac{1}{2}J_0^{\alpha_0,\beta_0}(\dot{\theta}^{n+1}+\dot{\theta}^n,v),\\
&=\frac{\rho}{\Delta t}\left(\dot{\theta}^{n+1}-\dot{\theta}^n,v\right)_{L_2(\Omega)}+\rho\left(\mathcal{E}_1^n,v\right)_{L_2(\Omega)}, \qquad (3.3.17)
\end{aligned}
$$

for $0\leq n\leq N-1$, where

$$
\mathcal{E}_1(t)=\frac{\ddot{u}(t+\Delta t)+\ddot{u}(t)}{2}-\frac{\dot{u}(t+\Delta t)-\dot{u}(t)}{\Delta t},
$$

by (3.3.11), (3.3.12) and (3.3.15). Note that we have the following identity equation

$$
\frac{\chi^{n+1}-\chi^n}{\Delta t}=\frac{\varpi^{n+1}+\varpi^n}{2}-\mathcal{E}_2^n-\mathcal{E}_3^n, \qquad (3.3.18)
$$

where

$$
\begin{aligned}
\mathcal{E}_2(t)&:=\frac{\dot{\theta}(t+\Delta t)+\dot{\theta}(t)}{2}-\frac{\theta(t+\Delta t)-\theta(t)}{\Delta t},\\
\mathcal{E}_3(t)&:=\frac{u(t+\Delta t)-u(t)}{\Delta t}-\frac{\dot{u}(t+\Delta t)+\dot{u}(t)}{2},
\end{aligned}
$$

by (3.3.1). Put $v = (\chi^{n+1} - \chi^n)/\Delta t$ into (3.3.17). Then (3.3.18) gives

$$
\begin{aligned}
&\frac{\rho}{\Delta t}\left(\varpi^{n+1} - \varpi^n, \frac{\chi^{n+1} - \chi^n}{\Delta t}\right)_{L_2(\Omega)} + \frac{1}{2}a_1\left(\chi^{n+1} + \chi^n, \frac{\chi^{n+1} - \chi^n}{\Delta t}\right) \\
&- \frac{1}{2}\sum_{q=1}^{N_\varphi} a_{-1}\left(\varsigma_q^{n+1} + \varsigma_q^n, \frac{\chi^{n+1} - \chi^n}{\Delta t}\right) + \frac{1}{2}J_0^{\alpha_0,\beta_0}\left(\varpi^{n+1} + \varpi^n, \frac{\chi^{n+1} - \chi^n}{\Delta t}\right), \\
=&\frac{\rho}{\Delta t}\left(\varpi^{n+1} - \varpi^n, \frac{\varpi^{n+1} + \varpi^n}{2}\right)_{L_2(\Omega)} - \frac{\rho}{\Delta t}\left(\varpi^{n+1} - \varpi^n, \mathcal{E}_2^n\right)_{L_2(\Omega)} \\
&- \frac{\rho}{\Delta t}\left(\varpi^{n+1} - \varpi^n, \mathcal{E}_3^n\right)_{L_2(\Omega)} + \frac{1}{2}a_1\left(\chi^{n+1} + \chi^n, \frac{\chi^{n+1} - \chi^n}{\Delta t}\right) \\
&- \frac{1}{2}\sum_{q=1}^{N_\varphi} a_{-1}\left(\varsigma_q^{n+1} + \varsigma_q^n, \frac{\chi^{n+1} - \chi^n}{\Delta t}\right) + \frac{1}{4}J_0^{\alpha_0,\beta_0}\left(\varpi^{n+1} + \varpi^n, \varpi^{n+1} + \varpi^n\right) \\
&- \frac{1}{2}J_0^{\alpha_0,\beta_0}\left(\varpi^{n+1} + \varpi^n, \mathcal{E}_2^n\right) - \frac{1}{2}J_0^{\alpha_0,\beta_0}\left(\varpi^{n+1} + \varpi^n, \mathcal{E}_3^n\right), \\
=&\frac{\rho}{2\Delta t}\left(\left\|\varpi^{n+1}\right\|_{L_2(\Omega)}^2 - \left\|\varpi^n\right\|_{L_2(\Omega)}^2\right) - \frac{\rho}{\Delta t}\left(\varpi^{n+1} - \varpi^n, \mathcal{E}_2^n\right)_{L_2(\Omega)} \\
&- \frac{\rho}{\Delta t}\left(\varpi^{n+1} - \varpi^n, \mathcal{E}_3^n\right)_{L_2(\Omega)} + \frac{1}{2\Delta t}\left(\left\|\chi^{n+1}\right\|_{\mathcal{V}}^2 - \left\|\chi^n\right\|_{\mathcal{V}}^2\right) \\
&- \frac{1}{2\Delta t}B(\chi^{n+1}, \chi^n) - \frac{1}{2\Delta t}\sum_{q=1}^{N_\varphi} a_{-1}\left(\varsigma_q^{n+1} + \varsigma_q^n, \chi^{n+1} - \chi^n\right) \\
&+ \frac{1}{4}J_0^{\alpha_0,\beta_0}\left(\varpi^{n+1} + \varpi^n, \varpi^{n+1} + \varpi^n\right) \\
&- \frac{1}{2}J_0^{\alpha_0,\beta_0}\left(\varpi^{n+1} + \varpi^n, \mathcal{E}_2^n\right) - \frac{1}{2}J_0^{\alpha_0,\beta_0}\left(\varpi^{n+1} + \varpi^n, \mathcal{E}_3^n\right), \\
=&\frac{\rho}{2\Delta t}\left(\dot{\theta}^{n+1} - \dot{\theta}^n, \varpi^{n+1} + \varpi^n\right)_{L_2(\Omega)} - \frac{\rho}{\Delta t}\left(\dot{\theta}^{n+1} - \dot{\theta}^n, \mathcal{E}_2^n\right)_{L_2(\Omega)} \\
&- \frac{\rho}{\Delta t}\left(\dot{\theta}^{n+1} - \dot{\theta}^n, \mathcal{E}_3^n\right)_{L_2(\Omega)} + \frac{\rho}{2}\left(\mathcal{E}_1^n, \varpi^{n+1} + \varpi^n\right)_{L_2(\Omega)} - \rho\left(\mathcal{E}_1^n, \mathcal{E}_2^n\right)_{L_2(\Omega)} \\
&- \rho\left(\mathcal{E}_1^n, \mathcal{E}_3^n\right)_{L_2(\Omega)}.
\end{aligned}
$$

Since continuity and homogeneous boundary condition imply

$$[\mathcal{E}_2^n] = 0, \ [\mathcal{E}_3^n] = 0,$$

we have

$$
\begin{aligned}
&\frac{\rho}{2\Delta t}\left(\left\|\varpi^{n+1}\right\|_{L_2(\Omega)}^2 - \left\|\varpi^n\right\|_{L_2(\Omega)}^2\right) - \frac{\rho}{\Delta t}\left(\varpi^{n+1} - \varpi^n, \mathcal{E}_2^n\right)_{L_2(\Omega)} \\
&- \frac{\rho}{\Delta t}\left(\varpi^{n+1} - \varpi^n, \mathcal{E}_3^n\right)_{L_2(\Omega)} + \frac{1}{2\Delta t}\left(\left\|\chi^{n+1}\right\|_{\mathcal{V}}^2 - \left\|\chi^n\right\|_{\mathcal{V}}^2\right) - \frac{1}{2\Delta t}B(\chi^{n+1}, \chi^n) \\
&- \frac{1}{2\Delta t}\sum_{q=1}^{N_\varphi} a_{-1}\left(\varsigma_q^{n+1} + \varsigma_q^n, \chi^{n+1} - \chi^n\right) + \frac{1}{4}J_0^{\alpha_0,\beta_0}\left(\varpi^{n+1} + \varpi^n, \varpi^{n+1} + \varpi^n\right)
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{\rho}{2\Delta t}\left(\dot{\theta}^{n+1}-\dot{\theta}^{n},\varpi^{n+1}+\varpi^{n}\right)_{L_2(\Omega)}-\frac{\rho}{\Delta t}\left(\dot{\theta}^{n+1}-\dot{\theta}^{n},\mathcal{E}_2^n\right)_{L_2(\Omega)}\\
&-\frac{\rho}{\Delta t}\left(\dot{\theta}^{n+1}-\dot{\theta}^{n},\mathcal{E}_3^n\right)_{L_2(\Omega)}+\frac{\rho}{2}\left(\mathcal{E}_1^n,\varpi^{n+1}+\varpi^{n}\right)_{L_2(\Omega)}-\rho\left(\mathcal{E}_1^n,\mathcal{E}_2^n\right)_{L_2(\Omega)}\\
&-\rho\left(\mathcal{E}_1^n,\mathcal{E}_3^n\right)_{L_2(\Omega)}
\end{aligned}
$$

hence on account of summation from $n=0$ to $n=m-1$ for $0<m\leq N$, we derive

$$
\begin{aligned}
&\frac{\rho}{2\Delta t}\left\|\varpi^{m}\right\|_{L_2(\Omega)}^2+\frac{1}{2\Delta t}\left\|\chi^{m}\right\|_{\mathcal{V}}^2-\frac{1}{2\Delta t}\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi}a_{-1}\left(\varsigma_q^{n+1}+\varsigma_q^{n},\chi^{n+1}-\chi^{n}\right)\\
&\quad+\frac{1}{4}\sum_{n=0}^{m-1}J_0^{\alpha_0,\beta_0}\left(\varpi^{n+1}+\varpi^{n},\varpi^{n+1}+\varpi^{n}\right)\\
&=\frac{\rho}{2\Delta t}\left\|\varpi^{0}\right\|_{L_2(\Omega)}^2+\frac{1}{2\Delta t}\left\|\chi^{0}\right\|_{\mathcal{V}}^2+\frac{\rho}{2\Delta t}\sum_{n=0}^{m-1}\left(\dot{\theta}^{n+1}-\dot{\theta}^{n},\varpi^{n+1}+\varpi^{n}\right)_{L_2(\Omega)}\\
&\quad-\frac{\rho}{\Delta t}\sum_{n=0}^{m-1}\left(\dot{\theta}^{n+1}-\dot{\theta}^{n},\mathcal{E}_2^n\right)_{L_2(\Omega)}-\frac{\rho}{\Delta t}\sum_{n=0}^{m-1}\left(\dot{\theta}^{n+1}-\dot{\theta}^{n},\mathcal{E}_3^n\right)_{L_2(\Omega)}\\
&\quad+\frac{\rho}{2}\sum_{n=0}^{m-1}\left(\mathcal{E}_1^n,\varpi^{n+1}+\varpi^{n}\right)_{L_2(\Omega)}-\rho\sum_{n=0}^{m-1}\left(\mathcal{E}_1^n,\mathcal{E}_2^n\right)_{L_2(\Omega)}-\rho\sum_{n=0}^{m-1}\left(\mathcal{E}_1^n,\mathcal{E}_3^n\right)_{L_2(\Omega)}\\
&\quad+\frac{\rho}{\Delta t}\sum_{n=0}^{m-1}\left(\varpi^{n+1}-\varpi^{n},\mathcal{E}_2^n\right)_{L_2(\Omega)}+\frac{\rho}{\Delta t}\sum_{n=0}^{m-1}\left(\varpi^{n+1}-\varpi^{n},\mathcal{E}_3^n\right)_{L_2(\Omega)}\\
&\quad+\frac{1}{2\Delta t}\sum_{n=0}^{m-1}B(\chi^{n+1},\chi^{n}). \qquad (3.3.19)
\end{aligned}
$$

On the other hand, let us consider the subtraction between (3.1.2) and (3.3.3) for each q. It gives

$$
\begin{aligned}
&\frac{\tau_q}{\Delta t}a_{-1}\left(\varsigma_q^{n+1}-\varsigma_q^{n},v\right)+\frac{1}{2}a_{-1}\left(\varsigma_q^{n+1}+\varsigma_q^{n},v\right)-\frac{\varphi_q}{2}a_{-1}\left(\chi^{n+1}+\chi^{n},v\right)\\
&=\frac{\tau_q}{\Delta t}a_{-1}\left(\vartheta_q^{n+1}-\vartheta_q^{n},v\right)+\frac{1}{2}a_{-1}\left(\vartheta_q^{n+1}+\vartheta_q^{n},v\right)-\frac{\varphi_q}{2}a_{-1}\left(\theta^{n+1}+\theta^{n},v\right)\\
&\quad+\tau_q a_{-1}\left(\frac{\dot{\psi}_q^{n+1}+\dot{\psi}_q^{n}}{2}-\frac{\psi_q^{n+1}-\psi_q^{n}}{\Delta t},v\right)\\
&=\tau_q a_{-1}\left(E_q^n,v\right)
\end{aligned}
$$

by (3.3.13) and (3.3.16), for any $v\in\mathcal{D}_k(\mathcal{E}_h)$, where

$$
E_q(t)=\frac{\dot{\psi}_q(t+\Delta t)+\dot{\psi}_q(t)}{2}-\frac{\psi_q(t+\Delta t)-\psi_q(t)}{\Delta t}\text{ for each } q.
$$

Inserting $v = (\varsigma_q^{n+1} - \varsigma_q^n)/\Delta t$ and taking summation for $n = 0, \ldots, m-1$ yield

$$\begin{aligned}
&\frac{\tau_q}{\Delta t^2} \sum_{n=0}^{m-1} a_{-1} \left(\varsigma_q^{n+1} - \varsigma_q^n, \varsigma_q^{n+1} - \varsigma_q^n\right) + \frac{1}{2\Delta t} \left(a_{-1} \left(\varsigma_q^m, \varsigma_q^m\right) - a_{-1} \left(\varsigma_q^0, \varsigma_q^0\right)\right) \\
&\quad - \frac{\varphi_q}{2\Delta t} \sum_{n=0}^{m-1} a_{-1} \left(\varsigma_q^{n+1} - \varsigma_q^n, \chi^{n+1} + \chi^n\right) \\
&= \frac{\tau_q}{\Delta t} \sum_{n=0}^{m-1} a_{-1} \left(E_q^n, \varsigma_q^{n+1} - \varsigma_q^n\right).
\end{aligned}$$

Since $\psi_q(0) = 0 = \psi_{hq}^0$, with applying summation by parts, we gain

$$\begin{aligned}
\sum_{n=0}^{m-1} a_{-1} \left(\varsigma_q^{n+1} - \varsigma_q^n, \chi^{n+1} + \chi^n\right) &= \sum_{n=0}^{m-1} a_{-1} \left(\chi^{n+1} + \chi^n, \varsigma_q^{n+1} - \varsigma_q^n\right) \\
&= 2a_{-1} \left(\chi^m, \varsigma_q^m\right) - \sum_{n=0}^{m-1} a_{-1} \left(\chi^{n+1} - \chi^n, \varsigma_q^{n+1} + \varsigma_q^n\right)
\end{aligned}$$

and

$$\sum_{n=0}^{m-1} a_{-1} \left(E_q^n, \varsigma_q^{n+1} - \varsigma_q^n\right) = a_{-1} \left(E_q^{m-1}, \varsigma_q^m\right) - \sum_{n=0}^{m-2} a_{-1} \left(E_q^{n+1} - E_q^n, \varsigma_q^{n+1}\right).$$

It implies

$$\begin{aligned}
\frac{\varphi_q}{2\Delta t} \sum_{n=0}^{m-1} a_{-1} \left(\chi^{n+1} - \chi^n, \varsigma_q^{n+1} + \varsigma_q^n\right) = &\frac{\varphi_q}{\Delta t} a_{-1} \left(\chi^m, \varsigma_q^m\right) - \frac{1}{2\Delta t} a_{-1} \left(\varsigma_q^m, \varsigma_q^m\right) \\
&- \frac{\tau_q}{\Delta t^2} \sum_{n=0}^{m-1} a_{-1} \left(\varsigma_q^{n+1} - \varsigma_q^n, \varsigma_q^{n+1} - \varsigma_q^n\right) \\
&+ \frac{\tau_q}{\Delta t} a_{-1} \left(E_q^{m-1}, \varsigma_q^m\right) \\
&- \frac{\tau_q}{\Delta t} \sum_{n=0}^{m-2} a_{-1} \left(E_q^{n+1} - E_q^n, \varsigma_q^{n+1}\right).
\end{aligned}$$

As a result, (3.3.19) can be written as

$$\begin{aligned}
&\frac{\rho}{2} \left\|\varpi^m\right\|_{L_2(\Omega)}^2 + \frac{1}{2} \left\|\chi^m\right\|_{\mathcal{V}}^2 + \frac{1}{2} \sum_{q=1}^{N_\varphi} \frac{1}{\varphi_q} a_{-1} \left(\varsigma_q^m, \varsigma_q^m\right) \\
&+ \Delta t \sum_{n=0}^{m-1} \sum_{q=1}^{N_\varphi} \frac{\tau_q}{\varphi_q} a_{-1} \left(\frac{\varsigma_q^{n+1} - \varsigma_q^n}{\Delta t}, \frac{\varsigma_q^{n+1} - \varsigma_q^n}{\Delta t}\right)
\end{aligned}$$

$$+\frac{\Delta t}{4}\sum_{n=0}^{m-1}J_0^{\alpha_0,\beta_0}\left(\varpi^{n+1}+\varpi^n,\varpi^{n+1}+\varpi^n\right)$$

$$=\frac{\rho}{2}\left\|\varpi^0\right\|_{L_2(\Omega)}^2+\frac{1}{2}\left\|\chi^0\right\|_{\mathcal{V}}^2+\frac{\rho}{2}\sum_{n=0}^{m-1}\left(\dot{\theta}^{n+1}-\dot{\theta}^n,\varpi^{n+1}+\varpi^n\right)_{L_2(\Omega)}$$

$$-\rho\sum_{n=0}^{m-1}\left(\dot{\theta}^{n+1}-\dot{\theta}^n,\mathcal{E}_2^n\right)_{L_2(\Omega)}-\rho\sum_{n=0}^{m-1}\left(\dot{\theta}^{n+1}-\dot{\theta}^n,\mathcal{E}_3^n\right)_{L_2(\Omega)}$$

$$+\frac{\rho}{2}\Delta t\sum_{n=0}^{m-1}\left(\mathcal{E}_1^n,\varpi^{n+1}+\varpi^n\right)_{L_2(\Omega)}-\rho\Delta t\sum_{n=0}^{m-1}\left(\mathcal{E}_1^n,\mathcal{E}_2^n\right)_{L_2(\Omega)}-\rho\Delta t\sum_{n=0}^{m-1}\left(\mathcal{E}_1^n,\mathcal{E}_3^n\right)_{L_2(\Omega)}$$

$$+\rho\sum_{n=0}^{m-1}\left(\varpi^{n+1}-\varpi^n,\mathcal{E}_2^n\right)_{L_2(\Omega)}+\rho\sum_{n=0}^{m-1}\left(\varpi^{n+1}-\varpi^n,\mathcal{E}_3^n\right)_{L_2(\Omega)}+\sum_{q=1}^{N_\varphi}a_{-1}\left(\chi^m,\varsigma_q^m\right)$$

$$+\sum_{q=1}^{N_\varphi}\frac{\tau_q}{\varphi_q}a_{-1}\left(E_q^{m-1},\varsigma_q^m\right)-\sum_{n=0}^{m-2}\sum_{q=1}^{N_\varphi}\frac{\tau_q}{\varphi_q}a_{-1}\left(E_q^{n+1}-E_q^n,\varsigma_q^{n+1}\right)+\frac{1}{2}\sum_{n=0}^{m-1}B(\chi^{n+1},\chi^n),$$

and applying coercivity and expanding SIPG also yield

$$\frac{\rho}{2}\left\|\varpi^m\right\|_{L_2(\Omega)}^2+\frac{1}{2}\left\|\chi^m\right\|_{\mathcal{V}}^2+\frac{1}{2}\sum_{q=1}^{N_\varphi}\frac{1}{\varphi_q}\left\|\varsigma_q^m\right\|_{\mathcal{V}}^2+\Delta t\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi}\frac{\kappa\tau_q}{\varphi_q}\left\|\frac{\varsigma_q^{n+1}-\varsigma_q^n}{\Delta t}\right\|_{\mathcal{V}}^2$$

$$+\frac{\Delta t}{4}\sum_{n=0}^{m-1}J_0^{\alpha_0,\beta_0}\left(\varpi^{n+1}+\varpi^n,\varpi^{n+1}+\varpi^n\right)$$

$$\leq\Bigg|\frac{\rho}{2}\left\|\varpi^0\right\|_{L_2(\Omega)}^2+\frac{1}{2}\left\|\chi^0\right\|_{\mathcal{V}}^2+\frac{\rho}{2}\sum_{n=0}^{m-1}\left(\dot{\theta}^{n+1}-\dot{\theta}^n,\varpi^{n+1}+\varpi^n\right)_{L_2(\Omega)}$$

$$-\rho\sum_{n=0}^{m-1}\left(\dot{\theta}^{n+1}-\dot{\theta}^n,\mathcal{E}_2^n\right)_{L_2(\Omega)}-\rho\sum_{n=0}^{m-1}\left(\dot{\theta}^{n+1}-\dot{\theta}^n,\mathcal{E}_3^n\right)_{L_2(\Omega)}$$

$$+\frac{\rho}{2}\Delta t\sum_{n=0}^{m-1}\left(\mathcal{E}_1^n,\varpi^{n+1}+\varpi^n\right)_{L_2(\Omega)}-\rho\Delta t\sum_{n=0}^{m-1}\left(\mathcal{E}_1^n,\mathcal{E}_2^n\right)_{L_2(\Omega)}-\rho\Delta t\sum_{n=0}^{m-1}\left(\mathcal{E}_1^n,\mathcal{E}_3^n\right)_{L_2(\Omega)}$$

$$+\rho\sum_{n=0}^{m-1}\left(\varpi^{n+1}-\varpi^n,\mathcal{E}_2^n\right)_{L_2(\Omega)}+\rho\sum_{n=0}^{m-1}\left(\varpi^{n+1}-\varpi^n,\mathcal{E}_3^n\right)_{L_2(\Omega)}+\sum_{q=1}^{N_\varphi}a_{-1}\left(\chi^m,\varsigma_q^m\right)$$

$$+\sum_{q=1}^{N_\varphi}\frac{\tau_q}{\varphi_q}a_{-1}\left(E_q^{m-1},\varsigma_q^m\right)-\sum_{n=0}^{m-2}\sum_{q=1}^{N_\varphi}\frac{\tau_q}{\varphi_q}a_{-1}\left(E_q^{n+1}-E_q^n,\varsigma_q^{n+1}\right)+\frac{1}{2}\sum_{n=0}^{m-1}B(\chi^{n+1},\chi^n)$$

$$+\sum_{q=1}^{N_\varphi}\frac{1}{\varphi_q}\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e\{D\nabla\varsigma_q^m\cdot\underline{n}_e\}[\varsigma_q^m]de\Bigg|. \tag{3.3.20}$$

Recall (2.3.19). The right hand side of (3.3.20) coincides with that of (2.3.19) except skew symmetric part of DG bilinear form $B(\cdot,\cdot)$. Also, instead of continuity of the bilinear form it is necessary to use (3.2.5). Using the same arguments in the proof of Lemma 2.10, we can obtain the upper bounds of the right hand side of(3.3.20) as follows.

- $\left\|\varpi^0\right\|^2_{L_2(\Omega)}$

$$\begin{aligned}\left\|\varpi^0\right\|^2_{L_2(\Omega)} &= \left(W_h^0 - R_1 w_0, W_h^0 - R_1 w_0\right)_{L_2(\Omega)}\\ &= \left(w_0 - R_1 w_0, W_h^0 - R_1 w_0\right)_{L_2(\Omega)}\\ &\quad (\because \text{ since } (w_0, v)_{L_2(\Omega)} = \left(W_h^0, v\right)_{L_2(\Omega)},\ \forall v \in V^h)\\ &= \left(\dot{\theta}^0, \varpi^0\right)_{L_2(\Omega)}\\ &\quad (\because \text{since } \dot{\theta}^0 = w_0 - R_1 w_0)\\ &\leq \left\|\dot{\theta}^0\right\|_{L_2(\Omega)} \left\|\varpi^0\right\|_{L_2(\Omega)}\\ &\quad (\because \text{by Cauchy-Schwarz inequality})\end{aligned}$$

so,

$$\left\|\varpi^0\right\|^2_{L_2(\Omega)} \leq \left\|\dot{\theta}^0\right\|^2_{L_2(\Omega)}.$$

- $\left\|\chi^0\right\|^2_{\mathcal{V}}$

$$\begin{aligned}a_1\left(U_h^0, v\right) &= a_1\left(u_0, v\right),\ \forall v \in \mathcal{D}_k(\mathcal{E}_h),\\ a_1\left(R_1 u_0, v\right) &= a_1\left(u_0, v\right),\ \forall v \in \mathcal{D}_k(\mathcal{E}_h),\end{aligned}$$

hence

$$a_1\left(U_h^0 - R_1 u_0, v\right) = 0,\ \forall v \in \mathcal{D}_k(\mathcal{E}_h),$$

and

$$\left\|\chi^0\right\|^2_{\mathcal{V}} = a_1\left(U_h^0 - R_1 u_0, U_h^0 - R_1 u_0\right) = 0.$$

- $\left|\sum_{n=0}^{m-1}\left(\dot{\theta}^{n+1} - \dot{\theta}^n, \varpi^{n+1} + \varpi^n\right)_{L_2(\Omega)}\right|$

$$\begin{aligned}&\left|\sum_{n=0}^{m-1}\left(\dot{\theta}^{n+1} - \dot{\theta}^n, \varpi^{n+1} + \varpi^n\right)_{L_2(\Omega)}\right|\\ &\quad = \left|\sum_{n=0}^{m-1}\int_{t_n}^{t_{n+1}}\left(\ddot{\theta}(t'), \varpi^{n+1} + \varpi^n\right)_{L_2(\Omega)} dt'\right|\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{n=0}^{m-1} \int_{t_n}^{t_{n+1}} \left\| \ddot{\theta}(t') \right\|_{L_2(\Omega)} \left\| \varpi^{n+1} + \varpi^n \right\|_{L_2(\Omega)} dt' \\
&\quad (\because \text{by Cauchy-Schwarz inequality}) \\
&\leq \frac{1}{2\epsilon_a} \sum_{n=0}^{m-1} \int_{t_n}^{t_{n+1}} \left\| \ddot{\theta}(t') \right\|_{L_2(\Omega)}^2 dt' + \frac{\epsilon_a}{2} \sum_{n=0}^{m-1} \int_{t_n}^{t_{m+1}} \left\| \varpi^{n+1} + \varpi^n \right\|_{L_2(\Omega)}^2 dt' \\
&\quad (\because \text{by Young's inequality for any positive } \epsilon_a) \\
&\leq \frac{1}{2\epsilon_a} \int_0^{t_m} \left\| \ddot{\theta}(t') \right\|_{L_2(\Omega)}^2 dt' + \frac{\epsilon_a}{2} \sum_{n=0}^{m-1} \left\| \varpi^{n+1} + \varpi^n \right\|_{L_2(\Omega)}^2 \Delta t \\
&\quad (\because \left\| \varpi^{n+1} + \varpi^n \right\|_{L_2(\Omega)} \text{ is independent of } s) \\
&\leq \frac{1}{2\epsilon_a} \int_0^{t_m} \left\| \ddot{\theta}(t') \right\|_{L_2(\Omega)}^2 dt' + 2\epsilon_a \sum_{n=0}^{m-1} \max_{0 \leq j \leq N} \left\| \varpi^j \right\|_{L_2(\Omega)}^2 \Delta t \\
&\quad (\because \ \left\| \varpi^{n+1} + \varpi^n \right\|_{L_2(\Omega)}^2 \leq 2 \left\| \varpi^{n+1} \right\|_{L_2(\Omega)}^2 + 2 \left\| \varpi^n \right\|_{L_2(\Omega)}^2 \leq 4 \max_{0 \leq j \leq N} \left\| \varpi^j \right\|_{L_2(\Omega)}^2) \\
&\leq \frac{1}{2\epsilon_a} \int_0^{T} \left\| \ddot{\theta}(t') \right\|_{L_2(\Omega)}^2 dt' + 2\epsilon_a \sum_{n=0}^{N-1} \max_{0 \leq j \leq N} \left\| \varpi^j \right\|_{L_2(\Omega)}^2 \Delta t \\
&\quad (\because m \leq N \ ) \\
&= \frac{1}{2\epsilon_a} \left\| \ddot{\theta} \right\|_{L_2(0,T;L_2(\Omega))}^2 + 2\epsilon_a T \max_{0 \leq j \leq N} \left\| \varpi^j \right\|_{L_2(\Omega)}^2 .
\end{aligned}
$$

- $| - \sum_{n=0}^{m-1} \left(\dot{\theta}^{n+1} - \dot{\theta}^n, \mathcal{E}_2^n \right)_{L_2(\Omega)} |, \ | - \sum_{n=0}^{m-1} \left(\dot{\theta}^{n+1} - \dot{\theta}^n, \mathcal{E}_3^n \right)_{L_2(\Omega)} |$

  In the same sense as the above,

$$
\begin{aligned}
\left| - \sum_{n=0}^{m-1} \left(\dot{\theta}^{n+1} - \dot{\theta}^n, \mathcal{E}_2^n \right)_{L_2(\Omega)} \right| =& \left| - \sum_{n=0}^{m-1} \int_{t_n}^{t_{n+1}} \left(\ddot{\theta}(t'), \mathcal{E}_2^n \right)_{L_2(\Omega)} dt' \right| \\
\leq& \frac{1}{2} \int_0^{t_m} \left\| \ddot{\theta}(t') \right\|_{L_2(\Omega)}^2 dt' + \frac{1}{2} \sum_{n=0}^{m-1} \left\| \mathcal{E}_2^n \right\|_{L_2(\Omega)}^2 \Delta t \\
\leq& \frac{1}{2} \left\| \ddot{\theta} \right\|_{L_2(0,T;L_2(\Omega))}^2 + \frac{T}{2} \max_{0 \leq j \leq N-1} \left\| \mathcal{E}_2^j \right\|_{L_2(\Omega)}^2 .
\end{aligned}
$$

Also,

$$
\left| - \sum_{n=0}^{m-1} \left(\dot{\theta}^{n+1} - \dot{\theta}^n, \mathcal{E}_3^n \right)_{L_2(\Omega)} \right| \leq \frac{1}{2} \left\| \ddot{\theta} \right\|_{L_2(0,T;L_2(\Omega))}^2 + \frac{T}{2} \max_{0 \leq j \leq N-1} \left\| \mathcal{E}_3^j \right\|_{L_2(\Omega)}^2 .
$$

- $\left|\Delta t\sum_{n=0}^{m-1}\left(\mathcal{E}_1^n,\varpi^{n+1}+\varpi^n\right)_{L_2(\Omega)}\right|$

$$\begin{aligned}
&\left|\Delta t\sum_{n=0}^{m-1}\left(\mathcal{E}_1^n,\varpi^{n+1}+\varpi^n\right)_{L_2(\Omega)}\right|\\
&\leq\frac{\Delta t}{2\epsilon_b}\sum_{n=0}^{m-1}\|\mathcal{E}_1^n\|^2_{L_2(\Omega)}+\frac{\epsilon_b\Delta t}{2}\sum_{n=0}^{m-1}\left\|\varpi^{n+1}+\varpi^n\right\|^2_{L_2(\Omega)}\\
&\quad(\because \text{ by Cauchy-Schwarz inequality and Young's inequality for some positive } \epsilon_b)\\
&\leq\frac{\Delta t}{2\epsilon_b}\sum_{n=0}^{m-1}\|\mathcal{E}_1^n\|^2_{L_2(\Omega)}+2\epsilon_b\Delta t\sum_{n=0}^{m-1}\max_{0\leq j\leq N}\left\|\varpi^j\right\|^2_{L_2(\Omega)}\\
&\quad(\because\ \left\|\varpi^{n+1}+\varpi^n\right\|^2_{L_2(\Omega)}\leq 2\left\|\varpi^{n+1}\right\|^2_{L_2(\Omega)}+2\left\|\varpi^n\right\|^2_{L_2(\Omega)}\leq 4\max_{0\leq j\leq N}\left\|\varpi^j\right\|^2_{L_2(\Omega)})\\
&\leq\frac{T}{2\epsilon_b}\max_{0\leq j\leq N-1}\left\|\mathcal{E}_1^j\right\|^2_{L_2(\Omega)}+2\epsilon_b T\max_{0\leq j\leq N}\left\|\varpi^j\right\|^2_{L_2(\Omega)}.\\
&\quad(\because\ m\leq N,\ T=N\Delta t)
\end{aligned}$$

- $\left|-\Delta t\sum_{n=0}^{m-1}(\mathcal{E}_1^n,\mathcal{E}_2^n)_{L_2(\Omega)}\right|$, $\left|-\Delta t\sum_{n=0}^{m-1}(\mathcal{E}_1^n,\mathcal{E}_3^n)_{L_2(\Omega)}\right|$

$$\begin{aligned}
&\left|-\Delta t\sum_{n=0}^{m-1}(\mathcal{E}_1^n,\mathcal{E}_2^n)_{L_2(\Omega)}\right|\\
&\leq\frac{\Delta t}{2}\sum_{n=0}^{m-1}\|\mathcal{E}_1^n\|^2_{L_2(\Omega)}+\frac{\Delta t}{2}\sum_{n=0}^{m-1}\|\mathcal{E}_2^n\|^2_{L_2(\Omega)}\\
&\leq\frac{\Delta t}{2}\sum_{n=0}^{m-1}\max_{0\leq j\leq N-1}\left\|\mathcal{E}_1^j\right\|^2_{L_2(\Omega)}+\frac{\Delta t}{2}\sum_{n=0}^{m-1}\max_{0\leq j\leq N-1}\left\|\mathcal{E}_2^j\right\|^2_{L_2(\Omega)}\\
&\leq\frac{T}{2}\max_{0\leq j\leq N-1}\left\|\mathcal{E}_1^j\right\|^2_{L_2(\Omega)}+\frac{T}{2}\max_{0\leq j\leq N-1}\left\|\mathcal{E}_2^j\right\|^2_{L_2(\Omega)}.
\end{aligned}$$

In the same way,

$$\left|-\Delta t\sum_{n=0}^{m-1}(\mathcal{E}_1^n,\mathcal{E}_3^n)_{L_2(\Omega)}\right|\leq\frac{T}{2}\max_{0\leq j\leq N-1}\left\|\mathcal{E}_1^j\right\|^2_{L_2(\Omega)}+\frac{T}{2}\max_{0\leq j\leq N-1}\left\|\mathcal{E}_3^j\right\|^2_{L_2(\Omega)}.$$

- $\left|\sum_{n=0}^{m-1}\left(\varpi^{n+1}-\varpi^n,\mathcal{E}_2^n\right)_{L_2(\Omega)}\right|$, $\left|\sum_{n=0}^{m-1}\left(\varpi^{n+1}-\varpi^n,\mathcal{E}_3^n\right)_{L_2(\Omega)}\right|$

By summation by parts,

$$\left|\sum_{n=0}^{m-1}\left(\varpi^{n+1}-\varpi^n,\mathcal{E}_2^n\right)_{L_2(\Omega)}\right|$$

$$
\begin{aligned}
=&\left| \left(\varpi^m, \mathcal{E}_2^m\right)_{L_2(\Omega)} - \left(\varpi^0, \mathcal{E}_2^0\right)_{L_2(\Omega)} - \sum_{n=0}^{m-1} \left(\varpi^{n+1}, \mathcal{E}_2^{n+1} - \mathcal{E}_2^n\right)_{L_2(\Omega)} \right| \\
=&\left| \left(\varpi^m, \mathcal{E}_2^m\right)_{L_2(\Omega)} - \left(\varpi^0, \mathcal{E}_2^0\right)_{L_2(\Omega)} - \sum_{n=0}^{m-1} \int_{t_n}^{t_{n+1}} \left(\varpi^{n+1}, \dot{\mathcal{E}}_2(t')\right)_{L_2(\Omega)} dt' \right| \\
\leq& \frac{\epsilon_c}{2} \left\|\varpi^m\right\|_{L_2(\Omega)}^2 + \frac{1}{2\epsilon_c} \left\|\mathcal{E}_2^m\right\|_{L_2(\Omega)}^2 + \frac{1}{2} \left\|\varpi^0\right\|_{L_2(\Omega)}^2 + \frac{1}{2} \left\|\mathcal{E}_2^0\right\|_{L_2(\Omega)}^2 \\
&+ \frac{\epsilon_d}{2} \sum_{n=0}^{m-1} \left\|\varpi^{n+1}\right\|_{L_2(\Omega)}^2 \Delta t + \frac{1}{2\epsilon_d} \int_0^{t_m} \left\|\dot{\mathcal{E}}_2(t')\right\|_{L_2(\Omega)}^2 dt' \\
\leq& \frac{\epsilon_c}{2} \max_{0\leq j\leq N} \left\|\varpi^j\right\|_{L_2(\Omega)}^2 + \frac{1}{2\epsilon_c} \max_{0\leq j\leq N-1} \left\|\mathcal{E}_2^j\right\|_{L_2(\Omega)}^2 + \frac{1}{2} \left\|\dot{\theta}^0\right\|_{L_2(\Omega)}^2 \\
&+ \frac{1}{2} \max_{0\leq j\leq N-1} \left\|\mathcal{E}_2^j\right\|_{L_2(\Omega)}^2 + \frac{\epsilon_d}{2} \sum_{n=0}^{m-1} \max_{0\leq j\leq N} \left\|\varpi^j\right\|_{L_2(\Omega)}^2 \Delta t \\
&+ \frac{1}{2\epsilon_d} \int_0^{t_m} \left\|\dot{\mathcal{E}}_2(t')\right\|_{L_2(\Omega)}^2 dt' \\
\leq& \frac{\epsilon_c}{2} \max_{0\leq j\leq N} \left\|\varpi^j\right\|_{L_2(\Omega)}^2 + \frac{1}{2\epsilon_c} \max_{0\leq j\leq N-1} \left\|\mathcal{E}_2^j\right\|_{L_2(\Omega)}^2 + \frac{1}{2} \left\|\dot{\theta}^0\right\|_{L_2(\Omega)}^2 \\
&+ \frac{1}{2} \max_{0\leq j\leq N-1} \left\|\mathcal{E}_2^j\right\|_{L_2(\Omega)}^2 + \frac{\epsilon_d}{2} T \max_{0\leq j\leq N} \left\|\varpi^j\right\|_{L_2(\Omega)}^2 + \frac{1}{2\epsilon_d} \left\|\dot{\mathcal{E}}_2\right\|_{L_2(0,T;L_2(\Omega))}^2
\end{aligned}
$$

for positive ϵ_c and ϵ_d. In this manner,

$$
\begin{aligned}
&\left| \sum_{n=0}^{m-1} \left(\varpi^{n+1} - \varpi^n, \mathcal{E}_3^n\right)_{L_2(\Omega)} \right| \\
&\quad\leq \frac{\epsilon_c}{2} \max_{0\leq j\leq N} \left\|\varpi^j\right\|_{L_2(\Omega)}^2 + \frac{1}{2\epsilon_c} \max_{0\leq j\leq N-1} \left\|\mathcal{E}_3^j\right\|_{L_2(\Omega)}^2 + \frac{1}{2} \left\|\dot{\theta}^0\right\|_{L_2(\Omega)}^2 \\
&\qquad+ \frac{1}{2} \max_{0\leq j\leq N-1} \left\|\mathcal{E}_3^j\right\|_{L_2(\Omega)}^2 + \frac{\epsilon_d}{2} T \max_{0\leq j\leq N} \left\|\varpi^j\right\|_{L_2(\Omega)}^2 + \frac{1}{2\epsilon_d} \left\|\dot{\mathcal{E}}_3\right\|_{L_2(0,T;L_2(\Omega))}^2 .
\end{aligned}
$$

- $\left| \sum_{q=1}^{N_\varphi} a_{-1}\left(\chi^m, \varsigma_q^m\right) \right|$, $\left| \sum_{q=1}^{N_\varphi} \frac{\tau_q}{\varphi_q} a_{-1}\left(E_q^{m-1}, \varsigma_q^m\right) \right|$

  Use of (3.2.5) makes

$$
\left| \sum_{q=1}^{N_\varphi} a_{-1}\left(\chi^m, \varsigma_q^m\right) \right| \leq \sum_{q=1}^{N_\varphi} \left(\frac{\epsilon_q}{2} + \frac{C}{\sqrt{\alpha_0}}\right) \left\|\chi^m\right\|_{\mathcal{V}}^2 + \sum_{q=1}^{N_\varphi} \left(\frac{1}{2\epsilon_q} + \frac{C}{\sqrt{\alpha_0}}\right) \left\|\varsigma_q^m\right\|_{\mathcal{V}}^2 ,
$$

for some positive $\{\epsilon_q\}$. And the continuity of SIPG form implies

$$
\left| \sum_{q=1}^{N_\varphi} \frac{\tau_q}{\varphi_q} a_{-1}\left(E_q^{m-1}, \varsigma_q^m\right) \right| \leq \sum_{q=1}^{N_\varphi} \frac{K^2 \tau_q}{\varphi_q} \frac{\tilde{\epsilon}_q}{2} \left\|E_q^{m-1}\right\|_{\mathcal{V}}^2 + \sum_{q=1}^{N_\varphi} \frac{\tau_q}{\varphi_q} \frac{1}{2\tilde{\epsilon}_q} \left\|\varsigma_q^m\right\|_{\mathcal{V}}^2 ,
$$

for some positive $\{\tilde{\epsilon}_q\}$.

- $\left| -\sum_{n=0}^{m-2}\sum_{q=1}^{N_\varphi}\frac{\tau_q}{\varphi_q}a_{-1}\left(E_q^{n+1}-E_q^n,\varsigma_q^{n+1}\right)\right|$

$$\begin{aligned}
&\left|-\sum_{n=0}^{m-2}\sum_{q=1}^{N_\varphi}\frac{\tau_q}{\varphi_q}a_{-1}\left(E_q^{n+1}-E_q^n,\varsigma_q^{n+1}\right)\right| \\
=&\left| -\sum_{q=1}^{N_\varphi}\frac{\tau_q}{\varphi_q}\sum_{n=0}^{m-2}\int_{t_n}^{t_{n+1}} a_{-1}\left(\dot{E}_q(t'),\varsigma_q^{n+1}\right)dt'\right| \\
\leq&\sum_{q=1}^{N_\varphi}\frac{K^2\tau_q}{\varphi_q}\frac{\check{\epsilon}_q}{2}\int_0^{t_m}\left\|\dot{E}_q(t')\right\|_{\mathcal{V}}^2 dt'+\sum_{q=1}^{N_\varphi}\frac{\tau_q}{\varphi_q}\frac{1}{2\check{\epsilon}_q}\sum_{n=0}^{m-1}\left\|\varsigma_q^{n+1}\right\|_{\mathcal{V}}^2\Delta t \\
\leq&\sum_{q=1}^{N_\varphi}\frac{K^2\tau_q}{\varphi_q}\frac{\check{\epsilon}_q}{2}\left\|\dot{E}_q\right\|_{L_2(0,T;\mathcal{V})}^2+\sum_{q=1}^{N_\varphi}\frac{\tau_q}{\varphi_q}\frac{T}{2\check{\epsilon}_q}\max_{0\leq j\leq N}\left\|\varsigma_q^j\right\|_{\mathcal{V}}^2
\end{aligned}$$

for some positive $\{\check{\epsilon}_q\}$.

- $\left|\frac{1}{2}\sum_{n=0}^{m-1}B(\chi^{n+1},\chi^n)\right|$

Since $B(v,v)=0$ for any $v\in\mathcal{D}_k(\mathcal{E}_h)$,

$$\sum_{n=0}^{m-1}B(\chi^{n+1},\chi^n)=\sum_{n=0}^{m-1}B(\chi^{n+1}-\chi^n,\chi^n).$$

Then the definition of skew symmetric $B(\cdot,\cdot)$ and (3.3.18) lead us to have

$$\begin{aligned}
&\sum_{n=0}^{m-1}B\left(\chi^{n+1}-\chi^n,\chi^n\right) \\
=&\Delta t\sum_{n=0}^{m-1}\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e\{D\nabla\chi^n\cdot\underline{n}_e\}[\varpi^{n+1}+\varpi^n-\mathcal{E}_2^n-\mathcal{E}_3^n]de \\
&-2\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e\{D\nabla\chi^m\cdot\underline{n}_e\}[\chi^{m-1}]de \\
&+2\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e\{D\nabla\chi^0\cdot\underline{n}_e\}[\chi^0]de \\
&+\Delta t\sum_{n=0}^{m-2}\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e\{D\nabla\chi^{n+1}\cdot\underline{n}_e\}[\varpi^{n+1}+\varpi^n-\mathcal{E}_2^n-\mathcal{E}_3^n]de.
\end{aligned}$$

Note that $[\mathcal{E}_2^n]=[\mathcal{E}_3^n]=0$ on $\Gamma_D\cup\Gamma_h$ and $\left\|\chi^0\right\|_{\mathcal{V}}=0$, hence (3.2.1) and (3.2.2) give

$$\left|\frac{1}{2}\sum_{n=0}^{m-1}B(\chi^{n+1},\chi^n)\right|\leq\frac{C\Delta t}{\sqrt{\alpha_0}}\sum_{n=0}^{m-1}\left\|\chi^n\right\|_{\mathcal{V}}^2+\frac{C}{\sqrt{\alpha_0}}\left\|\chi^m\right\|_{\mathcal{V}}^2+\frac{C}{\sqrt{\alpha_0}}\left\|\chi^{m-1}\right\|_{\mathcal{V}}^2$$

$$+\frac{C\Delta t}{\sqrt{\alpha_0}}\sum_{n=0}^{m-1}J_0^{\alpha_0,\beta_0}(\varpi^{n+1}+\varpi^n,\varpi^{n+1}+\varpi^n).$$

In addition, consider the maximum on the right hand side with respect to m except the jump penalty. Then we obtain

$$\left|\frac{1}{2}\sum_{n=0}^{m-1}B(\chi^{n+1},\chi^n)\right|\leq\frac{C(T+2)}{\sqrt{\alpha_0}}\max_{0\leq j\leq N}\|\chi^j\|_{\mathcal{V}}^2$$
$$+\frac{C\Delta t}{\sqrt{\alpha_0}}\sum_{n=0}^{m-1}J_0^{\alpha_0,\beta_0}(\varpi^{n+1}+\varpi^n,\varpi^{n+1}+\varpi^n).$$

- $|\sum_{q=1}^{N_\varphi}\frac{1}{\varphi_q}\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e\{D\nabla\varsigma_q^m\cdot\underline{n}_e\}[\varsigma_q^m]de|$

  From (3.2.1),

$$\left|\sum_{q=1}^{N_\varphi}\frac{1}{\varphi_q}\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e\{D\nabla\varsigma_q^m\cdot\underline{n}_e\}[\varsigma_q^m]de\right|\leq\sum_{q=1}^{N_\varphi}\frac{C}{\varphi_q\sqrt{\alpha_0}}\|\varsigma_q^m\|_{\mathcal{V}}^2.$$

Note that the terms of $\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3$ and E_q are bounded by Δt^2 by Crank-Nicolson method. So are their first time derivatives. For our sake, we set

$$\epsilon_a=\frac{1}{16(3+N_\varphi)T},\ \epsilon_b=\frac{1}{16(3+N_\varphi)T},\ \epsilon_c=\frac{1}{8(3+N_\varphi)},\ \epsilon_d=\frac{1}{8(3+N_\varphi)T},$$

and

$$\epsilon_q=\varphi_q+\frac{\varphi_0}{2N_\varphi},\ \tilde{\epsilon}_q=\frac{\tau_q(4\varphi_q^2N_\varphi+2\varphi_0\varphi_q)}{\varphi_0\varphi_q},\ \check{\epsilon}_q=(3+N_\varphi)\frac{T\tau_q}{\varphi_q}\frac{8\varphi_q^2N_\varphi+4\varphi_0\varphi_q}{\varphi_0}$$

for each q. Then,

$$\rho\left(\epsilon_aT+\epsilon_bT+\frac{\epsilon_c}{2}+\frac{\epsilon_d}{2}T\right)=\frac{\rho}{4(3+N_\varphi)}>0,$$

$$\frac{1}{2}-\sum_{q=1}^{N_\varphi}\frac{\epsilon_q}{2}=\frac{\varphi_0}{4}>0,$$

and

$$\sum_{q=1}^{N_\varphi}\frac{1}{2\varphi_q}-\sum_{q=1}^{N_\varphi}\frac{1}{2\epsilon_q}-\sum_{q=1}^{N_\varphi}\frac{\tau_q}{\varphi_q}\frac{1}{2\tilde{\epsilon}_q}=\sum_{q=1}^{N_\varphi}\frac{\varphi_0}{8\varphi_q^2N_\varphi+4\varphi_0\varphi_q}>0.$$

In the end, tidying up the above results with the elliptic approximation estimates,

$$\frac{\rho}{2}\|\varpi^m\|_{L_2(\Omega)}^2+\left(\frac{\varphi_0}{4}-\frac{CN_\varphi}{\sqrt{\alpha_0}}\right)\|\chi^m\|_{\mathcal{V}}^2$$

$$
\begin{aligned}
&+\sum_{q=1}^{N_\varphi}\left(\frac{\varphi_0}{8\varphi_q^2N_\varphi+4\varphi_0\varphi_q}-\frac{C(1+\varphi_q+\tau_q)}{\varphi_q\sqrt{\alpha_0}}\right)\left\|\varsigma_q^m\right\|_{\mathcal{V}}^2+\Delta t\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi}\frac{\kappa\tau_q}{\varphi_q}\left\|\frac{\varsigma_q^{n+1}-\varsigma_q^n}{\Delta t}\right\|_{\mathcal{V}}^2\\
&+\Delta t\sum_{n=0}^{m-1}(\frac{1}{4}-\frac{C}{\sqrt{\alpha_0}})J_0^{\alpha_0,\beta_0}\left(\varpi^{n+1}+\varpi^n,\varpi^{n+1}+\varpi^n\right)\\
\leq& C(h^{2(\min(k+1,s)-1)}+\Delta t^4)+\frac{\rho}{4(3+N_\varphi)}\max_{0\leq j\leq N}\left\|\varpi^j\right\|_{L_2(\Omega)}^2+\frac{C(T+2)}{\sqrt{\alpha_0}}\max_{0\leq j\leq N}\left\|\chi^j\right\|_{\mathcal{V}}^2\\
&+\sum_{q=1}^{N_\varphi}(\frac{1}{2(3+N_\varphi)}\frac{\varphi_0}{8\varphi_q^2N_\varphi+4\varphi_0\varphi_q}+\frac{CT\tau_q}{\varphi_q\sqrt{\alpha_0}})\max_{0\leq j\leq N}\left\|\varsigma_q^j\right\|_{\mathcal{V}}^2. \qquad (3.3.21)
\end{aligned}
$$

Whence taking into account maxima on (3.3.21), we can obtain

$$
\begin{aligned}
&\frac{\rho}{2}\max_{0\leq j\leq N}\left\|\varpi^j\right\|_{L_2(\Omega)}^2+\left(\frac{\varphi_0}{4}-\frac{CN_\varphi}{\sqrt{\alpha_0}}\right)\max_{0\leq j\leq N}\left\|\chi^j\right\|_{\mathcal{V}}^2\\
&+\sum_{q=1}^{N_\varphi}\left(\frac{\varphi_0}{8\varphi_q^2N_\varphi+4\varphi_0\varphi_q}-\frac{C(1+\varphi_q+\tau_q)}{\varphi_q\sqrt{\alpha_0}}\right)\max_{0\leq j\leq N}\left\|\varsigma_q^j\right\|_{\mathcal{V}}^2\\
&+\Delta t\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi}\frac{\kappa\tau_q}{\varphi_q}\left\|\frac{\varsigma_q^{n+1}-\varsigma_q^n}{\Delta t}\right\|_{\mathcal{V}}^2+\Delta t\sum_{n=0}^{m-1}(\frac{1}{4}-\frac{C}{\sqrt{\alpha_0}})J_0^{\alpha_0,\beta_0}\left(\varpi^{n+1}+\varpi^n,\varpi^{n+1}+\varpi^n\right)\\
\leq&(3+N_\varphi)\Bigg(C(h^{2(\min(k+1,s)-1)}+\Delta t^4)+\frac{\rho}{4(3+N_\varphi)}\max_{0\leq j\leq N}\left\|\varpi^j\right\|_{L_2(\Omega)}^2\\
&+\frac{C(2T+1)}{\sqrt{\alpha_0}}\max_{0\leq j\leq N}\left\|\chi^j\right\|_{\mathcal{V}}^2\\
&+\sum_{q=1}^{N_\varphi}\left(\frac{1}{2(3+N_\varphi)}\frac{\varphi_0}{8\varphi_q^2N_\varphi+4\varphi_0\varphi_q}+\frac{CT\tau_q}{\varphi_q\sqrt{\alpha_0}}\right)\max_{0\leq j\leq N}\left\|\varsigma_q^j\right\|_{\mathcal{V}}^2\Bigg),
\end{aligned}
$$

therefore

$$
\begin{aligned}
&\frac{\rho}{4}\max_{0\leq j\leq N}\left\|\varpi^j\right\|_{L_2(\Omega)}^2+\left(\frac{\varphi_0}{4}-\frac{C(T+2)(3+N_\varphi)+CN_\varphi}{\sqrt{\alpha_0}}\right)\max_{0\leq j\leq N}\left\|\chi^j\right\|_{\mathcal{V}}^2\\
&+\sum_{q=1}^{N_\varphi}\left(\frac{\varphi_0}{8\varphi_q^2N_\varphi+4\varphi_0\varphi_q}-\frac{C(1+\varphi_q+\tau_q(1+3T+N_\varphi T))}{\varphi_q\sqrt{\alpha_0}}\right)\max_{0\leq j\leq N}\left\|\varsigma_q^j\right\|_{\mathcal{V}}^2\\
&+\Delta t\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi}\frac{\kappa\tau_q}{\varphi_q}\left\|\frac{\varsigma_q^{n+1}-\varsigma_q^n}{\Delta t}\right\|_{\mathcal{V}}^2+\Delta t\sum_{n=0}^{m-1}(\frac{1}{4}-\frac{C}{\sqrt{\alpha_0}})J_0^{\alpha_0,\beta_0}\left(\varpi^{n+1}+\varpi^n,\varpi^{n+1}+\varpi^n\right)\\
\leq& C(h^{2(\min(k+1,s)-1)}+\Delta t^4).
\end{aligned}
$$

If we choose the large α_0 by

$$
\frac{\varphi_0}{4}-\frac{C(T+2)(3+N_\varphi)+CN_\varphi}{\sqrt{\alpha_0}}>0,
$$

$$\frac{1}{4} - \frac{C}{\sqrt{\alpha_0}} > 0$$

and

$$\frac{\varphi_0}{8\varphi_q^2 N_\varphi + 4\varphi_0\varphi_q} - \frac{C(1+\varphi_q+\tau_q(1+3T+N_\varphi T))}{\varphi_q\sqrt{\alpha_0}} > 0,$$

for each q, we can conclude that

$$\begin{aligned}
&\max_{0\le j\le N}\left\|\varpi^j\right\|_{L_2(\Omega)} + \max_{0\le j\le N}\left\|\chi^j\right\|_{\mathcal{V}} + \sum_{q=1}^{N_\varphi}\max_{0\le j\le N}\left\|\varsigma_q^j\right\|_{\mathcal{V}} + \Delta t\sum_{q=1}^{N_\varphi}\sum_{n=0}^{N-1}\left\|\frac{\varsigma_q^{n+1}-\varsigma_q^n}{\Delta t}\right\|_{\mathcal{V}} \\
&+ \left(\Delta t\sum_{n=0}^{N-1} J_0^{\alpha_0,\beta_0}\left(\varpi^{n+1}+\varpi^n, \varpi^{n+1}+\varpi^n\right)\right)^{1/2} \\
&\le C(h^{\min(k+1,s)-1} + \Delta t^2).
\end{aligned}$$

Furthermore, if Ω is convex and $\beta_0(d-1)\ge 3$ is given, (3.2.17) gives

$$\begin{aligned}
&\max_{0\le j\le N}\left\|\varpi^j\right\|_{L_2(\Omega)} + \max_{0\le j\le N}\left\|\chi^j\right\|_{\mathcal{V}} + \sum_{q=1}^{N_\varphi}\max_{0\le j\le N}\left\|\varsigma_q^j\right\|_{\mathcal{V}} + \Delta t\sum_{q=1}^{N_\varphi}\sum_{n=0}^{N-1}\left\|\frac{\varsigma_q^{n+1}-\varsigma_q^n}{\Delta t}\right\|_{\mathcal{V}} \\
&+ \left(\Delta t\sum_{n=0}^{N-1} J_0^{\alpha_0,\beta_0}\left(\varpi^{n+1}+\varpi^n, \varpi^{n+1}+\varpi^n\right)\right)^{1/2} \\
&\le C(h^{\min(k+1,s)} + \Delta t^2).
\end{aligned}$$

□

From Lemma 3.6, in a similar way with CGFEM cases and the semidiscrete problem, we can obatin the following error bounds for the fully discrete formulation of **(Q1)**.

Theorem 3.12. *Under the same condition on* Lemma 3.6*, we have*

$$\begin{aligned}
\max_{0\le j\le N}\left\|u(t_j) - U_h^j\right\|_{\mathcal{V}} &\le C(h^{\min(k+1,s)-1} + \Delta t^2), \\
\max_{0\le j\le N}\left\|u(t_j) - U_h^j\right\|_{L_2(\Omega)} &\le C(h^{\min(k+1,s)-1} + \Delta t^2), \\
\max_{0\le j\le N}\left\|\dot{u}(t_j) - W_h^j\right\|_{L_2(\Omega)} &\le C(h^{\min(k+1,s)-1} + \Delta t^2),
\end{aligned}$$

and if Ω *is convex and* $\beta_0(d-1)\ge 3$

$$\begin{aligned}
\max_{0\le j\le N}\left\|u(t_j) - U_h^j\right\|_{L_2(\Omega)} &\le C(h^{\min(k+1,s)} + \Delta t^2), \\
\max_{0\le j\le N}\left\|\dot{u}(t_j) - W_h^j\right\|_{L_2(\Omega)} &\le C(h^{\min(k+1,s)} + \Delta t^2).
\end{aligned}$$

Proof. It is easy to show the statement. We have already proved similar problems. The proof follows the same way with the proof of Theorem 2.14 and Corollary 2.1 but in $\|\cdot\|_{\mathcal{V}}$ rather than $\|\cdot\|_V$ by using the result in Lemma 3.6. □

As seen in theorems in terms of the stability bound and the error bounds for the fully discrete formulation of **(Q1)**, the discrete solution is bounded by data and converges without Grönwall's inequality but it requires sufficiently large α_0. In this manner, we can also deal with the fully discrete formulation of **(Q2)**.

3.3.2 Velocity Form

As following the notations, the fully discrete formulation for **(Q2)** is governed by Crank-Nicolson method:
Find U_h^n, W_h^n and $\mathcal{S}_{hq}^n \in \mathcal{D}_k(\mathcal{E}_h)$ for $n = 0, \dots, N$, $\forall q \in \{1, \dots, N_\varphi\}$ such that for any $v \in \mathcal{D}_k(\mathcal{E}_h)$

$$\left(\rho\frac{W_h^{n+1} - W_h^n}{\Delta t}, v\right)_{L_2(\Omega)} + \varphi_0 a_1\left(\frac{U_h^{n+1} + U_h^n}{2}, v\right) + \sum_{q=1}^{N_\varphi} a_{-1}\left(\frac{\mathcal{S}_{hq}^{n+1} + \mathcal{S}_{hq}^n}{2}, v\right)$$
$$+ J_0^{\alpha_0,\beta_0}\left(\frac{W_h^{n+1} + W_h^n}{2}, v\right) = \bar{F}_v^n(v), \qquad \text{for } n = 0, \dots, N-1, \tag{3.3.22}$$

$$\tau_q a_{-1}\left(\frac{\mathcal{S}_{hq}^{n+1} - \mathcal{S}_{hq}^n}{\Delta t}, v\right) + a_{-1}\left(\frac{\mathcal{S}_{hq}^{n+1} + \mathcal{S}_{hq}^n}{2}, v\right) = \tau_q\varphi_q a_{-1}\left(\frac{W_h^{n+1} + W_h^n}{2}, v\right), \tag{3.3.23}$$

for $n = 0, \dots, N-1$, $\forall q \in \{1, \dots, N_\varphi\}$,

$$a_{-1}(U_h^0, v) = a_{-1}(u_0, v)\ , \tag{3.3.24}$$
$$\left(W_h^0, v\right)_{L_2(\Omega)} = (w_0, v)_{L_2(\Omega)}\ , \tag{3.3.25}$$

with (3.3.1) and $\mathcal{S}_{hq} = 0$, $\forall q \in \{1, \dots, N_\varphi\}$. Whence we define

$$U_h^n = \sum_{i=1}^{N_{Vh}} \mathfrak{u}_i^n \phi_i,\ W_h^n = \sum_{i=1}^{N_{Vh}} \mathfrak{w}_i^n \phi_i,\ \text{and } \mathcal{S}_{hq}^n = \sum_{i=1}^{N_{Vh}} S_{hq,i}^n \phi_i \text{ for each } q,$$

the resulting linear system is given by

$$\underline{\mathfrak{u}}^{n+1} = \left(\frac{2\rho}{\Delta t^2}M + \frac{\varphi_0}{2}A + \mathcal{B} + \frac{1}{\Delta t}\mathcal{J}\right)^{-1}\left[\frac{2\rho}{\Delta t}M\underline{\mathfrak{w}}^n + \left(\frac{2\rho}{\Delta t^2}M - \frac{\varphi_0}{2}A + \mathcal{B} + \frac{1}{\Delta t}\mathcal{J}\right)\underline{\mathfrak{u}}^n\right.$$
$$\left. - \sum_{q=1}^{N_\varphi}\frac{2\tau_q}{2\tau_q + \Delta t}A^*\underline{S}_{hq}^n + \frac{1}{2}(\underline{\tilde{F}}^{n+1} + \underline{\tilde{F}}^n)\right], \tag{3.3.26}$$

$$\underline{\mathfrak{w}}^{n+1} = \frac{2}{\Delta t}(\underline{\mathfrak{u}}^{n+1} - \underline{\mathfrak{u}}^n) - \underline{\mathfrak{w}}^n, \tag{3.3.27}$$

$$\underline{S}_{hq}^{n+1} = \left(\frac{\tau_q}{\Delta t} + \frac{1}{2}\right)^{-1} \frac{\tau_q \varphi_q}{\Delta t}(\underline{\mathfrak{u}}^{n+1} - \underline{\mathfrak{u}}^n) + \left(\frac{\tau_q}{\Delta t} + \frac{1}{2}\right)^{-1} \left(\frac{\tau_q}{\Delta t} - \frac{1}{2}\right) \underline{S}_{hq}^n, \tag{3.3.28}$$

for $q = 1, \dots, N_\varphi$, where $\underline{S}_{hq}^0 = \underline{0}$, $\underline{\mathfrak{u}}^0$ and $\underline{\mathfrak{w}}^0$ are governed by (3.3.24) and (3.3.25), respectively with

$$M_{ij} = (\phi_j, \phi_i)_{L_2(\Omega)}, \; A_{ij} = a_1(\phi_j, \phi_i), \; A^*_{ij} = a_{-1}(\phi_j, \phi_i), \; \mathcal{J}_{i,j} = J_0^{\alpha_0, \beta_0}(\phi_j, \phi_i)$$

for $i, j = 1, \dots, N_{V^h}$ and $\mathcal{B} = \sum_{q=1}^{N_\varphi} \frac{\tau_q \varphi_q}{2\tau_q + \Delta t} A^*$. When we show the stability bound, we can also solve the resulting linear system uniquely.

We will refer to the proofs for the fully discrete formulation of **(P2)** to show the stability and error bounds for that of **(Q2)**. Basically, the difference between CGFEM and DGFEM on the proof will come from the skew symmetric part of DG bilinear form and DG energy norm. Hence we should deal with them carefully.

Lemma 3.7. *For each q and for any $m \in \mathbb{N}$ such that $1 \le m \le N$,*

$$\begin{aligned}\sum_{n=0}^{m-1} a_{-1}(\mathcal{S}_{hq}^{n+1} + \mathcal{S}_{hq}^n, W_h^{n+1} + W_h^n) =& \frac{2}{\varphi_q \Delta t} a_{-1}\left(\mathcal{S}_{hq}^m, \mathcal{S}_{hq}^m\right) \\ &+ \sum_{n=0}^{m-1} \frac{1}{\tau_q \varphi_q} a_{-1}\left(\mathcal{S}_{hq}^{n+1} + \mathcal{S}_{hq}^n, \mathcal{S}_{hq}^{n+1} + \mathcal{S}_{hq}^n\right).\end{aligned}$$

Proof. Consider $v = \mathcal{S}_{hq}^{n+1} + \mathcal{S}_{hq}^n$ for $0 \le n \le m-1$ on (3.3.23). We have

$$\begin{aligned}&\frac{\tau_q}{\Delta t}\left(a_{-1}\left(\mathcal{S}_{hq}^{n+1}, \mathcal{S}_{hq}^{n+1}\right) - a_{-1}\left(\mathcal{S}_{hq}^n, \mathcal{S}_{hq}^n\right)\right) + \frac{1}{2} a_{-1}\left(\mathcal{S}_{hq}^{n+1} + \mathcal{S}_{hq}^n, \mathcal{S}_{hq}^{n+1} + \mathcal{S}_{hq}^n\right) \\ &= \frac{\tau_q \varphi_q}{2} a_{-1}(W_h^{n+1} + W_h^n, \mathcal{S}_{hq}^{n+1} + \mathcal{S}_{hq}^n).\end{aligned}$$

Summing with respect to n,

$$\begin{aligned}&\frac{\tau_q}{\Delta t} a_{-1}\left(\mathcal{S}_{hq}^m, \mathcal{S}_{hq}^m\right) + \sum_{n=0}^{m-1} \frac{1}{2} a_{-1}\left(\mathcal{S}_{hq}^{n+1} + \mathcal{S}_{hq}^n, \mathcal{S}_{hq}^{n+1} + \mathcal{S}_{hq}^n\right) \\ &= \sum_{n=0}^{m-1} \frac{\tau_q \varphi_q}{2} a_{-1}(W_h^{n+1} + W_h^n, \mathcal{S}_{hq}^{n+1} + \mathcal{S}_{hq}^n),\end{aligned}$$

since $\mathcal{S}_{hq}^0 = 0$ for any q, thus it is observed that

$$\begin{aligned}\sum_{n=0}^{m-1} a_{-1}(\mathcal{S}_{hq}^{n+1} + \mathcal{S}_{hq}^n, W_h^{n+1} + W_h^n) =& \frac{2}{\varphi_q \Delta t} a_{-1}\left(\mathcal{S}_{hq}^m, \mathcal{S}_{hq}^m\right) \\ &+ \sum_{n=0}^{m-1} \frac{1}{\tau_q \varphi_q} a_{-1}\left(\mathcal{S}_{hq}^{n+1} + \mathcal{S}_{hq}^n, \mathcal{S}_{hq}^{n+1} + \mathcal{S}_{hq}^n\right).\end{aligned}$$

□

Theorem 3.13. *Suppose $\beta_0(d-1) \geq 1$. For a sufficiently large α_0, there exists a positive constant C such that*

$$\begin{aligned}
&\max_{0\leq j\leq N}\left\|W_h^j\right\|_{L_2(\Omega)}^2+\max_{0\leq j\leq N}\left\|U_h^j\right\|_{\mathcal{V}}^2+\sum_{q=1}^{N_\varphi}\max_{0\leq j\leq N}\left\|\mathcal{S}_{hq}^j\right\|_{\mathcal{V}}^2+\Delta t\sum_{n=0}^{N-1}\sum_{q=1}^{N_\varphi}\left\|\mathcal{S}_{hq}^{n+1}+\mathcal{S}_{hq}^n\right\|_{\mathcal{V}}^2\\
&+\Delta t\sum_{n=0}^{N-1}J_0^{\alpha_0,\beta_0}(W_h^{n+1}+W_h^n,W_h^{n+1}+W_h^n)\\
\leq& C\Bigg(\|w_0\|_{L_2(\Omega)}^2+\|u_0\|_{\mathcal{V}}^2+\|f\|_{L_\infty(0,T;L_2(\Omega))}^2+h^{-1}\|g_N\|_{L_\infty(0,T;L_2(\Gamma_N))}^2\\
&+h^{-1}\|\dot{g}_N\|_{L_2(0,T;L_2(\Gamma_N))}^2.\Bigg)
\end{aligned}$$

Proof. Let $m\in\{1,\ldots,N\}$ and $v=W_h^{n+1}+W_h^n$ for $n=0,\ldots,m-1$ and put it into (3.3.22) with (3.3.1). Then we have

$$\begin{aligned}
&\frac{\rho}{\Delta t}\left(\|W_h^{n+1}\|_{L_2(\Omega)}^2-\|W_h^n\|_{L_2(\Omega)}^2\right)+\frac{\varphi_0}{\Delta t}(\|U_h^{n+1}\|_{\mathcal{V}}^2-\|U_h^n\|_{\mathcal{V}}^2)-\frac{\varphi_0}{\Delta t}B(U_h^{n+1},U_h^n)\\
&+\frac{1}{2}\sum_{q=1}^{N_\varphi}a_{-1}(\mathcal{S}_{hq}^{n+1}+\mathcal{S}_{hq}^n,W_h^{n+1}+W_h^n)+\frac{1}{2}J_0^{\alpha_0,\beta_0}(W_h^{n+1}+W_h^n,W_h^{n+1}+W_h^n)\\
&=\bar{F}_v^n\left(W_h^{n+1}+W_h^n\right).
\end{aligned}$$

Summation from $n=0$ to $n=m-1$ and multiplying Δt give

$$\begin{aligned}
&\rho\|W_h^m\|_{L_2(\Omega)}^2+\varphi_0\|U_h^m\|_{\mathcal{V}}^2+\frac{\Delta t}{2}\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi}a_{-1}\left(\mathcal{S}_{hq}^{n+1}+\mathcal{S}_{hq}^n,W_h^{n+1}+W_h^n\right)\\
&+\frac{\Delta t}{2}\sum_{n=0}^{m-1}J_0^{\alpha_0,\beta_0}(W_h^{n+1}+W_h^n,W_h^{n+1}+W_h^n)\\
=&\rho\left\|W_h^0\right\|_{L_2(\Omega)}^2+\varphi_0\left\|U_h^0\right\|_{\mathcal{V}}^2+\Delta t\sum_{n=0}^{m-1}\bar{F}_v^n\left(W_h^{n+1}+W_h^n\right)+\varphi_0\sum_{n=0}^{m-1}B(U_h^{n+1},U_h^n).
\end{aligned}$$

By Lemma 3.7, we obtain

$$\begin{aligned}
&\rho\|W_h^m\|_{L_2(\Omega)}^2+\varphi_0\|U_h^m\|_{\mathcal{V}}^2+\sum_{q=1}^{N_\varphi}\frac{1}{\varphi_q}a_{-1}\left(\mathcal{S}_{hq}^m,\mathcal{S}_{hq}^m\right)\\
&+\frac{\Delta t}{2}\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi}\frac{1}{\tau_q\varphi_q}a_{-1}\left(\mathcal{S}_{hq}^{n+1}+\mathcal{S}_{hq}^n,\mathcal{S}_{hq}^{n+1}+\mathcal{S}_{hq}^n\right)\\
&+\frac{\Delta t}{2}\sum_{n=0}^{m-1}J_0^{\alpha_0,\beta_0}(W_h^{n+1}+W_h^n,W_h^{n+1}+W_h^n)
\end{aligned}$$

$$=\rho \left\|W_h^0\right\|_{L_2(\Omega)}^2 + \varphi_0 \left\|U_h^0\right\|_{\mathcal{V}}^2 + \Delta t \sum_{n=0}^{m-1} \bar{F}_v^n \left(W_h^{n+1} + W_h^n\right) + \varphi_0 \sum_{n=0}^{m-1} B(U_h^{n+1}, U_h^n),$$

so that

$$\begin{aligned}
&\rho \|W_h^m\|_{L_2(\Omega)}^2 + \varphi_0 \|U_h^m\|_{\mathcal{V}}^2 + \sum_{q=1}^{N_\varphi} \frac{\kappa}{\varphi_q} \|\mathcal{S}_{hq}^m\|_{\mathcal{V}}^2 + \frac{\Delta t}{2} \sum_{n=0}^{m-1} \sum_{q=1}^{N_\varphi} \frac{\kappa}{\tau_q \varphi_q} \left\|\mathcal{S}_{hq}^{n+1} + \mathcal{S}_{hq}^n\right\|_{\mathcal{V}}^2 \\
&+ \frac{\Delta t}{2} \sum_{n=0}^{m-1} J_0^{\alpha_0,\beta_0}(W_h^{n+1} + W_h^n, W_h^{n+1} + W_h^n) \\
\leq &\left|\rho \left\|W_h^0\right\|_{L_2(\Omega)}^2 + \varphi_0 \left\|U_h^0\right\|_{\mathcal{V}}^2 + \Delta t \sum_{n=0}^{m-1} \bar{F}_v^n \left(W_h^{n+1} + W_h^n\right) + \varphi_0 \sum_{n=0}^{m-1} B(U_h^{n+1}, U_h^n)\right|
\end{aligned} \tag{3.3.29}$$

by coercivity. Let us consider each component of the right hand side of (3.3.29).

- $|\Delta t \sum_{n=0}^{m-1} \bar{F}_v^n \left(W_h^{n+1} + W_h^n\right)|$
  Using the definition of F_v, (3.3.1) and summation by parts, we have

$$\begin{aligned}
&\left|\Delta t \sum_{n=0}^{m-1} \bar{F}_v^n \left(W_h^{n+1} + W_h^n\right)\right| \\
=&\Bigg|\Delta t \sum_{n=0}^{m-1} \left(\bar{f}^n, W_h^{n+1} + W_h^n\right)_{L_2(\Omega)} + 2\left(\bar{g}_N^{m-1}, U_h^m\right)_{L_2(\Gamma_N)} \\
&- 2\left(\bar{g}_N^0, U_h^0\right)_{L_2(\Gamma_N)} - \sum_{n=1}^{m-1} \int_{t_{n-1}}^{t_{n+1}} \left(\dot{g}_N(t'), U_h^n\right)_{L_2(\Gamma_N)} dt' \\
&+ 2 \sum_{n=0}^{m-1} \sum_{q=1}^{N_\varphi} \varphi_q \frac{e^{-t_{n+1}/\tau_q} + e^{-t_n/\tau_q}}{2} a_1\left(u_0, U_h^{n+1} - U_h^n\right)\Bigg| \\
\leq&\Delta t \sum_{n=0}^{m-1} |\left(\bar{f}^n, W_h^{n+1} + W_h^n\right)_{L_2(\Omega)}| + 2|\left(\bar{g}_N^{m-1}, U_h^m\right)_{L_2(\Gamma_N)}| \\
&+ 2|\left(\bar{g}_N^0, U_h^0\right)_{L_2(\Gamma_N)}| + \sum_{n=1}^{m-1} |\int_{t_{n-1}}^{t_{n+1}} \left(\dot{g}_N(t'), U_h^n\right)_{L_2(\Gamma_N)} dt'| \\
&2|a_1\left(u_0, U_h^m\right)| + 2|a_1\left(u_0, U_h^0\right)| + \sum_{q=1}^{N_\varphi} \varphi_q K \sum_{n=1}^{m-1} \int_{t_{n-1}}^{t_{n+1}} \left\|\frac{e^{-t'/\tau_q}}{\tau_q} u_0\right\|_{\mathcal{V}} \|U_h^n\|_{\mathcal{V}}\, dt'.
\end{aligned}$$

  Recall the similar arguments in the proof of Theorems 2.15 and 2.16 for the last term in the above. With applying (3.2.5), Cauchy-Schwarz inequality, Young's inequality, inverse polynomial trace theorem and Poincaré's inequality,

$$
\begin{aligned}
&\left|\Delta t \sum_{n=0}^{m-1} \bar{F}_v^n \left(W_h^{n+1}+W_h^n\right)\right| \\
\leq & \Delta t \sum_{n=0}^{m-1} \left\|\bar{f}^n\right\|_{L_2(\Omega)} \left\|W_h^{n+1}+W_h^n\right\|_{L_2(\Omega)} + 2\sum_{e\subset\Gamma_N} \left\|\bar{g}_N^{m-1}\right\|_{L_2(e)} \left\|U_h^m\right\|_{L_2(e)} \\
&+ 2\sum_{e\subset\Gamma_N} \left\|\bar{g}_N^0\right\|_{L_2(e)} \left\|U_h^0\right\|_{L_2(e)} + \sum_{n=1}^{m-1}\int_{t_{n-1}}^{t_{n+1}} \sum_{e\subset\Gamma_N} \left\|\dot{g}_N(t')\right\|_{L_2(e)} \left\|U_h^n\right\|_{L_2(e)} dt' \\
&+ 2\left\|u_0\right\|_{\mathcal{V}} \left\|U_h^m\right\|_{\mathcal{V}} + \frac{C}{\sqrt{\alpha_0}}\left\|u_0\right\|_{\mathcal{V}}^2 + \frac{C}{\sqrt{\alpha_0}}\left\|U_h^m\right\|_{\mathcal{V}}^2 + 2\left\|U_h^0\right\|_{\mathcal{V}}^2 \\
&+ \sum_{q=1}^{N_\varphi}\varphi_q \sum_{n=1}^{m-1}\int_{t_{n-1}}^{t_{n+1}} K\left\|\frac{e^{-t'/\tau_q}}{\tau_q}u_0\right\|_{\mathcal{V}} \left\|U_h^n\right\|_{\mathcal{V}} dt' \\
\leq & \frac{\Delta t}{\epsilon_a}\sum_{n=0}^{m-1}\left\|\bar{f}^n\right\|_{L_2(\Omega)}^2 + \frac{\Delta t\epsilon_a}{2}\sum_{n=0}^{m-1}\left\|W_h^{n+1}\right\|_{L_2(\Omega)}^2 + \frac{\Delta t\epsilon_a}{2}\sum_{n=0}^{m-1}\left\|W_h^n\right\|_{L_2(\Omega)}^2 \\
&+ \frac{1}{\epsilon_b}\left\|\bar{g}_N^{m-1}\right\|_{L_2(\Gamma_N)}^2 + Ch^{-1}\epsilon_b\left\|U_h^m\right\|_{\mathcal{V}}^2 + h^{-1}\left\|\bar{g}_N^0\right\|_{L_2(\Gamma_N)}^2 + C\left\|U_h^0\right\|_{\mathcal{V}}^2 \\
&+ \frac{1}{\epsilon_c}\int_0^{t_m}\left\|\dot{g}_N(t')\right\|_{L_2(\Gamma_N)}^2 dt' + Ch^{-1}\Delta t\epsilon_c\sum_{n=1}^{m-1}\left\|U_h^n\right\|_{\mathcal{V}}^2 + \frac{1}{\epsilon_d}\left\|u_0\right\|_{\mathcal{V}}^2 \\
&+ \epsilon_d\left\|U_h^m\right\|_{\mathcal{V}}^2 + \frac{C}{\sqrt{\alpha_0}}\left\|u_0\right\|_{\mathcal{V}}^2 + \frac{C}{\sqrt{\alpha_0}}\left\|U_h^m\right\|_{\mathcal{V}}^2 + 2\left\|U_h^0\right\|_{\mathcal{V}}^2 \\
&+ \sum_{q=1}^{N_\varphi}\frac{K^2}{4\tau_q\epsilon_e}\left\|u_0\right\|_{\mathcal{V}}^2 + \Delta t\frac{\epsilon_e}{2}\sum_{n=0}^{m-1}\left\|U_h^n\right\|_{\mathcal{V}}^2 \\
\leq & \frac{\Delta t}{\epsilon_a}\sum_{n=0}^{N-1}\left\|\bar{f}^n\right\|_{L_2(\Omega)}^2 + T\epsilon_a\max_{0\leq j\leq N}\left\|W_h^j\right\|_{L_2(\Omega)}^2 \\
&+ (\frac{1}{\epsilon_b}+h^{-1})\max_{0\leq j\leq N-1}\left\|\bar{g}_N^j\right\|_{L_2(\Gamma_N)}^2 + Ch^{-1}\epsilon_b\left\|U_h^m\right\|_{\mathcal{V}}^2 + C\left\|U_h^0\right\|_{\mathcal{V}}^2 \\
&+ \frac{1}{\epsilon_c}\left\|\dot{g}_N\right\|_{L_2(0,T;L_2(\Gamma_N))}^2 + Ch^{-1}T\epsilon_c\max_{0\leq j\leq N}\left\|U_h^j\right\|_{\mathcal{V}}^2 + \frac{1}{\epsilon_d}\left\|u_0\right\|_{\mathcal{V}}^2 \\
&+ \epsilon_d\left\|U_h^m\right\|_{\mathcal{V}}^2 + \frac{C}{\sqrt{\alpha_0}}\left\|u_0\right\|_{\mathcal{V}}^2 + \frac{C}{\sqrt{\alpha_0}}\left\|U_h^m\right\|_{\mathcal{V}}^2 + 2\left\|U_h^0\right\|_{\mathcal{V}}^2 \\
&+ \sum_{q=1}^{N_\varphi}\frac{K^2}{4\tau_q\epsilon_e}\left\|u_0\right\|_{\mathcal{V}}^2 + \frac{T\epsilon_e}{2}\max_{0\leq j\leq N}\left\|U_h^j\right\|_{\mathcal{V}}^2
\end{aligned}
$$

for any positive $\epsilon_a,\ \epsilon_b,\ \epsilon_c,\ \epsilon_d$ and ϵ_e.

- $\left|\varphi_0 \sum_{n=0}^{m-1} B(U_h^{n+1}, U_h^n)\right|$

Since $B(v,v)=0,\ \forall v \in \mathcal{D}_k(\mathcal{E}_h)$,

$$B(U_h^{n+1},U_h^n)=B(U_h^{n+1}-U_h^n,U_h^n).$$

(3.3.1) allows us to have

$$B(U_h^{n+1}-U_h^n,U_h^n)=\frac{\Delta t}{2}B(W_h^{n+1}+W_h^n,U_h^n).$$

As seen in the proof of Theorem 3.11 by (3.2.1), it yields

$$\begin{aligned}\left|\varphi_0\sum_{n=0}^{m-1}B\left(U_h^{n+1},U_h^n\right)\right| \leq & \frac{C\varphi_0K}{\sqrt{\alpha_0}}\left\|u_0\right\|_{\mathcal{V}}^2+\frac{2C\varphi_0(T+1)}{\sqrt{\alpha_0}}\max_{0\leq n\leq N}\left\|U_h^n\right\|_{\mathcal{V}}^2\\ &+\frac{2C\varphi_0\Delta t}{\sqrt{\alpha_0}}\sum_{n=0}^{m-1}J_0^{\alpha_0,\beta_0}(W_h^{n+1}+W_h^n,W_h^{n+1}+W_h^n).\end{aligned}$$

Tidying up the above results, (3.3.29) yields

$$\begin{aligned}&\rho\left\|W_h^m\right\|_{L_2(\Omega)}^2+(\varphi_0-Ch^{-1}\epsilon_b-\epsilon_d-\frac{C}{\sqrt{\alpha_0}})\left\|U_h^m\right\|_{\mathcal{V}}^2+\sum_{q=1}^{N_\varphi}\frac{\kappa}{\varphi_q}\left\|\mathcal{S}_{hq}^m\right\|_{\mathcal{V}}^2\\ &+\frac{\Delta t}{2}\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi}\frac{\kappa}{\tau_q\varphi_q}\left\|\mathcal{S}_{hq}^{n+1}+\mathcal{S}_{hq}^n\right\|_{\mathcal{V}}^2\\ &+\Delta t(\frac{1}{2}-\frac{2C\varphi_0}{\sqrt{\alpha_0}})\sum_{n=0}^{m-1}J_0^{\alpha_0,\beta_0}(W_h^{n+1}+W_h^n,W_h^{n+1}+W_h^n)\\ \leq&\rho\left\|W_h^0\right\|_{L_2(\Omega)}^2+(C+2+\frac{C}{\sqrt{\alpha_0}})\left\|U_h^0\right\|_{\mathcal{V}}^2+\frac{\Delta t}{\epsilon_a}\sum_{n=0}^{N-1}\left\|\bar{f}^n\right\|_{L_2(\Omega)}^2\\ &+T\epsilon_a\max_{0\leq j\leq N}\left\|W_h^j\right\|_{L_2(\Omega)}^2+(\frac{1}{\epsilon_b}+h^{-1})\max_{0\leq j\leq N-1}\left\|\bar{g}_N^j\right\|_{L_2(\Gamma_N)}^2\\ &+\frac{1}{\epsilon_c}\left\|\dot{g}_N\right\|_{L_2(0,T;L_2(\Gamma_N))}^2+Ch^{-1}T\epsilon_c\max_{0\leq j\leq N}\left\|U_h^j\right\|_{\mathcal{V}}^2\\ &+\left(\frac{1}{\epsilon_d}+\frac{C(1+\varphi_0K)}{\sqrt{\alpha_0}}+\sum_{q=1}^{N_\varphi}\frac{K^2}{4\tau_q\epsilon_e}\right)\left\|u_0\right\|_{\mathcal{V}}^2+\frac{T}{2\epsilon_e}\max_{0\leq n\leq N}\left\|U_h^n\right\|_{\mathcal{V}}^2\\ &+\frac{2C\varphi_0(T+1)}{\sqrt{\alpha_0}}\max_{0\leq n\leq N}\left\|U_h^n\right\|_{\mathcal{V}}^2.\end{aligned}$$

Whence taking into account maxima, we have

$$\rho\max_{0\leq j\leq N}\left\|W_h^j\right\|_{L_2(\Omega)}^2+(\varphi_0-Ch^{-1}\epsilon_b-\epsilon_d-\frac{C}{\sqrt{\alpha_0}})\max_{0\leq j\leq N}\left\|U_h^j\right\|_{\mathcal{V}}^2+\sum_{q=1}^{N_\varphi}\frac{\kappa}{\varphi_q}\left\|\mathcal{S}_{hq}^m\right\|_{\mathcal{V}}^2$$

$$
\begin{aligned}
&+ \frac{\Delta t}{2} \sum_{n=0}^{m-1} \sum_{q=1}^{N_\varphi} \frac{\kappa}{\tau_q \varphi_q} \left\| \mathcal{S}_{hq}^{n+1} + \mathcal{S}_{hq}^{n} \right\|_{\mathcal{V}}^2 \\
&+ \Delta t \big(\frac{1}{2} - \frac{2C\varphi_0}{\sqrt{\alpha_0}} \big) \sum_{n=0}^{m-1} J_0^{\alpha_0, \beta_0} (W_h^{n+1} + W_h^n, W_h^{n+1} + W_h^n) \\
\leq & 3 \Bigg(\rho \left\| W_h^0 \right\|_{L_2(\Omega)}^2 + (C + 1 + \frac{C}{\sqrt{\alpha_0}}) \left\| U_h^0 \right\|_{\mathcal{V}}^2 + \frac{\Delta t}{\epsilon_a} \sum_{n=0}^{N-1} \left\| \bar{f}^n \right\|_{L_2(\Omega)}^2 \\
&+ T \epsilon_a \max_{0 \leq j \leq N} \left\| W_h^j \right\|_{L_2(\Omega)}^2 + (\frac{1}{\epsilon_b} + h^{-1}) \max_{0 \leq j \leq N-1} \left\| \bar{g}_N^j \right\|_{L_2(\Gamma_N)}^2 \\
&+ \frac{1}{\epsilon_c} \left\| \dot{g}_N \right\|_{L_2(0,T;L_2(\Gamma_N))}^2 + C h^{-1} T \epsilon_c \max_{0 \leq j \leq N} \left\| U_h^j \right\|_{\mathcal{V}}^2 \\
&+ \left(\frac{1}{\epsilon_d} + \frac{C(1 + \varphi_0 K)}{\sqrt{\alpha_0}} + \sum_{q=1}^{N_\varphi} \frac{K^2}{4 \tau_q \epsilon_e} \right) \left\| u_0 \right\|_{\mathcal{V}}^2 + \frac{T}{2\epsilon_e} \max_{0 \leq n \leq N} \left\| U_h^n \right\|_{\mathcal{V}}^2 \\
&+ \frac{2C\varphi_0(T+1)}{\sqrt{\alpha_0}} \max_{0 \leq n \leq N} \left\| U_h^n \right\|_{\mathcal{V}}^2 \Bigg).
\end{aligned}
$$

Set

$$
\epsilon_a = \frac{\rho}{6T}, \ \epsilon_b = \frac{\varphi_0 h}{8C}, \ \epsilon_c = \frac{\varphi_0 h}{48CT}, \ \epsilon_d = \frac{\varphi_0}{2}, \ \epsilon_e = \frac{\varphi_0}{24T}
$$

then it implies

$$
\begin{aligned}
&\frac{\rho}{2} \max_{0 \leq j \leq N} \left\| W_h^j \right\|_{L_2(\Omega)}^2 + \left(\frac{\varphi_0}{4} - \frac{6C\varphi_0(T+1)}{\sqrt{\alpha_0}} \right) \max_{0 \leq j \leq N} \left\| U_h^j \right\|_{\mathcal{V}}^2 + \sum_{q=1}^{N_\varphi} \frac{\kappa}{\varphi_q} \left\| \mathcal{S}_{hq}^m \right\|_{\mathcal{V}}^2 \\
&\quad + \frac{\Delta t}{2} \sum_{n=0}^{m-1} \sum_{q=1}^{N_\varphi} \frac{\kappa}{\tau_q \varphi_q} \left\| \mathcal{S}_{hq}^{n+1} + \mathcal{S}_{hq}^{n} \right\|_{\mathcal{V}}^2 \\
&\quad + \Delta t \big(\frac{1}{2} - \frac{2C\varphi_0}{\sqrt{\alpha_0}} \big) \sum_{n=0}^{m-1} J_0^{\alpha_0, \beta_0} (W_h^{n+1} + W_h^n, W_h^{n+1} + W_h^n) \\
&\leq 3 \Bigg(\rho \left\| W_h^0 \right\|_{L_2(\Omega)}^2 + (\varphi_0 + C + 1 + \frac{C}{\sqrt{\alpha_0}}) \left\| U_h^0 \right\|_{\mathcal{V}}^2 \\
&\quad + \left(\frac{2}{\varphi_0} + \frac{C(1 + \varphi_0 K)}{\sqrt{\alpha_0}} + \sum_{q=1}^{N_\varphi} \frac{6K^2 T}{\tau_q \varphi_0} \right) \left\| u_0 \right\|_{\mathcal{V}}^2 + \Delta t \frac{6T}{\rho} \sum_{n=0}^{N-1} \left\| \bar{f}^n \right\|_{L_2(\Omega)}^2 \\
&\quad + (\frac{8C}{\varphi_0} + 1) h^{-1} \max_{0 \leq j \leq N-1} \left\| \bar{g}_N^j \right\|_{L_2(\Gamma_N)}^2 + \frac{48CT}{\varphi_0} h^{-1} \left\| \dot{g}_N \right\|_{L_2(0,T;L_2(\Gamma_N))}^2 \Bigg).
\end{aligned}
$$

If we assume α_0 is large enough as

$$
\frac{\varphi_0}{4} - \frac{6C\varphi_0(T+1)}{\sqrt{\alpha_0}} > 0, \ \frac{1}{2} - \frac{2C\varphi_0}{\sqrt{\alpha_0}} > 0,
$$

we can conclude that

$$\max_{0\le j\le N}\left\|W_h^j\right\|_{L_2(\Omega)}^2+\max_{0\le j\le N}\left\|U_h^j\right\|_{\mathcal{V}}^2+\sum_{q=1}^{N_\varphi}\max_{0\le j\le N}\left\|\mathcal{S}_{hq}^j\right\|_{\mathcal{V}}^2+\Delta t\sum_{n=0}^{N-1}\sum_{q=1}^{N_\varphi}\left\|\mathcal{S}_{hq}^{n+1}+\mathcal{S}_{hq}^n\right\|_{\mathcal{V}}^2$$
$$+\Delta t\sum_{n=0}^{N-1}J_0^{\alpha_0,\beta_0}(W_h^{n+1}+W_h^n,W_h^{n+1}+W_h^n)$$
$$\le C\big(\|w_0\|_{L_2(\Omega)}^2+\|u_0\|_{\mathcal{V}}^2+\Delta t\sum_{n=0}^{N-1}\left\|\bar{f}^n\right\|_{L_2(\Omega)}^2+h^{-1}\max_{0\le j\le N-1}\left\|\bar{g}_N^j\right\|_{L_2(\Gamma_N)}^2$$
$$+h^{-1}\|\dot{g}_N\|_{L_2(0,T;L_2(\Gamma_N))}^2\big)$$
$$\le C\big(\|w_0\|_{L_2(\Omega)}^2+\|u_0\|_{\mathcal{V}}^2+\|f\|_{L_\infty(0,T;L_2(\Omega))}^2+h^{-1}\|g_N\|_{L_\infty(0,T;L_2(\Gamma_N))}^2$$
$$+h^{-1}\|\dot{g}_N\|_{L_2(0,T;L_2(\Gamma_N))}^2\big)$$

by Cauchy-Schwarz inequalities, since m is arbitrary, $\left\|W_h^0\right\|_{L_2(\Omega)}\le\|w_0\|_{L_2(\Omega)}$, and $\left\|U_h^0\right\|_{\mathcal{V}}\le K\|u_0\|_{\mathcal{V}}$. □

Theorem 3.13 indicates that the fully discrete problem of **(Q2)** can be solved uniquely. Hence (3.3.26)-(3.3.28) could be determined for any $n=0,\dots,N-1$. With similar techniques, we can obtain the error bounds for the fully discrete formulation of **(Q2)**.

Let us define

$$\theta(t):=u(t)-R_1u(t),\qquad \chi^n:=U_h^n-R_1u(t_n),\qquad \varpi^n:=W_h^n-R_1\dot{u}(t_n),$$
$$\nu_q(t):=\zeta_q(t)-R_{-1}\zeta_q(t),\qquad \Upsilon_q^n:=\mathcal{S}_{hq}^n-R_{-1}\zeta_q(t_n),\ \forall q\in\{1,\dots,N_\varphi\},$$

for $n=0,\dots,N$, $\forall t\in[0,T]$. Recall Galerkin orthogonality and its properties such as (3.3.15) and (3.3.16).

Lemma 3.8. *Suppose $u\in H^4(0,T;C^2(\Omega))\cap W_\infty^1(0,T;H^s(\mathcal{E}_h))$ and $\beta_0(d-1)\ge 1$ for $s>3/2$. For large enough α_0, there exists a positive constant C such that*

$$\max_{0\le j\le N}\left\|\varpi^j\right\|_{L_2(\Omega)}+\max_{0\le j\le N}\left\|\chi^j\right\|_{\mathcal{V}}+\sum_{q=1}^{N_\varphi}\max_{0\le j\le N}\left\|\Upsilon_q^j\right\|_{\mathcal{V}}+\Delta t\sum_{q=1}^{N_\varphi}\sum_{n=0}^{N-1}\left\|\Upsilon_q^{n+1}+\Upsilon_q^n\right\|_{\mathcal{V}}$$
$$+\Delta t\sum_{n=0}^{N-1}J_0^{\alpha_0,\beta_0}(\varpi^{n+1}+\varpi^n,\varpi^{n+1}+\varpi^n)^{1/2}$$
$$\le C(h^{\min(k+1,s)-1}+\Delta t^2).$$

Furthermore, if Ω is convex and $\beta_0(d-1)\ge 3$, we have

$$\max_{0\le j\le N}\left\|\varpi^j\right\|_{L_2(\Omega)}+\max_{0\le j\le N}\left\|\chi^j\right\|_{\mathcal{V}}+\sum_{q=1}^{N_\varphi}\max_{0\le j\le N}\left\|\Upsilon_q^j\right\|_{\mathcal{V}}+\Delta t\sum_{q=1}^{N_\varphi}\sum_{n=0}^{N-1}\left\|\Upsilon_q^{n+1}+\Upsilon_q^n\right\|_{\mathcal{V}}$$

$$+\Delta t\sum_{n=0}^{N-1} J_0^{\alpha_0,\beta_0}(\varpi^{n+1}+\varpi^n,\varpi^{n+1}+\varpi^n)^{1/2}$$
$$\leq C(h^{\min(k+1,s)}+\Delta t^2).$$

Proof. Let $n\in\{0,\dots,N-1\}$. For average between $t=t_{n+1}$ and $t=t_n$, subtracting (3.3.22) from (3.1.5) gives

$$\begin{aligned}
&\rho\left(\frac{\ddot{u}^{n+1}+\ddot{u}^n}{2}-\frac{W_h^{n+1}-W_h^n}{\Delta t},v\right)_{L_2(\Omega)}+\varphi_0 a_1\left(\frac{u^{n+1}+u^n}{2}-\frac{U_h^{n+1}+U_h^n}{2},v\right)\\
&+\sum_{q=1}^{N_\varphi} a_{-1}\left(\frac{\zeta_q^{n+1}+\zeta_q^n}{2}-\frac{\mathcal{S}_{hq}^{n+1}+\mathcal{S}_{hq}^n}{2},v\right)+\frac{1}{2}J_0^{\alpha_0,\beta_0}(\dot{u}^{n+1}+\dot{u}^n-(W_h^{n+1}+W_h^n),v)\\
=&0,
\end{aligned}$$

for any $v\in\mathcal{D}_k(\mathcal{E}_h)$. By adding zeros and Galerkin orthogonality, we have

$$\begin{aligned}
&\frac{\rho}{\Delta t}\left(\varpi^{n+1}-\varpi^n,v\right)_{L_2(\Omega)}+\frac{\varphi_0}{2}a_1\left(\chi^{n+1}+\chi^n,v\right)+\frac{1}{2}\sum_{q=1}^{N_\varphi}a_{-1}\left(\Upsilon_q^{n+1}+\Upsilon_q^n,v\right)\\
&\quad+\frac{1}{2}J_0^{\alpha_0,\beta_0}(\varpi^{n+1}+\varpi^n,v)\\
&=\frac{\rho}{\Delta t}\left(\dot{\theta}^{n+1}-\dot{\theta}^n,v\right)_{L_2(\Omega)}+\rho\left(\mathcal{E}_1^n,v\right)_{L_2(\Omega)}
\end{aligned}$$

for any $v\in\mathcal{D}_k(\mathcal{E}_h)$, where $\mathcal{E}_1(t):=\frac{\ddot{u}(t+\Delta t)+\ddot{u}(t)}{2}-\frac{\dot{u}(t+\Delta t)-\dot{u}(t)}{\Delta t}$. If we put $v=\frac{\chi^{n+1}-\chi^n}{\Delta t}$ with (3.3.18) here, we can obtain

$$\begin{aligned}
&\frac{\rho}{2\Delta t}\left(\left\|\varpi^{n+1}\right\|_{L_2(\Omega)}^2-\left\|\varpi^n\right\|_{L_2(\Omega)}^2\right)+\frac{\varphi_0}{2\Delta t}\left(\left\|\chi^{n+1}\right\|_{\mathcal{V}}^2-\left\|\chi^n\right\|_{\mathcal{V}}^2\right)\\
&\quad+\frac{1}{2\Delta t}\sum_{q=1}^{N_\varphi}a_{-1}\left(\Upsilon_q^{n+1}+\Upsilon_q^n,\chi^{n+1}-\chi^n\right)+\frac{1}{4}J_0^{\alpha_0,\beta_0}(\varpi^{n+1}+\varpi^n,\varpi^{n+1}+\varpi^n)\\
&=\frac{\rho}{2\Delta t}\left(\dot{\theta}^{n+1}-\dot{\theta}^n,\varpi^{n+1}+\varpi^n\right)_{L_2(\Omega)}+\frac{\rho}{2}\left(\mathcal{E}_1^n,\varpi^{n+1}+\varpi^n\right)_{L_2(\Omega)}\\
&\quad-\frac{\rho}{\Delta t}\left(\dot{\theta}^{n+1}-\dot{\theta}^n,\mathcal{E}_2^n\right)_{L_2(\Omega)}-\frac{\rho}{\Delta t}\left(\dot{\theta}^{n+1}-\dot{\theta}^n,\mathcal{E}_3^n\right)_{L_2(\Omega)}\\
&\quad-\rho\left(\mathcal{E}_1^n,\mathcal{E}_2^n\right)_{L_2(\Omega)}-\rho\left(\mathcal{E}_1^n,\mathcal{E}_3^n\right)_{L_2(\Omega)}+\frac{\varphi_0}{2\Delta t}B(\chi^{n+1},\chi^n),
\end{aligned}\tag{3.3.30}$$

where

$$\begin{aligned}
\mathcal{E}_2(t)&:=\frac{\dot{\theta}(t+\Delta t)+\dot{\theta}(t)}{2}-\frac{\theta(t+\Delta t)-\theta(t)}{\Delta t},\\
\mathcal{E}_3(t)&:=\frac{u(t+\Delta t)-u(t)}{\Delta t}-\frac{\dot{u}(t+\Delta t)+\dot{u}(t)}{2}.
\end{aligned}$$

In a similar way, subtracting (3.3.23) from (3.1.6) yields

$$
\begin{aligned}
&\tau_q a_{-1}\left(\frac{\dot{\zeta}_q^{n+1}+\dot{\zeta}_q^{n}}{2}-\frac{\mathcal{S}_{hq}^{n+1}-\mathcal{S}_{hq}^{n}}{\Delta t},v\right)+a_{-1}\left(\frac{\zeta_q^{n+1}+\zeta_q^{n}}{2}-\frac{\mathcal{S}_{hq}^{n+1}+\mathcal{S}_{hq}^{n}}{2},v\right)\\
=&\tau_q\varphi_q a_{-1}\left(\frac{\dot{u}^{n+1}+\dot{u}^{n}}{2}-\frac{U_h^{n+1}-U_h^{n}}{\Delta t},v\right)
\end{aligned}
$$

for any $q\in\{1,\dots,N_\varphi\}$, $\forall v\in\mathcal{D}_k(\mathcal{E}_h)$ and so

$$
\begin{aligned}
&\frac{\tau_q}{\Delta t}a_{-1}\left(\Upsilon_q^{n+1}-\Upsilon_q^{n},v\right)+\frac{1}{2}a_{-1}\left(\Upsilon_q^{n+1}+\Upsilon_q^{n},v\right)-\frac{\tau_q\varphi_q}{\Delta t}a_{-1}\left(\chi^{n+1}-\chi^{n},v\right)\\
&=\tau_q a_{-1}\left(E_q^n,v\right)-\tau_q\varphi_q a_{-1}\left(\mathcal{E}_3^n,v\right)
\end{aligned}
$$

by Garlerkin orthogonality where

$$
E_q(t):=\frac{\dot{\zeta}_q(t+\Delta t)+\dot{\zeta}_q(t)}{2}-\frac{\zeta_q(t+\Delta t)-\zeta_q(t)}{\Delta t}\qquad\text{for each } q.
$$

Whence we set $v=\frac{\Upsilon_q^{n+1}+\Upsilon_q^{n}}{2}$, we have

$$
\begin{aligned}
&\frac{1}{2\Delta t}a_{-1}\left(\Upsilon_q^{n+1}+\Upsilon_q^{n},\chi^{n+1}-\chi^{n}\right)\\
&=\frac{1}{2\Delta t}a_{-1}\left(\chi^{n+1}-\chi^{n},\Upsilon_q^{n+1}+\Upsilon_q^{n}\right)\\
&=\frac{1}{2\Delta t\varphi_q}\left(a_{-1}\left(\Upsilon_q^{n+1},\Upsilon_q^{n+1}\right)-a_{-1}\left(\Upsilon_q^{n},\Upsilon_q^{n}\right)\right)+\frac{1}{2\tau_q\varphi_q}a_{-1}\left(\Upsilon_q^{n+1}+\Upsilon_q^{n},\Upsilon_q^{n+1}+\Upsilon_q^{n}\right)\\
&\quad-\frac{1}{2\varphi_q}a_{-1}\left(E_q^n,\Upsilon_q^{n+1}+\Upsilon_q^{n}\right)+\frac{1}{2}a_{-1}\left(\mathcal{E}_3^n,\Upsilon_q^{n+1}+\Upsilon_q^{n}\right),
\end{aligned}\tag{3.3.31}
$$

for $q\in\{1,\dots,N_\varphi\}$.

By substitution of (3.3.31) into (3.3.30) with multiplying Δt,

$$
\begin{aligned}
&\frac{\rho}{2}\left(\|\varpi^{n+1}\|_{L_2(\Omega)}^2-\|\varpi^{n}\|_{L_2(\Omega)}^2\right)+\frac{\varphi_0}{2}\left(\|\chi^{n+1}\|_{\mathcal{V}}^2-\|\chi^{n}\|_{\mathcal{V}}^2\right)\\
&\quad+\frac{1}{2}\sum_{q=1}^{N_\varphi}\frac{1}{\varphi_q}\left(a_{-1}\left(\Upsilon_q^{n+1},\Upsilon_q^{n+1}\right)-a_{-1}\left(\Upsilon_q^{n},\Upsilon_q^{n}\right)\right)\\
&\quad+\Delta t\sum_{q=1}^{N_\varphi}\frac{1}{2\tau_q\varphi_q}a_{-1}\left(\Upsilon_q^{n+1}+\Upsilon_q^{n},\Upsilon_q^{n+1}+\Upsilon_q^{n}\right)\\
&\quad+\frac{\Delta t}{4}J_0^{\alpha_0,\beta_0}(\varpi^{n+1}+\varpi^{n},\varpi^{n+1}+\varpi^{n})\\
&=\frac{\rho}{2}\left(\dot{\theta}^{n+1}-\dot{\theta}^{n},\varpi^{n+1}+\varpi^{n}\right)_{L_2(\Omega)}+\frac{\rho}{2}\Delta t\left(\mathcal{E}_1^n,\varpi^{n+1}+\varpi^{n}\right)_{L_2(\Omega)}\\
&\quad-\rho\left(\dot{\theta}^{n+1}-\dot{\theta}^{n},\mathcal{E}_2^n\right)_{L_2(\Omega)}-\rho\left(\dot{\theta}^{n+1}-\dot{\theta}^{n},\mathcal{E}_3^n\right)_{L_2(\Omega)}-\rho\Delta t\left(\mathcal{E}_1^n,\mathcal{E}_2^n\right)_{L_2(\Omega)}
\end{aligned}
$$

$$- \rho \Delta t \left(\mathcal{E}_1^n, \mathcal{E}_3^n\right)_{L_2(\Omega)} + \frac{\Delta t}{2} \sum_{q=1}^{N_\varphi} \frac{1}{\varphi_q} a_{-1}\left(E_q^n, \Upsilon_q^{n+1} + \Upsilon_q^n\right)$$

$$- \frac{\Delta t}{2} \sum_{q=1}^{N_\varphi} a_{-1}\left(\mathcal{E}_3^n, \Upsilon_q^{n+1} + \Upsilon_q^n\right) + \frac{\varphi_0}{2} B(\chi^{n+1}, \chi^n).$$

Summation of this for $n = 0, \ldots, m-1$ for $1 \leq m \leq N$ implies

$$\begin{aligned}
&\frac{\rho}{2} \|\varpi^m\|_{L_2(\Omega)}^2 + \frac{\varphi_0}{2} \|\chi^m\|_{\mathcal{V}}^2 + \frac{1}{2} \sum_{q=1}^{N_\varphi} \frac{1}{\varphi_q} a_{-1}\left(\Upsilon_q^m, \Upsilon_q^m\right) \\
&+ \Delta t \sum_{n=0}^{m-1} \sum_{q=1}^{N_\varphi} \frac{1}{2\tau_q \varphi_q} a_{-1}\left(\Upsilon_q^{n+1} + \Upsilon_q^n, \Upsilon_q^{n+1} + \Upsilon_q^n\right) \\
&+ \frac{\Delta t}{4} \sum_{n=0}^{m-1} J_0^{\alpha_0, \beta_0}(\varpi^{n+1} + \varpi^n, \varpi^{n+1} + \varpi^n) \\
=& \frac{\rho}{2} \left\|\varpi^0\right\|_{L_2(\Omega)}^2 + \frac{\varphi_0}{2} \left\|\chi^0\right\|_{\mathcal{V}}^2 + \frac{\rho}{2} \sum_{n=0}^{m-1} \left(\dot{\theta}^{n+1} - \dot{\theta}^n, \varpi^{n+1} + \varpi^n\right)_{L_2(\Omega)} \\
&+ \frac{\rho}{2} \Delta t \sum_{n=0}^{m-1} \left(\mathcal{E}_1^n, \varpi^{n+1} + \varpi^n\right)_{L_2(\Omega)} - \rho \sum_{n=0}^{m-1} \left(\dot{\theta}^{n+1} - \dot{\theta}^n, \mathcal{E}_2^n\right)_{L_2(\Omega)} \\
&- \rho \sum_{n=0}^{m-1} \left(\dot{\theta}^{n+1} - \dot{\theta}^n, \mathcal{E}_3^n\right)_{L_2(\Omega)} - \rho \Delta t \sum_{n=0}^{m-1} \left(\mathcal{E}_1^n, \mathcal{E}_2^n\right)_{L_2(\Omega)} - \rho \Delta t \sum_{n=0}^{m-1} \left(\mathcal{E}_1^n, \mathcal{E}_3^n\right)_{L_2(\Omega)} \\
&+ \frac{\Delta t}{2} \sum_{n=0}^{m-1} \sum_{q=1}^{N_\varphi} \frac{1}{\varphi_q} a_{-1}\left(E_q^n, \Upsilon_q^{n+1} + \Upsilon_q^n\right) - \frac{\Delta t}{2} \sum_{n=0}^{m-1} \sum_{q=1}^{N_\varphi} a_{-1}\left(\mathcal{E}_3^n, \Upsilon_q^{n+1} + \Upsilon_q^n\right) \\
&+ \frac{\varphi_0}{2} \sum_{n=0}^{m-1} B(\chi^{n+1}, \chi^n),
\end{aligned}$$

hence coercivity leads us to obtain

$$\begin{aligned}
&\frac{\rho}{2} \|\varpi^m\|_{L_2(\Omega)}^2 + \frac{\varphi_0}{2} \|\chi^m\|_{\mathcal{V}}^2 + \frac{1}{2} \sum_{q=1}^{N_\varphi} \frac{\kappa}{\varphi_q} \left\|\Upsilon_q^m\right\|_{\mathcal{V}}^2 + \Delta t \sum_{n=0}^{m-1} \sum_{q=1}^{N_\varphi} \frac{\kappa}{2\tau_q \varphi_q} \left\|\Upsilon_q^{n+1} + \Upsilon_q^n\right\|_{\mathcal{V}}^2 \\
&+ \frac{\Delta t}{4} \sum_{n=0}^{m-1} J_0^{\alpha_0, \beta_0}(\varpi^{n+1} + \varpi^n, \varpi^{n+1} + \varpi^n) \qquad (3.3.32) \\
\leq& \left| \frac{\rho}{2} \left\|\varpi^0\right\|_{L_2(\Omega)}^2 + \frac{\varphi_0}{2} \left\|\chi^0\right\|_{\mathcal{V}}^2 + \frac{1}{2} \sum_{q=1}^{N_\varphi} \frac{1}{\varphi_q} \left\|\Upsilon_q^0\right\|_{\mathcal{V}}^2 + \frac{\rho}{2} \sum_{n=0}^{m-1} \left(\dot{\theta}^{n+1} - \dot{\theta}^n, \varpi^{n+1} + \varpi^n\right)_{L_2(\Omega)} \right. \\
&+ \frac{\rho}{2} \Delta t \sum_{n=0}^{m-1} \left(\mathcal{E}_1^n, \varpi^{n+1} + \varpi^n\right)_{L_2(\Omega)} - \rho \sum_{n=0}^{m-1} \left(\dot{\theta}^{n+1} - \dot{\theta}^n, \mathcal{E}_2^n\right)_{L_2(\Omega)}
\end{aligned}$$

$$- \rho \sum_{n=0}^{m-1} \left(\dot{\theta}^{n+1} - \dot{\theta}^n, \mathcal{E}_3^n \right)_{L_2(\Omega)} - \rho \Delta t \sum_{n=0}^{m-1} \left(\mathcal{E}_1^n, \mathcal{E}_2^n \right)_{L_2(\Omega)} - \rho \Delta t \sum_{n=0}^{m-1} \left(\mathcal{E}_1^n, \mathcal{E}_3^n \right)_{L_2(\Omega)}$$
$$+ \frac{\Delta t}{2} \sum_{n=0}^{m-1} \sum_{q=1}^{N_\varphi} \frac{1}{\varphi_q} a_{-1} \left(E_q^n, \Upsilon_q^{n+1} + \Upsilon_q^n \right) - \frac{\Delta t}{2} \sum_{n=0}^{m-1} \sum_{q=1}^{N_\varphi} a_{-1} \left(\mathcal{E}_3^n, \Upsilon_q^{n+1} + \Upsilon_q^n \right)$$
$$+ \frac{\varphi_0}{2} \sum_{n=0}^{m-1} B(\chi^{n+1}, \chi^n) \Bigg|. \tag{3.3.33}$$

Comparing (3.3.33) with (2.3.34), we can derive the bounds for the right hand side of (3.3.33) except the skew symmetric terms $B(\cdot,\cdot)$ but in $\|\cdot\|_{\mathcal{V}}$ rather than in $\|\cdot\|_V$. In other words, as following the same arguments in bounds for (2.3.34) in Lemma 2.13, for example using Cauchy-Schwarz inequality, Young's inequality, Theorem 3.7, (3.2.5) and the result of Crank-Nicolson methods, we can obtain

$$\frac{\rho}{2} \|\varpi^m\|_{L_2(\Omega)}^2 + \frac{\varphi_0}{2} \|\chi^m\|_{\mathcal{V}}^2 + \frac{\kappa}{2} \sum_{q=1}^{N_\varphi} \frac{1}{\varphi_q} \left\|\Upsilon_q^m\right\|_{\mathcal{V}}^2 + \Delta t \sum_{n=0}^{m-1} \sum_{q=1}^{N_\varphi} \frac{\kappa}{2\tau_q \varphi_q} \left\|\Upsilon_q^{n+1} + \Upsilon_q^n\right\|_{\mathcal{V}}^2$$
$$+ \frac{\Delta t}{4} \sum_{n=0}^{m-1} J_0^{\alpha_0,\beta_0}(\varpi^{n+1} + \varpi^n, \varpi^{n+1} + \varpi^n)$$
$$\leq \frac{\rho}{12} \max_{0 \leq j \leq N} \left\|\varpi^j\right\|_{L_2(\Omega)}^2 + O(h^{2(\min(k+1,s)-1)} + \Delta t^4)$$
$$+ \frac{\Delta t}{2} \sum_{n=0}^{m-1} \sum_{q=1}^{N_\varphi} \frac{1}{\varphi_q} |a_{-1} \left(E_q^n, \Upsilon_q^{n+1} + \Upsilon_q^n \right)| + \frac{\Delta t}{2} \sum_{n=0}^{m-1} \sum_{q=1}^{N_\varphi} |a_{-1} \left(\mathcal{E}_3^n, \Upsilon_q^{n+1} + \Upsilon_q^n \right)|$$
$$+ \frac{\varphi_0}{2} |B(\chi^{n+1}, \chi^n)|. \tag{3.3.34}$$

Here, instead of (1.4.8), Theorem 3.7 is applied for approximations. Now, we shall consider the skew symmetric terms and SIPG terms. (3.2.4) and the continuity of SIPG imply the bounds for these skew symmetric terms as below

- $\frac{\Delta t}{2} \sum_{n=0}^{m-1} \sum_{q=1}^{N_\varphi} \frac{1}{\varphi_q} |a_{-1} \left(E_q^n, \Upsilon_q^{n+1} + \Upsilon_q^n \right)|$, $\frac{\Delta t}{2} \sum_{n=0}^{m-1} \sum_{q=1}^{N_\varphi} |a_{-1} \left(\mathcal{E}_3^n, \Upsilon_q^{n+1} + \Upsilon_q^n \right)|$
  Use of continuity of SIPG and Young's inequality provides

$$\frac{\Delta t}{2} \sum_{n=0}^{m-1} \sum_{q=1}^{N_\varphi} \frac{1}{\varphi_q} |a_{-1} \left(E_q^n, \Upsilon_q^{n+1} + \Upsilon_q^n \right)|$$
$$\leq \frac{\Delta t}{2} \sum_{n=0}^{m-1} \sum_{q=1}^{N_\varphi} \frac{K}{\varphi_q} \left\|E_q^n\right\|_{\mathcal{V}} \left\|\Upsilon_q^{n+1} + \Upsilon_q^n\right\|_{\mathcal{V}}$$
$$\leq \frac{\Delta t}{2} \sum_{n=0}^{m-1} \sum_{q=1}^{N_\varphi} \frac{K^2}{2\varphi_q^2 \epsilon_q} \left\|E_q^n\right\|_{\mathcal{V}}^2 + \frac{\Delta t}{2} \sum_{n=0}^{m-1} \sum_{q=1}^{N_\varphi} \frac{\epsilon_q}{2} \left\|\Upsilon_q^{n+1} + \Upsilon_q^n\right\|_{\mathcal{V}}^2$$

$$\leq \frac{\Delta t}{2} \sum_{n=0}^{N-1} \sum_{q=1}^{N_\varphi} \frac{K^2}{2\varphi_q^2 \epsilon_q} \left\| E_q^n \right\|_{\mathcal{V}}^2 + \frac{\Delta t}{2} \sum_{n=0}^{m-1} \sum_{q=1}^{N_\varphi} \frac{\epsilon_q}{2} \left\| \Upsilon_q^{n+1} + \Upsilon_q^n \right\|_{\mathcal{V}}^2$$

for any positive ϵ_q for each q. In the same sense,

$$\begin{aligned}
&\frac{\Delta t}{2} \sum_{n=0}^{m-1} \sum_{q=1}^{N_\varphi} |a_{-1}\left(\mathcal{E}_3^n, \Upsilon_q^{n+1} + \Upsilon_q^n\right)| \\
&\leq \frac{\Delta t}{2} \sum_{n=0}^{N-1} \sum_{q=1}^{N_\varphi} \frac{K^2}{2\epsilon_q} \|\mathcal{E}_3^n\|_{\mathcal{V}}^2 + \frac{\Delta t}{2} \sum_{n=0}^{m-1} \sum_{q=1}^{N_\varphi} \frac{\epsilon_q}{2} \left\| \Upsilon_q^{n+1} + \Upsilon_q^n \right\|_{\mathcal{V}}^2
\end{aligned}$$

While we take $\epsilon_q = \frac{\kappa}{2\tau_q \varphi_q}$ for each q, since finite difference approximations used, we gain

$$\begin{aligned}
&\frac{\Delta t}{2} \sum_{n=0}^{m-1} \sum_{q=1}^{N_\varphi} \frac{1}{\varphi_q} |a_{-1}\left(E_q^n, \Upsilon_q^{n+1} + \Upsilon_q^n\right)| + \frac{\Delta t}{2} \sum_{q=1}^{N_\varphi} |a_{-1}\left(\mathcal{E}_3^n, \Upsilon_q^{n+1} + \Upsilon_q^n\right)| \\
&\leq O(\Delta t^4) + \Delta t \sum_{n=0}^{m-1} \sum_{q=1}^{N_\varphi} \frac{\kappa}{4\tau_q \varphi_q} \left\| \Upsilon_q^{n+1} + \Upsilon_q^n \right\|_{\mathcal{V}}^2 .
\end{aligned}$$

- $\frac{\varphi_0}{2} \sum_{n=0}^{m-1} |B(\chi^{n+1}, \chi^n)|$

  Since $B(v, v) = 0$ for any $v \in \mathcal{D}_k(\mathcal{E}_h)$,

$$\begin{aligned}
\sum_{n=0}^{m-1} B(\chi^{n+1}, \chi^n) &= \sum_{n=0}^{m-1} B(\chi^{n+1} - \chi^n, \chi^n) \\
&= \sum_{n=0}^{m-1} \frac{\Delta t}{2} B(\varpi^{n+1} + \varpi^n, \chi^n) - \sum_{n=0}^{m-1} \Delta t B(\mathcal{E}_2^n, \chi^n) \\
&\quad - \sum_{n=0}^{m-1} \Delta t B(\mathcal{E}_3^n, \chi^n)
\end{aligned}$$

by (3.3.18). Then the definition of skew symmetric $B(\cdot, \cdot)$, (3.2.1) and (3.2.2) lead us to have

$$\begin{aligned}
\left| \frac{\varphi_0}{2} \sum_{n=0}^{m-1} B(\chi^{n+1}, \chi^n) \right| =& \varphi_0 \left| \sum_{n=0}^{m-1} \frac{\Delta t}{4} B(\varpi^{n+1} + \varpi^n, \chi^n) - \sum_{n=0}^{m-1} \frac{\Delta t}{2} B(\mathcal{E}_2^n, \chi^n) \right. \\
& \left. - \sum_{n=0}^{m-1} \frac{\Delta t}{2} B(\mathcal{E}_3^n, \chi^n) \right| \\
\leq& \varphi_0 \left(\frac{2C\Delta t}{\sqrt{\alpha_0}} \sum_{n=0}^{m-1} \|\chi^n\|_{\mathcal{V}}^2 + \frac{C}{2\sqrt{\alpha_0}} \|\chi^m\|_{\mathcal{V}}^2 + \frac{C}{2\sqrt{\alpha_0}} \left\| \chi^{m-1} \right\|_{\mathcal{V}}^2 \right.
\end{aligned}$$

$$+\frac{C\Delta t}{2\sqrt{\alpha_0}}\sum_{n=0}^{m-1}\|\mathcal{E}_2^n\|_{\mathcal{V}}^2+\frac{C\Delta t}{2\sqrt{\alpha_0}}\sum_{n=0}^{m-1}\|\mathcal{E}_3^n\|_{\mathcal{V}}^2$$

$$+\frac{C\Delta t}{\sqrt{\alpha_0}}\sum_{n=0}^{m-1}J_0^{\alpha_0,\beta_0}(\varpi^{n+1}+\varpi^n,\varpi^{n+1}+\varpi^n)\Big).$$

In addition, consider the maximum on the right hand side with respect to m, except the jump penalty. Then we obtain

$$\begin{aligned}\left|\frac{\varphi_0}{2}\sum_{n=0}^{m-1}B(\chi^{n+1},\chi^n)\right| \leq &\varphi_0\Big(\frac{C\Delta t}{2\sqrt{\alpha_0}}\sum_{n=0}^{N-1}\|\mathcal{E}_2^n\|_{\mathcal{V}}^2+\frac{C\Delta t}{2\sqrt{\alpha_0}}\sum_{n=0}^{N-1}\|\mathcal{E}_3^n\|_{\mathcal{V}}^2\\
&+\frac{C(2T+1)}{\sqrt{\alpha_0}}\max_{0\leq j\leq N}\left\|\chi^j\right\|_{\mathcal{V}}^2\\
&+\frac{C\Delta t}{\sqrt{\alpha_0}}\sum_{n=0}^{m-1}J_0^{\alpha_0,\beta_0}(\varpi^{n+1}+\varpi^n,\varpi^{n+1}+\varpi^n)\Big)\\
\leq &O(\Delta t^4)+\frac{C\varphi_0(2T+1)}{\sqrt{\alpha_0}}\max_{0\leq j\leq N}\left\|\chi^j\right\|_{\mathcal{V}}^2\\
&+\frac{C\varphi_0\Delta t}{\sqrt{\alpha_0}}\sum_{n=0}^{m-1}J_0^{\alpha_0,\beta_0}(\varpi^{n+1}+\varpi^n,\varpi^{n+1}+\varpi^n)\end{aligned}$$

Hence (3.3.34) can be rewritten by

$$\begin{aligned}&\frac{\rho}{2}\|\varpi^m\|_{L_2(\Omega)}^2+\frac{\varphi_0}{2}\|\chi^m\|_{\mathcal{V}}^2+\frac{\kappa}{2}\sum_{q=1}^{N_\varphi}\frac{1}{\varphi_q}\left\|\Upsilon_q^m\right\|_{\mathcal{V}}^2+\Delta t\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi}\frac{\kappa}{4\tau_q\varphi_q}\left\|\Upsilon_q^{n+1}+\Upsilon_q^n\right\|_{\mathcal{V}}^2\\
&+\Delta t\sum_{n=0}^{m-1}(\frac{1}{4}-\frac{C\varphi_0}{\sqrt{\alpha_0}})J_0^{\alpha_0,\beta_0}(\varpi^{n+1}+\varpi^n,\varpi^{n+1}+\varpi^n)\\
\leq&\frac{\rho}{12}\max_{0\leq j\leq N}\left\|\varpi^j\right\|_{L_2(\Omega)}^2+\frac{C\varphi_0(2T+1)}{\sqrt{\alpha_0}}\max_{0\leq j\leq N}\left\|\chi^j\right\|_{\mathcal{V}}^2+O(h^{2(\min(k+1,s)-1)}+\Delta t^4)\end{aligned}$$

so that when taking into account maxima on the left hand side, we obtain

$$\begin{aligned}&\frac{\rho}{2}\max_{0\leq j\leq N}\left\|\varpi^j\right\|_{L_2(\Omega)}^2+\frac{\varphi_0}{2}\max_{0\leq j\leq N}\left\|\chi^j\right\|_{\mathcal{V}}^2+\frac{\kappa}{2}\sum_{q=1}^{N_\varphi}\frac{1}{\varphi_q}\left\|\Upsilon_q^m\right\|_{\mathcal{V}}^2\\
&+\Delta t\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi}\frac{\kappa}{4\tau_q\varphi_q}\left\|\Upsilon_q^{n+1}+\Upsilon_q^n\right\|_{\mathcal{V}}^2\\
&+\Delta t\sum_{n=0}^{m-1}(\frac{1}{4}-\frac{C\varphi_0}{\sqrt{\alpha_0}})J_0^{\alpha_0,\beta_0}(\varpi^{n+1}+\varpi^n,\varpi^{n+1}+\varpi^n)\\
\leq&3\Big(\frac{\rho}{12}\max_{0\leq j\leq N}\left\|\varpi^j\right\|_{L_2(\Omega)}^2+\frac{C\varphi_0(2T+1)}{\sqrt{\alpha_0}}\max_{0\leq j\leq N}\left\|\chi^j\right\|_{\mathcal{V}}^2+O(h^{2(\min(k+1,s)-1)}+\Delta t^4)\Big).\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\frac{\rho}{4}\max_{0\le j\le N}\left\|\varpi^j\right\|_{L_2(\Omega)}^2+\frac{\varphi_0}{2}\left(1-\frac{C(12T+6)}{\sqrt{\alpha_0}}\right)\max_{0\le j\le N}\left\|\chi^j\right\|_{\mathcal{V}}^2+\sum_{q=1}^{N_\varphi}\frac{\kappa}{2\varphi_q}\left\|\Upsilon_q^m\right\|_{\mathcal{V}}^2\\
&+\Delta t\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi}\frac{\kappa}{4\tau_q\varphi_q}\left\|\Upsilon_q^{n+1}+\Upsilon_q^n\right\|_{\mathcal{V}}^2\\
&+\Delta t\sum_{n=0}^{m-1}\left(\frac{1}{4}-\frac{C\varphi_0}{\sqrt{\alpha_0}}\right)J_0^{\alpha_0,\beta_0}(\varpi^{n+1}+\varpi^n,\varpi^{n+1}+\varpi^n)\\
\le&O(h^{2(\min(k+1,s)-1)}+\Delta t^4).
\end{aligned}$$

If we assume α_0 is large enough such that

$$1-\frac{C(12T+6)}{\sqrt{\alpha_0}}>0,\ \frac{1}{4}-\frac{C\varphi_0}{\sqrt{\alpha_0}}>0$$

then

$$\begin{aligned}
&\max_{0\le j\le N}\left\|\varpi^j\right\|_{L_2(\Omega)}^2+\max_{0\le j\le N}\left\|\chi^j\right\|_{\mathcal{V}}^2+\sum_{q=1}^{N_\varphi}\left\|\Upsilon_q^m\right\|_{\mathcal{V}}^2+\Delta t\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi}\left\|\Upsilon_q^{n+1}+\Upsilon_q^n\right\|_{\mathcal{V}}^2\\
&+\Delta t\sum_{n=0}^{m-1}J_0^{\alpha_0,\beta_0}(\varpi^{n+1}+\varpi^n,\varpi^{n+1}+\varpi^n)\\
\le&O(h^{\min(k+1,s)-1}+\Delta t^2)
\end{aligned}$$

and since m is arbitrary

$$\begin{aligned}
&\max_{0\le j\le N}\left\|\varpi^j\right\|_{L_2(\Omega)}^2+\max_{0\le j\le N}\left\|\chi^j\right\|_{\mathcal{V}}^2+\sum_{q=1}^{N_\varphi}\max_{0\le j\le N}\left\|\Upsilon_q^j\right\|_{\mathcal{V}}^2+\Delta t\sum_{n=0}^{N-1}\sum_{q=1}^{N_\varphi}\left\|\Upsilon_q^{n+1}+\Upsilon_q^n\right\|_{\mathcal{V}}^2\\
&+\Delta t\sum_{n=0}^{N-1}J_0^{\alpha_0,\beta_0}(\varpi^{n+1}+\varpi^n,\varpi^{n+1}+\varpi^n)\\
\le&O(h^{2(\min(k+1,s)-1)}+\Delta t^4).
\end{aligned}$$

In addition, whence Ω is convex and $\beta_0(d-1)\ge 3$, (3.2.17) could be applied thus we can derive higher order result as

$$\begin{aligned}
&\max_{0\le j\le N}\left\|\varpi^j\right\|_{L_2(\Omega)}^2+\max_{0\le j\le N}\left\|\chi^j\right\|_{\mathcal{V}}^2+\sum_{q=1}^{N_\varphi}\max_{0\le j\le N}\left\|\Upsilon_q^j\right\|_{\mathcal{V}}^2+\Delta t\sum_{n=0}^{N-1}\sum_{q=1}^{N_\varphi}\left\|\Upsilon_q^{n+1}+\Upsilon_q^n\right\|_{\mathcal{V}}^2\\
&+\Delta t\sum_{n=0}^{N-1}J_0^{\alpha_0,\beta_0}(\varpi^{n+1}+\varpi^n,\varpi^{n+1}+\varpi^n)\\
\le&O(h^{2\min(k+1,s)}+\Delta t^4).
\end{aligned}$$

□

As shown in the proof of Lemma 3.8, we followed very similar way for CGFEM but we used the DG bilinear form so we should deal with skew symmetric part $B(\cdot,\cdot)$. This term can be controlled by the penalty parameter α_0 so that large α_0 is required for our claim. Also, in the same manner in CGFEM, Lemma 3.8 will imply the following error bounds of the fully discrete solution for **(Q2)**.

Theorem 3.14. *Under the same condition on* Lemma 3.8*, we have*

$$\max_{0\leq j\leq N} \left\|u(t_j) - U_h^j\right\|_{\mathcal{V}} \leq C(h^{\min(k+1,s)-1} + \Delta t^2),$$
$$\max_{0\leq j\leq N} \left\|u(t_j) - U_h^j\right\|_{L_2(\Omega)} \leq C(h^{\min(k+1,s)-1} + \Delta t^2),$$
$$\max_{0\leq j\leq N} \left\|\dot{u}(t_j) - W_h^j\right\|_{L_2(\Omega)} \leq C(h^{\min(k+1,s)-1} + \Delta t^2),$$

and if Ω *is convex*

$$\max_{0\leq j\leq N} \left\|u(t_j) - U_h^j\right\|_{L_2(\Omega)} \leq C(h^{\min(k+1,s)} + \Delta t^2),$$
$$\max_{0\leq j\leq N} \left\|\dot{u}(t_j) - W_h^j\right\|_{L_2(\Omega)} \leq C(h^{\min(k+1,s)} + \Delta t^2).$$

Proof. It is easy to show our claim. The proof follows the exactly same way with the proof of Theorem 2.17 and Corollary 2.2 but in $\|\cdot\|_{\mathcal{V}}$ rather than $\|\cdot\|_V$ by using the result in Lemma 3.8 and Theorem 3.7. □

As seen in Theorem 3.14, the fully discrete solution for **(Q2)** has also same order accuracy as that of the displacement form **(Q1)**. Both formulations, the displacement form and the velocity form, require a large penalty term α_0 for the stability bounds and error bounds. From (3.3.6)-(3.3.8) and (3.3.26)-(3.3.28), we can construct computational forms with based on FEniCS and so we will implement numerical experiments regarding **(Q1)** and **(Q2)** in next section.

3.4 Numerical Experiments

In the same sense with CGFEM, we can construct numerical simulation codes in FEniCS with use of the fully discrete formulations for DGFEM. Here, we make some numerical experiments to verify error estimates as seen in Theorems 3.12 and 3.14. Our examples of exact solutions are sufficiently smooth in other words let us take $s = \infty$. Then as following error estimates theorems the order of error bounds depends only on a degree of polynomials k. First of all, we assume that $\rho = 1,\ D = 1,\ T = 1$ and $\Omega = [0,1]\times[0,1]$. We choose α_0 and β_0 to fulfil the conditions of stability bounds. Note that Ω is convex and so elliptic regularity has been equipped. That is, optimised L_2 estimates would be observed. Also, let us define

$$e_h^n := u(t_n) - U_h^n \text{ and } \tilde{e}_h^n := \dot{u}(t_n) - W_h^n$$

where $N \in \mathbb{N}$, $\Delta t = T/N$ and $t_n = n\Delta t$ for $n = 0, \dots, N$.

We want to show the exactness to verify our code implementation satisfying the error estimates theorems. Note that in order to see the exactness in time we should ignore internal variables since the internal variables are defined as integral forms with exponential functions in time.

Example 3.1.

Suppose $u = (1+t^2)xy$ and there is no internal variable, which leads us to solve a simple wave equation. From the exact solution u, we can obtain f and g_N, respectively then we have the following Table 3.1.

h	Δt	$\lVert e_h^N \rVert_{\mathcal{V}}$	$\lVert \tilde{e}_h^N \rVert_{L_2(\Omega)}$	$\lVert e_h^N \rVert_{L_2(\Omega)}$
1/2	1/2	4.3704×10^{-14}	1.1106×10^{-14}	3.3440×10^{-15}
1/4	1/4	1.0365×10^{-13}	2.4790×10^{-14}	1.6028×10^{-14}

Table 3.1: Errors for $u = xy$ on $k = 2$

As shown in Table 3.1, for coarse mesh sizes in time and space, the errors are sufficiently small about $10^{-13} \sim 10^{-15}$. These small values are kind of round-off errors and so they are negligible hence we can conclude that the numerical solutions are exact to the strong solutions. Therefore, we can say that our codes satisfy the exactness.

Example 3.2.

Let $u = e^{-t}xy$ with two internal variables where $\varphi_1 = 0.1$, $\varphi_2 = 0.4$, $\tau_1 = 0.5$ and $\tau_2 = 1.5$. With respect to the spatial domain Ω, for fixed time, u is a quadratic polynomial. Hence, as described in Tables 3.2, the errors are independent of a spatial mesh size h. To be specific, when we observe the errors column-wisely, the convergence is not shown. However, as Δt decreasing, the errors approach to zero. So, the error convergence rates show

$$\lVert e_h^N \rVert_{\mathcal{V}}, \lVert \tilde{e}_h^N \rVert_{L_2(\Omega)}, \lVert e_h^N \rVert_{L_2(\Omega)} = O(\Delta t^2).$$

$\lVert e_h^N \rVert_{\mathcal{V}}$

h \ Δt	1/4	1/8	1/16	1/32	1/64
1/4	2.6451×10^{-3}	6.8075×10^{-4}	1.7091×10^{-4}	4.2735×10^{-5}	1.0687×10^{-5}
1/8	2.6444×10^{-3}	6.7987×10^{-4}	1.7105×10^{-4}	4.2791×10^{-5}	1.0700×10^{-5}
1/16	2.6425×10^{-3}	6.7980×10^{-4}	1.7098×10^{-4}	4.2793×10^{-5}	1.0702×10^{-5}
1/32	2.6423×10^{-3}	6.7984×10^{-4}	1.7097×10^{-4}	4.2781×10^{-5}	1.0696×10^{-5}
1/64	2.6423×10^{-3}	6.7981×10^{-4}	1.7095×10^{-4}	4.2820×10^{-5}	1.0409×10^{-5}

$\|\tilde{e}_h^N\|_{L_2(\Omega)}$

h \ Δt	1/4	1/8	1/16	1/32	1/64
1/4	3.6021×10^{-3}	9.0223×10^{-4}	2.2576×10^{-4}	5.6484×10^{-5}	1.4123×10^{-5}
1/8	3.5969×10^{-3}	9.0127×10^{-4}	2.2548×10^{-4}	5.6386×10^{-5}	1.4099×10^{-5}
1/16	3.5967×10^{-3}	9.0121×10^{-4}	2.2547×10^{-4}	5.6381×10^{-5}	1.4096×10^{-5}
1/32	3.5967×10^{-3}	9.0121×10^{-4}	2.2546×10^{-4}	5.6378×10^{-5}	1.4093×10^{-5}
1/64	3.5967×10^{-3}	9.0121×10^{-4}	2.2546×10^{-4}	5.6385×10^{-5}	1.4097×10^{-5}

$\|e_h^N\|_{L_2(\Omega)}$

h \ Δt	1/4	1/8	1/16	1/32	1/64
1/4	1.0067×10^{-3}	2.6644×10^{-4}	6.7531×10^{-5}	1.6940×10^{-5}	4.2386×10^{-6}
1/8	1.0053×10^{-3}	2.6609×10^{-4}	6.7449×10^{-5}	1.6920×10^{-5}	4.2335×10^{-6}
1/16	1.0052×10^{-3}	2.6606×10^{-4}	6.7442×10^{-5}	1.6918×10^{-5}	4.2329×10^{-6}
1/32	1.0052×10^{-3}	2.6606×10^{-4}	6.7441×10^{-5}	1.6913×10^{-5}	4.2300×10^{-6}
1/64	1.0052×10^{-3}	2.6606×10^{-4}	6.7434×10^{-5}	1.6922×10^{-5}	4.2306×10^{-6}

Table 3.2: Errors for $u = e^{-t}xy$ on $k = 2$

Example 3.3.

If we set $u = t\sin(xy)$ without internal variables, u is a linear in time on each $(x, y) \in \Omega$. As following error estimates, the error convergence rates are given by

$$\|e_h\|_{\mathcal{V}} = O(h^k) \text{ and } \|\tilde{e}_h\|_{L_2(\Omega)}, \|e_h\|_{L_2(\Omega)} = O(h^{k+1})$$

since Ω is convex. It is also indicated that for $k = 1$

$$\|e_h^N\|_{\mathcal{V}} = O(h), \qquad \|\tilde{e}_h^N\|_{L_2(\Omega)}, \|e_h^N\|_{L_2(\Omega)} = O(h^2)$$

in Table 3.3. More precisely, when we consider the table row-wisely it is seen that the time step size Δt has no effect on the errors between the exact solution and our numerical solution.

$\|e_h^N\|_{\mathcal{V}}$

h \ Δt	1/4	1/8	1/16	1/32	1/64
1/4	1.1179×10^{-1}	1.1176×10^{-1}	1.1175×10^{-1}	1.1174×10^{-1}	1.1174×10^{-1}
1/8	5.9704×10^{-2}	5.9700×10^{-2}	5.9698×10^{-2}	5.9698×10^{-2}	5.9698×10^{-2}
1/16	3.0372×10^{-2}	3.0371×10^{-2}	3.0371×10^{-2}	3.0371×10^{-2}	3.0371×10^{-2}
1/32	1.5253×10^{-2}	1.5253×10^{-2}	1.5253×10^{-2}	1.5253×10^{-2}	1.5253×10^{-2}
1/64	7.6348×10^{-3}	7.6348×10^{-3}	7.6348×10^{-3}	7.6348×10^{-3}	7.6348×10^{-3}

$\|\tilde{e}_h^N\|_{L_2(\Omega)}$

h \ Δt	1/4	1/8	1/16	1/32	1/64
1/4	8.6095×10^{-3}	8.5555×10^{-3}	8.6335×10^{-3}	8.7479×10^{-3}	8.7848×10^{-3}
1/8	2.5642×10^{-3}	2.4664×10^{-3}	2.4078×10^{-3}	2.4107×10^{-3}	2.4311×10^{-3}
1/16	6.8260×10^{-4}	6.5654×10^{-4}	6.4071×10^{-4}	6.3238×10^{-4}	6.3120×10^{-4}
1/32	1.7364×10^{-4}	1.6761×10^{-4}	1.6362×10^{-4}	1.6172×10^{-4}	1.6074×10^{-4}
1/64	4.3610×10^{-5}	4.2179×10^{-5}	4.1142×10^{-5}	4.0757×10^{-5}	4.0534×10^{-5}

$\|e_h^N\|_{L_2(\Omega)}$

h \ Δt	1/4	1/8	1/16	1/32	1/64
1/4	5.9528×10^{-3}	5.9162×10^{-3}	5.8862×10^{-3}	5.8770×10^{-3}	5.8747×10^{-3}
1/8	1.7732×10^{-3}	1.7825×10^{-3}	1.7789×10^{-3}	1.7761×10^{-3}	1.7755×10^{-3}
1/16	4.6509×10^{-4}	4.6868×10^{-4}	4.6901×10^{-4}	4.6882×10^{-4}	4.6862×10^{-4}
1/32	1.1777×10^{-4}	1.1874×10^{-4}	1.1887×10^{-4}	1.1888×10^{-4}	1.1886×10^{-4}
1/64	2.9545×10^{-5}	2.9791×10^{-5}	2.9827×10^{-5}	2.9837×10^{-5}	2.9840×10^{-5}

Table 3.3: Errors $u = t\sin(xy)$ on $k = 1$

Example 3.4.

From Examples 3.1, 3.2 and 3.3, we can find the exactness of our code implements. To see more general cases, let us consider the following numerical experiment such that $u = e^{-t}\sin(xy)$ with two internal variables where $\varphi_1 = 0.1$, $\varphi_2 = 0.4$, $\tau_1 = 0.5$ and $\tau_2 = 1.5$. Then the internal variables and data terms can be computed.

Numerical errors for $u = e^{-t}\sin(xy)$ have been shown in the following tables which vary with the parameter β_0, the degree of polynomials k and the form of internal variables:

- Standard penalisation ($\beta_0 = 1$): Tables 3.4, 3.6, 3.8, 3.10
- Super-penalisation ($\beta_0 = 3$): Tables 3.5, 3.7, 3.9, 3.11
- Linear polynomial basis ($k = 1$): Tables 3.4, 3.5, 3.8, 3.9
- Quadratic polynomial basis ($k = 2$): Tables 3.6, 3.7, 3.10, 3.11
- Displacement form **(Q1)**: Tables 3.4, 3.5, 3.6, 3.7
- Velocity form **(Q2)**: Tables 3.8, 3.9, 3.10, 3.11

Due to Crank-Nicolson method for the time discretisation, it is observed that the fixed order of accuracy Δt^2 for any case. The convergence orders with respect to the spatial mesh h only depend on various settings such as β_0 and k. They thus show the evidence of our error estimates theorems. Regardless of the form of internal variables, the error analysis theorems provide the same convergence rate where condition parameters α_0 and

β_0 are sufficiently large. It is shown in Tables 3.4 - 3.11. The convergence rates for the displacement form and the velocity form are given by

$$\|e_h^N\|_{\mathcal{V}} = O(h + \Delta t^2), \qquad \|\tilde{e}_h^N\|_{L_2(\Omega)}, \|e_h^N\|_{L_2(\Omega)} = O(h^2 + \Delta t^2),$$

for $k = 1$, respectively. While degree of polynomials k increasing, the order accuracy in time does not change but that of the spatial mesh becomes higher. For example, when we set $k = 2$, we can see the following error estimates for both forms

$$\|e_h^N\|_{\mathcal{V}} = O(h^2 + \Delta t^2), \qquad \|\tilde{e}_h^N\|_{L_2(\Omega)}, \|e_h^N\|_{L_2(\Omega)} = O(h^3 + \Delta t^2),$$

in Tables 3.7 and 3.11. The exact errors have no big difference between **(Q1)** and **(Q2)** but convergence rates depend only on k under the certain conditions such as sufficiently large α_0 and $\beta_0(d-1) \geq 3$. Interestingly, as in Theorems 3.12, 3.14 the super-penalisation is an essential condition for optimal L_2 estimations but in practice Table 3.4 and 3.8 describe that the standard penalisation also provides the optimality for linear basis. L_2 optimality of the standard penalisation is observed only in odd k. Hence, we shall take account into benefits of the two penalisation in detail.

Remark We have two penalty parameters, α_0 and β_0. For a stability analysis and an error analysis, it is essential that α_0 is sufficiently large. On the other hand, we would take into account the standard penalisation $\beta_0(d-1) = 1$ and the super-penalisation $\beta_0(d-1) \geq 3$. The standard penalisation leads us to obtain the existence and uniqueness as well as suboptimal error estimates in L_2. If it is super-penalised, L_2 optimality will be given. To be specific, according to [24], super-penalisation is a necessary condition for optimal L_2 error estimate in elliptic problems with NIPG. In our error estimates theorems, we used these elliptic approximation estimates, therefore, we need to use super-penalised NIPG for the sake of elliptic regularity estimates.

$\|e_h^N\|_{\mathcal{V}}$

h \ Δt	1/2	1/4	1/8	1/16	1/32	1/64	1/128	1/256
1/2	1.1713×10^{-1}	1.1474×10^{-1}	1.1306×10^{-1}	1.1252×10^{-1}	1.1238×10^{-1}	1.1234×10^{-1}	1.1232×10^{-1}	1.1231×10^{-1}
1/4	5.9207×10^{-2}	5.7903×10^{-2}	5.7477×10^{-2}	5.7310×10^{-2}	5.7247×10^{-2}	5.7232×10^{-2}	5.7226×10^{-2}	5.7223×10^{-2}
1/8	2.9878×10^{-2}	2.8513×10^{-2}	2.8307×10^{-2}	2.8261×10^{-2}	2.8241×10^{-2}	2.8233×10^{-2}	2.8231×10^{-2}	2.8230×10^{-2}
1/16	1.6245×10^{-2}	1.4236×10^{-2}	1.3999×10^{-2}	1.3972×10^{-2}	1.3967×10^{-2}	1.3965×10^{-2}	1.3964×10^{-2}	1.3964×10^{-2}
1/32	1.0540×10^{-2}	7.3568×10^{-3}	6.9752×10^{-3}	6.9435×10^{-3}	6.9401×10^{-3}	6.9394×10^{-3}	6.9391×10^{-3}	6.9390×10^{-3}
1/64	8.5710×10^{-3}	4.1871×10^{-3}	3.5159×10^{-3}	3.4631×10^{-3}	3.4590×10^{-3}	3.4586×10^{-3}	3.4585×10^{-3}	3.4585×10^{-3}
1/128	8.0082×10^{-3}	2.9076×10^{-3}	1.8322×10^{-3}	1.7338×10^{-3}	1.7270×10^{-3}	1.7265×10^{-3}	1.7264×10^{-3}	1.7264×10^{-3}
1/256	7.8618×10^{-3}	2.4888×10^{-3}	1.0556×10^{-3}	8.7629×10^{-4}	8.6344×10^{-4}	8.6259×10^{-4}	8.6252×10^{-4}	8.6252×10^{-4}

$\|\tilde{e}_h^N\|_{L_2(\Omega)}$

h \ Δt	1/2	1/4	1/8	1/16	1/32	1/64	1/128	1/256
1/2	3.0251×10^{-2}	2.0250×10^{-2}	1.7577×10^{-2}	1.6803×10^{-2}	1.6640×10^{-2}	1.6598×10^{-2}	1.6588×10^{-2}	1.6586×10^{-2}
1/4	1.7876×10^{-2}	8.2162×10^{-3}	6.0155×10^{-3}	5.3461×10^{-3}	5.1737×10^{-3}	5.1459×10^{-3}	5.1387×10^{-3}	5.1372×10^{-3}
1/8	1.4483×10^{-2}	4.5881×10^{-3}	2.1824×10^{-3}	1.5944×10^{-3}	1.4121×10^{-3}	1.3622×10^{-3}	1.3553×10^{-3}	1.3536×10^{-3}
1/16	1.3626×10^{-2}	3.6952×10^{-3}	1.1739×10^{-3}	5.5720×10^{-4}	4.0750×10^{-4}	3.6043×10^{-4}	3.4753×10^{-4}	3.4569×10^{-4}
1/32	1.3409×10^{-2}	3.4770×10^{-3}	9.3066×10^{-4}	2.9518×10^{-4}	1.4035×10^{-4}	1.0261×10^{-4}	9.0829×10^{-5}	8.7500×10^{-5}
1/64	1.3355×10^{-2}	3.4225×10^{-3}	8.7201×10^{-4}	2.3301×10^{-4}	7.3953×10^{-5}	3.5180×10^{-5}	2.5721×10^{-5}	2.2753×10^{-5}
1/128	1.3341×10^{-2}	3.4089×10^{-3}	8.5753×10^{-4}	2.1815×10^{-4}	5.8293×10^{-5}	1.8501×10^{-5}	8.8024×10^{-6}	6.4350×10^{-6}
1/256	1.3338×10^{-2}	3.4055×10^{-3}	8.5391×10^{-4}	2.1449×10^{-4}	5.4551×10^{-5}	1.4576×10^{-5}	4.6265×10^{-6}	2.1992×10^{-6}

$\|e_h^N\|_{L_2(\Omega)}$

h \ Δt	1/2	1/4	1/8	1/16	1/32	1/64	1/128	1/256
1/2	7.6041×10^{-3}	7.7669×10^{-3}	7.6371×10^{-3}	7.6095×10^{-3}	7.6027×10^{-3}	7.5996×10^{-3}	7.5981×10^{-3}	7.5974×10^{-3}
1/4	2.8425×10^{-3}	2.4123×10^{-3}	2.4522×10^{-3}	2.4857×10^{-3}	2.4950×10^{-3}	2.4969×10^{-3}	2.4971×10^{-3}	2.4969×10^{-3}
1/8	2.6171×10^{-3}	9.1137×10^{-4}	6.5459×10^{-4}	6.8995×10^{-4}	7.0528×10^{-4}	7.0941×10^{-4}	7.1036×10^{-4}	7.1055×10^{-4}
1/16	2.8088×10^{-3}	8.7184×10^{-4}	2.3712×10^{-4}	1.6952×10^{-4}	1.8079×10^{-4}	1.8528×10^{-4}	1.8648×10^{-4}	1.8677×10^{-4}
1/32	2.8741×10^{-3}	9.1747×10^{-4}	2.3181×10^{-4}	5.9841×10^{-5}	4.3025×10^{-5}	4.6029×10^{-5}	4.7208×10^{-5}	4.7523×10^{-5}
1/64	2.8916×10^{-3}	9.3228×10^{-4}	2.4393×10^{-4}	5.8807×10^{-5}	1.5005×10^{-5}	1.0829×10^{-5}	1.1595×10^{-5}	1.1894×10^{-5}
1/128	2.8961×10^{-3}	9.3620×10^{-4}	2.4771×10^{-4}	6.1902×10^{-5}	1.4754×10^{-5}	3.7557×10^{-6}	2.7159×10^{-6}	2.9088×10^{-6}
1/256	2.8972×10^{-3}	9.3720×10^{-4}	2.4871×10^{-4}	6.2857×10^{-5}	1.5533×10^{-5}	3.6915×10^{-6}	9.3967×10^{-7}	6.8069×10^{-7}

Table 3.4: Errors of **(Q1)** for linear polynomial basis; Example 3.4 where $\alpha_0 = 10, \beta_0 = 1$

$\|e_h^N\|_{\mathcal{V}}$

h \ Δt	1/2	1/4	1/8	1/16	1/32	1/64	1/128	1/256
1/2	1.0898×10^{-1}	1.0729×10^{-1}	1.0601×10^{-1}	1.0552×10^{-1}	1.0539×10^{-1}	1.0536×10^{-1}	1.0534×10^{-1}	1.0533×10^{-1}
1/4	4.8538×10^{-2}	4.7914×10^{-2}	4.7746×10^{-2}	4.7693×10^{-2}	4.7673×10^{-2}	4.7663×10^{-2}	4.7660×10^{-2}	4.7659×10^{-2}
1/8	2.4052×10^{-2}	2.2924×10^{-2}	2.2809×10^{-2}	2.2799×10^{-2}	2.2797×10^{-2}	2.2797×10^{-2}	2.2797×10^{-2}	2.2796×10^{-2}
1/16	1.3682×10^{-2}	1.1510×10^{-2}	1.1290×10^{-2}	1.1275×10^{-2}	1.1274×10^{-2}	1.1274×10^{-2}	1.1274×10^{-2}	1.1274×10^{-2}
1/32	9.6127×10^{-3}	6.0860×10^{-3}	5.6553×10^{-3}	5.6254×10^{-3}	5.6236×10^{-3}	5.6235×10^{-3}	5.6235×10^{-3}	5.6235×10^{-3}
1/64	8.2987×10^{-3}	3.6514×10^{-3}	2.8746×10^{-3}	2.8143×10^{-3}	2.8105×10^{-3}	2.8102×10^{-3}	2.8102×10^{-3}	2.8102×10^{-3}

$\|\tilde{e}_h^N\|_{L_2(\Omega)}$

h \ Δt	1/2	1/4	1/8	1/16	1/32	1/64	1/128	1/256
1/2	3.2167×10^{-2}	2.2092×10^{-2}	1.9363×10^{-2}	1.8661×10^{-2}	1.8546×10^{-2}	1.8509×10^{-2}	1.8500×10^{-2}	1.8498×10^{-2}
1/4	1.8968×10^{-2}	8.9226×10^{-3}	6.5453×10^{-3}	5.9124×10^{-3}	5.6356×10^{-3}	5.5971×10^{-3}	5.6239×10^{-3}	5.6225×10^{-3}
1/8	1.4765×10^{-2}	4.8213×10^{-3}	2.3224×10^{-3}	1.6953×10^{-3}	1.5443×10^{-3}	1.4877×10^{-3}	1.4509×10^{-3}	1.4451×10^{-3}
1/16	1.3692×10^{-2}	3.7542×10^{-3}	1.2162×10^{-3}	5.8429×10^{-4}	4.2848×10^{-4}	3.9042×10^{-4}	3.7989×10^{-4}	3.7392×10^{-4}
1/32	1.3425×10^{-2}	3.4912×10^{-3}	9.4210×10^{-4}	3.0409×10^{-4}	1.4609×10^{-4}	1.0734×10^{-4}	9.7994×10^{-5}	9.5601×10^{-5}
1/64	1.3359×10^{-2}	3.4260×10^{-3}	8.7489×10^{-4}	2.3556×10^{-4}	7.5981×10^{-5}	3.6356×10^{-5}	2.6691×10^{-5}	2.4371×10^{-5}

$\|e_h^N\|_{L_2(\Omega)}$

h \ Δt	1/2	1/4	1/8	1/16	1/32	1/64	1/128	1/256
1/2	6.9346×10^{-3}	7.6602×10^{-3}	7.9200×10^{-3}	8.0123×10^{-3}	8.0372×10^{-3}	8.0432×10^{-3}	8.0444×10^{-3}	8.0446×10^{-3}
1/4	1.2587×10^{-3}	2.2791×10^{-3}	2.8254×10^{-3}	2.9858×10^{-3}	3.0276×10^{-3}	3.0381×10^{-3}	3.0407×10^{-3}	3.0414×10^{-3}
1/8	2.0565×10^{-3}	3.8793×10^{-4}	6.4345×10^{-4}	8.0387×10^{-4}	8.4702×10^{-4}	8.5786×10^{-4}	8.6057×10^{-4}	8.6125×10^{-4}
1/16	2.6738×10^{-3}	7.3670×10^{-4}	1.0347×10^{-4}	1.6729×10^{-4}	2.0828×10^{-4}	2.1904×10^{-4}	2.2175×10^{-4}	2.2243×10^{-4}
1/32	2.8408×10^{-3}	8.8562×10^{-4}	1.9914×10^{-4}	2.6434×10^{-5}	4.2293×10^{-5}	5.2597×10^{-5}	5.5271×10^{-5}	5.5947×10^{-5}
1/64	2.8833×10^{-3}	9.2445×10^{-4}	2.3625×10^{-4}	5.0709×10^{-5}	6.6418×10^{-6}	1.0753×10^{-5}	1.3382×10^{-5}	1.4017×10^{-5}

Table 3.5: Errors of **(Q1)** for linear polynomial basis; Example 3.4 where $\alpha_0 = 10, \beta_0 = 3$

$\|e_h^N\|_{\mathcal{V}}$

h \ Δt	1/2	1/4	1/8	1/16	1/32	1/64	1/128	1/256
1/2	1.1537×10^{-2}	9.3151×10^{-3}	9.5031×10^{-3}	9.5653×10^{-3}	9.5955×10^{-3}	9.6001×10^{-3}	9.6008×10^{-3}	9.6010×10^{-3}
1/4	7.6279×10^{-3}	3.1716×10^{-3}	2.7503×10^{-3}	2.8078×10^{-3}	2.8353×10^{-3}	2.8406×10^{-3}	2.8419×10^{-3}	2.8422×10^{-3}
1/8	7.6320×10^{-3}	2.2528×10^{-3}	8.5506×10^{-4}	7.6106×10^{-4}	7.7841×10^{-4}	7.8541×10^{-4}	7.8693×10^{-4}	7.8749×10^{-4}
1/16	7.7559×10^{-3}	2.2850×10^{-3}	5.8643×10^{-4}	2.2280×10^{-4}	2.0102×10^{-4}	2.0546×10^{-4}	2.0719×10^{-4}	2.0767×10^{-4}
1/32	7.7974×10^{-3}	2.3189×10^{-3}	5.9498×10^{-4}	1.4814×10^{-4}	5.6842×10^{-5}	5.1665×10^{-5}	5.2796×10^{-5}	5.3228×10^{-5}
1/64	7.8087×10^{-3}	2.3292×10^{-3}	6.0357×10^{-4}	1.5013×10^{-4}	3.7141×10^{-5}	1.4357×10^{-5}	1.3097×10^{-5}	1.3381×10^{-5}

$\|\tilde{e}_h^N\|_{L_2(\Omega)}$

h \ Δt	1/2	1/4	1/8	1/16	1/32	1/64	1/128	1/256
1/2	1.3916×10^{-2}	4.1086×10^{-3}	1.5857×10^{-3}	1.0351×10^{-3}	8.7919×10^{-4}	8.5231×10^{-4}	8.4630×10^{-4}	8.4488×10^{-4}
1/4	1.3412×10^{-2}	3.4905×10^{-3}	9.4733×10^{-4}	3.2155×10^{-4}	1.8270×10^{-4}	1.5516×10^{-4}	1.4956×10^{-4}	1.4818×10^{-4}
1/8	1.3350×10^{-2}	3.4188×10^{-3}	8.6781×10^{-4}	2.3045×10^{-4}	7.6050×10^{-5}	4.4247×10^{-5}	3.8907×10^{-5}	3.7790×10^{-5}
1/16	1.3340×10^{-2}	3.4074×10^{-3}	8.5579×10^{-4}	2.1647×10^{-4}	5.7080×10^{-5}	1.8735×10^{-5}	1.1152×10^{-5}	9.9576×10^{-6}
1/32	1.3337×10^{-2}	3.4051×10^{-3}	8.5342×10^{-4}	2.1398×10^{-4}	5.4072×10^{-5}	1.4223×10^{-5}	4.6612×10^{-6}	2.7998×10^{-6}
1/64	1.3337×10^{-2}	3.4045×10^{-3}	8.5288×10^{-4}	2.1344×10^{-4}	5.3497×10^{-5}	1.3513×10^{-5}	3.5509×10^{-6}	1.1630×10^{-6}

$\|e_h^N\|_{L_2(\Omega)}$

h \ Δt	1/2	1/4	1/8	1/16	1/32	1/64	1/128	1/256
1/2	1.3250×10^{-3}	9.6746×10^{-4}	1.5999×10^{-3}	1.7756×10^{-3}	1.8210×10^{-3}	1.8323×10^{-3}	1.8351×10^{-3}	1.8357×10^{-3}
1/4	2.4414×10^{-3}	4.8102×10^{-4}	2.2573×10^{-4}	4.0590×10^{-4}	4.5251×10^{-4}	4.6420×10^{-4}	4.6712×10^{-4}	4.6785×10^{-4}
1/8	2.7813×10^{-3}	8.2074×10^{-4}	1.3371×10^{-4}	5.7283×10^{-5}	1.0292×10^{-4}	1.1460×10^{-4}	1.1752×10^{-4}	1.1826×10^{-4}
1/16	2.8684×10^{-3}	9.0817×10^{-4}	2.1980×10^{-4}	3.4350×10^{-5}	1.4520×10^{-5}	2.5910×10^{-5}	2.8823×10^{-5}	2.9554×10^{-5}
1/32	2.8903×10^{-3}	9.3018×10^{-4}	2.4171×10^{-4}	5.5880×10^{-5}	8.6472×10^{-6}	3.6529×10^{-6}	6.4957×10^{-6}	7.2233×10^{-6}
1/64	2.8958×10^{-3}	9.3569×10^{-4}	2.4721×10^{-4}	6.1359×10^{-5}	1.4028×10^{-5}	2.1656×10^{-6}	9.1573×10^{-7}	1.6258×10^{-6}

Table 3.6: Errors of **(Q1)** for quadratic polynomial basis; Example 3.4 where $\alpha_0 = 10, \beta_0 = 1$

$\|e_h^N\|_{\mathcal{V}}$

h \ Δt	1/2	1/4	1/8	1/16	1/32	1/64	1/128	1/256
1/2	1.1488×10^{-2}	9.0056×10^{-3}	9.0435×10^{-3}	9.0725×10^{-3}	9.0940×10^{-3}	9.0967×10^{-3}	9.0970×10^{-3}	9.0970×10^{-3}
1/4	8.0441×10^{-3}	3.2534×10^{-3}	2.4698×10^{-3}	2.4308×10^{-3}	2.4346×10^{-3}	2.4353×10^{-3}	2.4354×10^{-3}	2.4355×10^{-3}
1/8	7.8243×10^{-3}	2.4024×10^{-3}	8.5827×10^{-4}	6.3589×10^{-4}	6.2090×10^{-4}	6.2042×10^{-4}	6.2048×10^{-4}	6.2052×10^{-4}
1/16	7.8132×10^{-3}	2.3374×10^{-3}	6.2612×10^{-4}	2.1868×10^{-4}	1.6138×10^{-4}	1.5722×10^{-4}	1.5699×10^{-4}	1.5698×10^{-4}
1/32	7.8126×10^{-3}	2.3332×10^{-3}	6.0830×10^{-4}	1.5815×10^{-4}	5.5116×10^{-5}	4.0749×10^{-5}	3.9685×10^{-5}	3.9619×10^{-5}

$\|\tilde{e}_h^N\|_{L_2(\Omega)}$

h \ Δt	1/2	1/4	1/8	1/16	1/32	1/64	1/128	1/256
1/2	1.3733×10^{-2}	3.8846×10^{-3}	1.3658×10^{-3}	8.3552×10^{-4}	6.8306×10^{-4}	6.6706×10^{-4}	6.6364×10^{-4}	6.6288×10^{-4}
1/4	1.3356×10^{-2}	3.4242×10^{-3}	8.7622×10^{-4}	2.4847×10^{-4}	1.0043×10^{-4}	7.2509×10^{-5}	7.3936×10^{-5}	7.4375×10^{-5}
1/8	1.3338×10^{-2}	3.4052×10^{-3}	8.5370×10^{-4}	2.1464×10^{-4}	5.5927×10^{-5}	1.8351×10^{-5}	9.5664×10^{-6}	9.6140×10^{-6}
1/16	1.3337×10^{-2}	3.4044×10^{-3}	8.5277×10^{-4}	2.1333×10^{-4}	5.3410×10^{-5}	1.3509×10^{-5}	3.8040×10^{-6}	1.7037×10^{-6}
1/32	1.3337×10^{-2}	3.4043×10^{-3}	8.5271×10^{-4}	2.1327×10^{-4}	5.3332×10^{-5}	1.3338×10^{-5}	3.3487×10^{-6}	8.6729×10^{-7}

$\|e_h^N\|_{L_2(\Omega)}$

h \ Δt	1/2	1/4	1/8	1/16	1/32	1/64	1/128	1/256
1/2	1.6765×10^{-3}	5.7620×10^{-4}	1.1570×10^{-3}	1.3287×10^{-3}	1.3738×10^{-3}	1.3850×10^{-3}	1.3878×10^{-3}	1.3884×10^{-3}
1/4	2.7693×10^{-3}	8.1112×10^{-4}	1.3364×10^{-4}	8.6344×10^{-5}	1.2570×10^{-4}	1.3643×10^{-4}	1.3916×10^{-4}	1.3985×10^{-4}
1/8	2.8884×10^{-3}	9.2850×10^{-4}	2.4018×10^{-4}	5.4895×10^{-5}	1.1183×10^{-5}	1.0180×10^{-5}	1.1933×10^{-5}	1.2444×10^{-5}
1/16	2.8970×10^{-3}	9.3695×10^{-4}	2.4846×10^{-4}	6.2624×10^{-5}	1.5323×10^{-5}	3.5899×10^{-6}	1.2551×10^{-6}	1.2302×10^{-6}
1/32	2.8975×10^{-3}	9.3749×10^{-4}	2.4900×10^{-4}	6.3156×10^{-5}	1.5824×10^{-5}	3.9363×10^{-6}	9.7334×10^{-7}	2.6049×10^{-7}

Table 3.7: Errors of **(Q1)** for quadratic polynomial basis; Example 3.4 where $\alpha_0 = 10, \beta_0 = 3$

$\|e_h^N\|_{\mathcal{V}}$

h \ Δt	1/2	1/4	1/8	1/16	1/32	1/64	1/128	1/256
1/2	1.1623×10^{-1}	1.1461×10^{-1}	1.1335×10^{-1}	1.1297×10^{-1}	1.1291×10^{-1}	1.1290×10^{-1}	1.1289×10^{-1}	1.1289×10^{-1}
1/4	5.8410×10^{-2}	5.7675×10^{-2}	5.7407×10^{-2}	5.7296×10^{-2}	5.7255×10^{-2}	5.7248×10^{-2}	5.7247×10^{-2}	5.7247×10^{-2}
1/8	2.9007×10^{-2}	2.8391×10^{-2}	2.8282×10^{-2}	2.8254×10^{-2}	2.8242×10^{-2}	2.8237×10^{-2}	2.8236×10^{-2}	2.8236×10^{-2}
1/16	1.4849×10^{-2}	1.4090×10^{-2}	1.3986×10^{-2}	1.3972×10^{-2}	1.3969×10^{-2}	1.3967×10^{-2}	1.3967×10^{-2}	1.3967×10^{-2}
1/32	8.3170×10^{-3}	7.1043×10^{-3}	6.9564×10^{-3}	6.9425×10^{-3}	6.9408×10^{-3}	6.9404×10^{-3}	6.9402×10^{-3}	6.9402×10^{-3}
1/64	5.6469×10^{-3}	3.7392×10^{-3}	3.4813×10^{-3}	3.4609×10^{-3}	3.4591×10^{-3}	3.4589×10^{-3}	3.4588×10^{-3}	3.4588×10^{-3}
1/128	4.7578×10^{-3}	2.2200×10^{-3}	1.7662×10^{-3}	1.7294×10^{-3}	1.7268×10^{-3}	1.7266×10^{-3}	1.7265×10^{-3}	1.7265×10^{-3}
1/256	4.5092×10^{-3}	1.6352×10^{-3}	9.3718×10^{-4}	8.6766×10^{-4}	8.6290×10^{-4}	8.6257×10^{-4}	8.6254×10^{-4}	8.6254×10^{-4}

$\|\tilde{e}_h^N\|_{L_2(\Omega)}$

h \ Δt	1/2	1/4	1/8	1/16	1/32	1/64	1/128	1/256
1/2	2.9173×10^{-2}	2.0582×10^{-2}	1.8183×10^{-2}	1.7538×10^{-2}	1.7403×10^{-2}	1.7366×10^{-2}	1.7356×10^{-2}	1.7354×10^{-2}
1/4	1.5567×10^{-2}	7.7461×10^{-3}	5.8329×10^{-3}	5.2038×10^{-3}	5.0444×10^{-3}	5.0189×10^{-3}	5.0118×10^{-3}	5.0100×10^{-3}
1/8	1.1904×10^{-2}	3.9657×10^{-3}	2.0112×10^{-3}	1.5112×10^{-3}	1.3427×10^{-3}	1.2963×10^{-3}	1.2901×10^{-3}	1.2884×10^{-3}
1/16	1.1000×10^{-2}	3.0369×10^{-3}	1.0070×10^{-3}	5.0815×10^{-4}	3.8254×10^{-4}	3.3962×10^{-4}	3.2759×10^{-4}	3.2600×10^{-4}
1/32	1.0775×10^{-2}	2.8113×10^{-3}	7.6397×10^{-4}	2.5249×10^{-4}	1.2740×10^{-4}	9.5957×10^{-5}	8.5305×10^{-5}	8.2221×10^{-5}
1/64	1.0718×10^{-2}	2.7553×10^{-3}	7.0535×10^{-4}	1.9119×10^{-4}	6.3175×10^{-5}	3.1869×10^{-5}	2.4019×10^{-5}	2.1349×10^{-5}
1/128	1.0704×10^{-2}	2.7414×10^{-3}	6.9089×10^{-4}	1.7646×10^{-4}	4.7817×10^{-5}	1.5795×10^{-5}	7.9668×10^{-6}	6.0053×10^{-6}
1/256	1.0701×10^{-2}	2.7379×10^{-3}	6.8728×10^{-4}	1.7283×10^{-4}	4.4126×10^{-5}	1.1955×10^{-5}	3.9484×10^{-6}	1.9886×10^{-6}

$\|e_h^N\|_{L_2(\Omega)}$

h \ Δt	1/2	1/4	1/8	1/16	1/32	1/64	1/128	1/256
1/2	7.5253×10^{-3}	7.3057×10^{-3}	7.0495×10^{-3}	6.9860×10^{-3}	6.9724×10^{-3}	6.9691×10^{-3}	6.9683×10^{-3}	6.9681×10^{-3}
1/4	2.2590×10^{-3}	2.2604×10^{-3}	2.2732×10^{-3}	2.2828×10^{-3}	2.2861×10^{-3}	2.2872×10^{-3}	2.2874×10^{-3}	2.2875×10^{-3}
1/8	1.3695×10^{-3}	6.6056×10^{-4}	6.1039×10^{-4}	6.3809×10^{-4}	6.4729×10^{-4}	6.4982×10^{-4}	6.5047×10^{-4}	6.5064×10^{-4}
1/16	1.4808×10^{-3}	4.9589×10^{-4}	1.6937×10^{-4}	1.5846×10^{-4}	1.6753×10^{-4}	1.7041×10^{-4}	1.7116×10^{-4}	1.7136×10^{-4}
1/32	1.5374×10^{-3}	5.2925×10^{-4}	1.3362×10^{-4}	4.2661×10^{-5}	4.0318×10^{-5}	4.2739×10^{-5}	4.3502×10^{-5}	4.3702×10^{-5}
1/64	1.5533×10^{-3}	5.4266×10^{-4}	1.4307×10^{-4}	3.3980×10^{-5}	1.0704×10^{-5}	1.0164×10^{-5}	1.0781×10^{-5}	1.0975×10^{-5}
1/128	1.5574×10^{-3}	5.4632×10^{-4}	1.4655×10^{-4}	3.6443×10^{-5}	8.5284×10^{-6}	2.6811×10^{-6}	2.5514×10^{-6}	2.7073×10^{-6}
1/256	1.5585×10^{-3}	5.4726×10^{-4}	1.4748×10^{-4}	3.7326×10^{-5}	9.1521×10^{-6}	2.1335×10^{-6}	6.7090×10^{-7}	6.4040×10^{-7}

Table 3.8: Errors of **(Q2)** for linear polynomial basis; Example 3.4 where $\alpha_0 = 10, \beta_0 = 1$

$\left\|e_h^N\right\|_{\mathcal{V}}$

h \ Δt	1/2	1/4	1/8	1/16	1/32	1/64	1/128	1/256
1/2	1.0865×10^{-1}	1.0755×10^{-1}	1.0660×10^{-1}	1.0624×10^{-1}	1.0617×10^{-1}	1.0616×10^{-1}	1.0616×10^{-1}	1.0616×10^{-1}
1/4	4.8157×10^{-2}	4.7922×10^{-2}	4.7815×10^{-2}	4.7778×10^{-2}	4.7764×10^{-2}	4.7757×10^{-2}	4.7755×10^{-2}	4.7755×10^{-2}
1/8	2.3219×10^{-2}	2.2858×10^{-2}	2.2812×10^{-2}	2.2807×10^{-2}	2.2805×10^{-2}	2.2805×10^{-2}	2.2805×10^{-2}	2.2804×10^{-2}
1/16	1.2096×10^{-2}	1.1360×10^{-2}	1.1280×10^{-2}	1.1275×10^{-2}	1.1275×10^{-2}	1.1275×10^{-2}	1.1275×10^{-2}	1.1275×10^{-2}
1/32	7.1473×10^{-3}	5.7916×10^{-3}	5.6350×10^{-3}	5.6242×10^{-3}	5.6236×10^{-3}	5.6235×10^{-3}	5.6235×10^{-3}	5.6235×10^{-3}
1/64	5.2380×10^{-3}	3.1336×10^{-3}	2.8336×10^{-3}	2.8117×10^{-3}	2.8103×10^{-3}	2.8102×10^{-3}	2.8102×10^{-3}	2.8102×10^{-3}

$\left\|\tilde{e}_h^N\right\|_{L_2(\Omega)}$

h \ Δt	1/2	1/4	1/8	1/16	1/32	1/64	1/128	1/256
1/2	3.0732×10^{-2}	2.2320×10^{-2}	1.9956×10^{-2}	1.9399×10^{-2}	1.9313×10^{-2}	1.9283×10^{-2}	1.9274×10^{-2}	1.9272×10^{-2}
1/4	1.6488×10^{-2}	8.3678×10^{-3}	6.4446×10^{-3}	5.9173×10^{-3}	5.6702×10^{-3}	5.6397×10^{-3}	5.6679×10^{-3}	5.6668×10^{-3}
1/8	1.2148×10^{-2}	4.1743×10^{-3}	2.1672×10^{-3}	1.6595×10^{-3}	1.5374×10^{-3}	1.4881×10^{-3}	1.4532×10^{-3}	1.4478×10^{-3}
1/16	1.1058×10^{-2}	3.0904×10^{-3}	1.0525×10^{-3}	5.4459×10^{-4}	4.1891×10^{-4}	3.8816×10^{-4}	3.7944×10^{-4}	3.7392×10^{-4}
1/32	1.0789×10^{-2}	2.8243×10^{-3}	7.7597×10^{-4}	2.6301×10^{-4}	1.3610×10^{-4}	1.0492×10^{-4}	9.7421×10^{-5}	9.5547×10^{-5}
1/64	1.0722×10^{-2}	2.7585×10^{-3}	7.0830×10^{-4}	1.9396×10^{-4}	6.5651×10^{-5}	3.3679×10^{-5}	2.5815×10^{-5}	2.4157×10^{-5}

$\left\|e_h^N\right\|_{L_2(\Omega)}$

h \ Δt	1/2	1/4	1/8	1/16	1/32	1/64	1/128	1/256
1/2	7.4176×10^{-3}	7.4240×10^{-3}	7.4678×10^{-3}	7.5018×10^{-3}	7.5127×10^{-3}	7.5156×10^{-3}	7.5163×10^{-3}	7.5165×10^{-3}
1/4	1.9401×10^{-3}	2.5797×10^{-3}	2.8873×10^{-3}	2.9809×10^{-3}	3.0057×10^{-3}	3.0120×10^{-3}	3.0136×10^{-3}	3.0140×10^{-3}
1/8	7.7685×10^{-4}	4.4274×10^{-4}	7.2967×10^{-4}	8.2645×10^{-4}	8.5248×10^{-4}	8.5900×10^{-4}	8.6063×10^{-4}	8.6104×10^{-4}
1/16	1.3372×10^{-3}	3.5390×10^{-4}	1.0799×10^{-4}	1.8907×10^{-4}	2.1405×10^{-4}	2.2053×10^{-4}	2.2216×10^{-4}	2.2256×10^{-4}
1/32	1.5023×10^{-3}	4.9597×10^{-4}	9.9252×10^{-5}	2.6964×10^{-5}	4.7742×10^{-5}	5.4035×10^{-5}	5.5612×10^{-5}	5.5948×10^{-5}
1/64	1.5446×10^{-3}	5.3450×10^{-4}	1.3500×10^{-4}	2.5424×10^{-5}	6.7725×10^{-6}	1.2327×10^{-5}	1.4137×10^{-5}	1.4219×10^{-5}

Table 3.9: Errors of **(Q2)** for linear polynomial basis; Example 3.4 where $\alpha_0 = 10, \beta_0 = 3$

$\|e_h^N\|_{\mathcal{V}}$

h \ Δt	1/2	1/4	1/8	1/16	1/32	1/64	1/128	1/256
1/2	1.1623×10^{-1}	1.1461×10^{-1}	1.1335×10^{-1}	1.1297×10^{-1}	1.1291×10^{-1}	1.1290×10^{-1}	1.1289×10^{-1}	1.1289×10^{-1}
1/4	5.8410×10^{-2}	5.7675×10^{-2}	5.7407×10^{-2}	5.7296×10^{-2}	5.7255×10^{-2}	5.7248×10^{-2}	5.7247×10^{-2}	5.7247×10^{-2}
1/8	2.9007×10^{-2}	2.8391×10^{-2}	2.8282×10^{-2}	2.8254×10^{-2}	2.8242×10^{-2}	2.8237×10^{-2}	2.8236×10^{-2}	2.8236×10^{-2}
1/16	1.4849×10^{-2}	1.4090×10^{-2}	1.3986×10^{-2}	1.3972×10^{-2}	1.3969×10^{-2}	1.3967×10^{-2}	1.3967×10^{-2}	1.3967×10^{-2}
1/32	8.3170×10^{-3}	7.1043×10^{-3}	6.9564×10^{-3}	6.9425×10^{-3}	6.9408×10^{-3}	6.9404×10^{-3}	6.9402×10^{-3}	6.9402×10^{-3}
1/64	5.6469×10^{-3}	3.7392×10^{-3}	3.4813×10^{-3}	3.4609×10^{-3}	3.4591×10^{-3}	3.4589×10^{-3}	3.4588×10^{-3}	3.4588×10^{-3}
1/128	4.7578×10^{-3}	2.2200×10^{-3}	1.7662×10^{-3}	1.7294×10^{-3}	1.7268×10^{-3}	1.7266×10^{-3}	1.7265×10^{-3}	1.7265×10^{-3}
1/256	4.5092×10^{-3}	1.6352×10^{-3}	9.3718×10^{-4}	8.6766×10^{-4}	8.6290×10^{-4}	8.6257×10^{-4}	8.6254×10^{-4}	8.6254×10^{-4}

$\|\tilde{e}_h^N\|_{L_2(\Omega)}$

h \ Δt	1/2	1/4	1/8	1/16	1/32	1/64	1/128	1/256
1/2	2.9173×10^{-2}	2.0582×10^{-2}	1.8183×10^{-2}	1.7538×10^{-2}	1.7403×10^{-2}	1.7366×10^{-2}	1.7356×10^{-2}	1.7354×10^{-2}
1/4	1.5567×10^{-2}	7.7461×10^{-3}	5.8329×10^{-3}	5.2038×10^{-3}	5.0444×10^{-3}	5.0189×10^{-3}	5.0118×10^{-3}	5.0100×10^{-3}
1/8	1.1904×10^{-2}	3.9657×10^{-3}	2.0112×10^{-3}	1.5112×10^{-3}	1.3427×10^{-3}	1.2963×10^{-3}	1.2901×10^{-3}	1.2884×10^{-3}
1/16	1.1000×10^{-2}	3.0369×10^{-3}	1.0070×10^{-3}	5.0815×10^{-4}	3.8254×10^{-4}	3.3962×10^{-4}	3.2759×10^{-4}	3.2600×10^{-4}
1/32	1.0775×10^{-2}	2.8113×10^{-3}	7.6397×10^{-4}	2.5249×10^{-4}	1.2740×10^{-4}	9.5957×10^{-5}	8.5305×10^{-5}	8.2221×10^{-5}
1/64	1.0718×10^{-2}	2.7553×10^{-3}	7.0535×10^{-4}	1.9119×10^{-4}	6.3175×10^{-5}	3.1869×10^{-5}	2.4019×10^{-5}	2.1349×10^{-5}
1/128	1.0704×10^{-2}	2.7414×10^{-3}	6.9089×10^{-4}	1.7646×10^{-4}	4.7817×10^{-5}	1.5795×10^{-5}	7.9668×10^{-6}	6.0053×10^{-6}
1/256	1.0701×10^{-2}	2.7379×10^{-3}	6.8728×10^{-4}	1.7283×10^{-4}	4.4126×10^{-5}	1.1955×10^{-5}	3.9484×10^{-6}	1.9886×10^{-6}

$\|e_h^N\|_{L_2(\Omega)}$

h \ Δt	1/2	1/4	1/8	1/16	1/32	1/64	1/128	1/256
1/2	7.5253×10^{-3}	7.3057×10^{-3}	7.0495×10^{-3}	6.9860×10^{-3}	6.9724×10^{-3}	6.9691×10^{-3}	6.9683×10^{-3}	6.9681×10^{-3}
1/4	2.2590×10^{-3}	2.2604×10^{-3}	2.2732×10^{-3}	2.2828×10^{-3}	2.2861×10^{-3}	2.2872×10^{-3}	2.2874×10^{-3}	2.2875×10^{-3}
1/8	1.3695×10^{-3}	6.6056×10^{-4}	6.1039×10^{-4}	6.3809×10^{-4}	6.4729×10^{-4}	6.4982×10^{-4}	6.5047×10^{-4}	6.5064×10^{-4}
1/16	1.4808×10^{-3}	4.9589×10^{-4}	1.6937×10^{-4}	1.5846×10^{-4}	1.6753×10^{-4}	1.7041×10^{-4}	1.7116×10^{-4}	1.7136×10^{-4}
1/32	1.5374×10^{-3}	5.2925×10^{-4}	1.3362×10^{-4}	4.2661×10^{-5}	4.0318×10^{-5}	4.2739×10^{-5}	4.3502×10^{-5}	4.3702×10^{-5}
1/64	1.5533×10^{-3}	5.4266×10^{-4}	1.4307×10^{-4}	3.3980×10^{-5}	1.0704×10^{-5}	1.0164×10^{-5}	1.0781×10^{-5}	1.0975×10^{-5}
1/128	1.5574×10^{-3}	5.4632×10^{-4}	1.4655×10^{-4}	3.6443×10^{-5}	8.5284×10^{-6}	2.6811×10^{-6}	2.5514×10^{-6}	2.7073×10^{-6}
1/256	1.5585×10^{-3}	5.4726×10^{-4}	1.4748×10^{-4}	3.7326×10^{-5}	9.1521×10^{-6}	2.1335×10^{-6}	6.7090×10^{-7}	6.4040×10^{-7}

Table 3.10: Errors of **(Q2)** for quadratic polynomial basis; Example 3.4 where $\alpha_0 = 10, \beta_0 = 1$

$\|e_h^N\|_{\mathcal{V}}$

h \ Δt	1/2	1/4	1/8	1/16	1/32	1/64	1/128	1/256
1/2	9.8172×10^{-3}	8.9173×10^{-3}	8.9616×10^{-3}	8.9695×10^{-3}	8.9832×10^{-3}	8.9840×10^{-3}	8.9840×10^{-3}	8.9839×10^{-3}
1/4	4.9597×10^{-3}	2.7376×10^{-3}	2.4396×10^{-3}	2.4276×10^{-3}	2.4296×10^{-3}	2.4296×10^{-3}	2.4295×10^{-3}	2.4295×10^{-3}
1/8	4.4579×10^{-3}	1.5119×10^{-3}	7.1448×10^{-4}	6.2576×10^{-4}	6.2046×10^{-4}	6.2033×10^{-4}	6.2035×10^{-4}	6.2037×10^{-4}
1/16	4.4256×10^{-3}	1.3954×10^{-3}	3.9630×10^{-4}	1.8171×10^{-4}	1.5857×10^{-4}	1.5706×10^{-4}	1.5698×10^{-4}	1.5698×10^{-4}
1/32	4.4235×10^{-3}	1.3877×10^{-3}	3.6665×10^{-4}	1.0028×10^{-4}	4.5835×10^{-5}	4.0029×10^{-5}	3.9639×10^{-5}	3.9618×10^{-5}

$\|\tilde{e}_h^N\|_{L_2(\Omega)}$

h \ Δt	1/2	1/4	1/8	1/16	1/32	1/64	1/128	1/256
1/2	1.1353×10^{-2}	3.4701×10^{-3}	1.4274×10^{-3}	9.7365×10^{-4}	8.3742×10^{-4}	8.2060×10^{-4}	8.1684×10^{-4}	8.1594×10^{-4}
1/4	1.0748×10^{-2}	2.7874×10^{-3}	7.4124×10^{-4}	2.3788×10^{-4}	1.1462×10^{-4}	8.6711×10^{-5}	8.5795×10^{-5}	8.5576×10^{-5}
1/8	1.0703×10^{-2}	2.7398×10^{-3}	6.8933×10^{-4}	1.7535×10^{-4}	4.8046×10^{-5}	1.8047×10^{-5}	1.0309×10^{-5}	1.0133×10^{-5}
1/16	1.0700×10^{-2}	2.7369×10^{-3}	6.8629×10^{-4}	1.7184×10^{-4}	4.3151×10^{-5}	1.1079×10^{-5}	3.3604×10^{-6}	1.6862×10^{-6}
1/32	1.0699×10^{-2}	2.7367×10^{-3}	6.8610×10^{-4}	1.7164×10^{-4}	4.2928×10^{-5}	1.0733×10^{-5}	2.6941×10^{-6}	7.8729×10^{-7}

$\|e_h^N\|_{L_2(\Omega)}$

h \ Δt	1/2	1/4	1/8	1/16	1/32	1/64	1/128	1/256
1/2	6.2959×10^{-4}	7.5763×10^{-4}	1.1082×10^{-3}	1.2073×10^{-3}	1.2339×10^{-3}	1.2406×10^{-3}	1.2422×10^{-3}	1.2426×10^{-3}
1/4	1.4463×10^{-3}	4.3957×10^{-4}	6.6655×10^{-5}	9.1956×10^{-5}	1.1551×10^{-4}	1.2180×10^{-4}	1.2341×10^{-4}	1.2381×10^{-4}
1/8	1.5508×10^{-3}	5.3979×10^{-4}	1.4029×10^{-4}	3.1118×10^{-5}	8.8690×10^{-6}	1.0246×10^{-5}	1.1278×10^{-5}	1.1565×10^{-5}
1/16	1.5583×10^{-3}	5.4707×10^{-4}	1.4730×10^{-4}	3.7163×10^{-5}	9.0326×10^{-6}	2.2079×10^{-6}	1.1828×10^{-6}	1.2355×10^{-6}
1/32	1.5588×10^{-3}	5.4754×10^{-4}	1.4777×10^{-4}	3.7610×10^{-5}	9.4203×10^{-6}	2.3404×10^{-6}	5.6276×10^{-7}	2.0866×10^{-7}

Table 3.11: Errors of **(Q2)** for quadratic polynomial basis; Example 3.4 where $\alpha_0 = 10, \beta_0 = 3$

As seen in the proofs of the error estimates theorems, the parameters α_0 and β_0 should satisfy the given condition to get optimal error estimates. Let Δt be sufficiently small to become negligible and d_0, d_1, d_2 be a convergence order such that

$$\|e_h\|_{\mathcal{V}} = O(h^{d_0}),\ \|\tilde{e}_h\|_{L_2(\Omega)} = O(h^{d_1}),\ \|e_h\|_{L_2(\Omega)} = O(h^{d_2})$$

at the final time T. We can determine d_0 experimentally by

$$d_0 = \ln(\|e_h\|_{\mathcal{V}}/\|e_{h/2}\|_{\mathcal{V}})/\ln 2.$$

It varies with the degree of polynomials k and the choice of the parameters. In a similar way, d_1 and d_2 can be also computed. Theoretically, $d_0 = k$, $d_1 = k+1$, and $d_2 = k+1$ if α_0 is large enough and $\beta_0(d-1) \geq 3$. In Table 3.12, it shows that the numerical rates are very close to the theoretical convergence rates when the penalty parameters α_0 and β_0 are sufficiently large. However, if the penalty parameters do not satisfy the conditions of the stability bound and the error bound, the optimal error estimate may not be observed. To be specific, in a discontinuous piecewise linear polynomial basis, the numerical approximations exist such that either the error increases as $h \to h/2$ or the numerical convergence rates are less than their theoretical results. But it is not able to solve the equivalent linear systems to the fully discrete formulations in $k = 2$ whence α_0 and β_0 do not fulfil the condition of the existence and uniqueness, so any result is not seen at the second and third row on Table 3.12 for $k = 2$. Interestingly, regardless of super-penalisation, L_2 error estimates show optimal results for linear polynomials. In rectangular meshes or 1D problems with odd degree of polynomials, the optimal L_2 error estimates are theoretically proved in [63, 62, 61]. Also, Table 3.12 indicates the standard penalisation is able to have the optimal convergence rates for $k = 1$. Moreover, for uniform meshes, the convergence rates become optimal if k is odd, however, this is not theoretically shown [24].

$k=1$		**(Q1)**			**(Q2)**		
α_0	β_0	d_0	d_1	d_2	d_0	d_1	d_2
5	3	1.005	2.002	1.973	1.005	2.003	1.976
5	0.1	-47.64	-51.99	-48.94	-49.72	-51.19	-50.50
0.01	3	0.712	1.826	1.283	0.754	1.943	1.336
5	1	1.011	1.976	1.990	1.010	1.987	1.982

$k=2$		**(Q1)**			**(Q2)**		
α_0	β_0	d_0	d_1	d_2	d_0	d_1	d_2
5	3	2.000	3.115	3.564	2.003	3.237	3.507
5	0.1	N/A	N/A	N/A	N/A	N/A	N/A
0.01	3	N/A	N/A	N/A	N/A	N/A	N/A
5	1	1.937	2.045	2.009	1.940	2.091	2.018

Table 3.12: Numerical convergence rates

Recall (3.3.6) and (3.3.26), let us consider the global matrices. Since solving fully discrete formulations is equivalent to solving the resulting linear systems, the global matrices are so important. Note that our global matrix is non-symmetric hence we have to deal with solving the linear systems carefully. More precisely, it consists of a mass matrix, symmetric/nonsymmetric stiffness matrices and a jump matrix. We would use iterative methods to solve the linear system hence the global matrix such as either $\frac{2\rho}{\Delta t^2}M + \mathcal{A} + \frac{1}{\Delta t}\mathcal{J}$ in (3.3.6) or $\frac{2\rho}{\Delta t^2}M + \frac{\varphi_0}{2}A + \mathcal{B} + \frac{1}{\Delta t}\mathcal{J}$ in (3.3.26), must be good. If a condition number of the global matrix is too big, which means ill-conditioned, the numerical result would not be appropriate. As in [64], a condition number of a global matrix of NIPG for elliptic PDEs depends on a spatial mesh size and penalty parameters. For example, the condition number is of order $O(h^{-(1+\beta_0)})$ for $\beta_0 = 1, 3$. In our case, the condition numbers of our DG global matrix with and without the super-penalisation are shown in Table 3.13. As same as elliptic problems, the condition numbers have increased when the spatial mesh size becomes small. Furthermore, the super-penalised β_0 yields more sharply increasing graphs of condition numbers in Figure 3.1. More precisely, the condition number of the standard penalisation is of order $O(h^{-2})$ but that of the super-penalisation is of order $O(h^{-4})$, which means the global matrix of the standard penalisation is more ill-conditioned than the standard penalised one. On the other hand, our model problem is time-dependant hence time discretisation would have effect on the condition number. A fine time step size gives a better condition number in contrast with a spatial mesh size. Regardless of β_0 for fixed h, the condition number depends only on Δt with first order. Consequently, the condition number has an numerical order $O(h^{-(\beta_0+1)} + \Delta t)$ for $\beta_0 = 1, 3$. Unfortunately, this is not proved theoretically but we could observe the results numerically. Only the theoretical analysis of SIPG for elliptic problems is given in [64, 65], however our problem may also be dealt in a similar way, and this would be a future question.

$h \backslash \Delta t$	1	1/2	1/4	1/8	1/16	1/32
1	6.16×10	5.17×10	3.94×10	2.70×10	1.93×10	1.66×10
1/2	1.43×10^2	1.06×10^2	9.04×10	7.64×10	5.50×10	3.64×10
1/4	4.89×10^2	3.16×10^2	1.93×10^2	1.63×10^2	1.46×10^2	1.11×10^2
1/8	1.89×10^3	1.18×10^3	6.30×10^2	3.61×10^2	3.09×10^2	2.83×10^2
1/16	7.49×10^3	4.64×10^3	2.40×10^3	1.21×10^3	6.93×10^2	6.00×10^2
1/32	2.99×10^4	1.85×10^4	9.49×10^3	4.65×10^3	2.35×10^3	1.35×10^3

$h \backslash \Delta t$	1	1/2	1/4	1/8	1/16	1/32
1	3.89×10	2.98×10	2.30×10	1.80×10	1.61×10	1.62×10
1/2	2.51×10^2	1.98×10^2	1.73×10^2	1.46×10^2	1.01×10^2	6.14×10
1/4	3.18×10^3	2.27×10^3	1.46×10^3	1.26×10^3	1.13×10^3	8.27×10^2
1/8	4.84×10^4	3.36×10^4	1.90×10^4	1.12×10^4	9.69×10^3	8.87×10^3
1/16	7.64×10^5	5.28×10^5	2.90×10^5	1.51×10^5	8.72×10^4	7.60×10^4
1/32	1.22×10^7	8.40×10^6	4.58×10^6	2.31×10^6	1.18×10^6	6.87×10^5

Table 3.13: Condition numbers of a global matrix; $k = 1$, $\beta_0 = 1$(top), $\beta_0 = 3$(bottom)

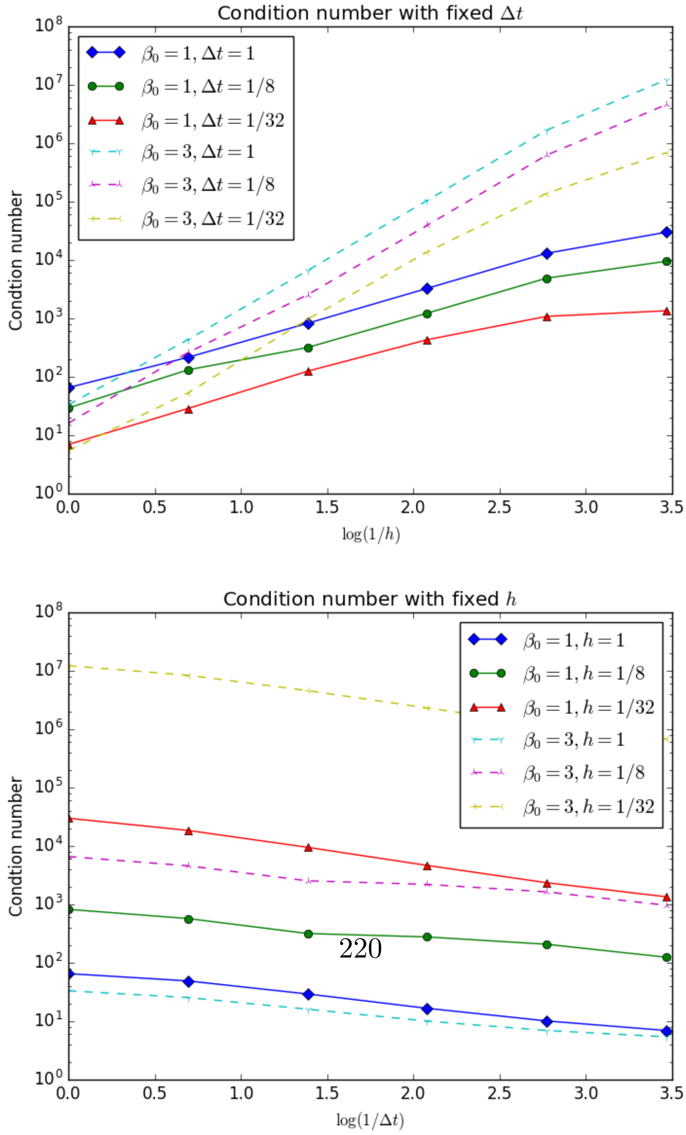

Figure 3.1: Graph of condition numbers; $\beta_0 = 1$(solid line) and $\beta_0 = 3$(dash line)

<u>Remark</u> CG vs DG

(a) *Bilinear forms*
Regardless of finite elements methods, we can observe coercivity and continuity on finite dimensional spaces. However, DG bilinear forms are conditionally coercive and continuous, which need large penalty parameters. Also it is only valid on the finite dimensional test spaces of polynomials but CG bilinear form is unconditionally coercive and continuous on the subspaces of H^1 space. One more difference is that we consider also non-symmetric DG bilinear form (NIPG).

(b) *Well-posedness*
Both finite element methods show stability bounds so that they imply the existence and uniqueness of solutions. Discrete solutions are bounded by data terms but stability bounds of DG have h^{-1} terms due to use of inverse polynomial trace theorem. However, it has no effect on stability analysis as well as error analysis.

(c) *Boundary conditions*
In general, DGFEM has imposed boundary conditions weakly, whereas CGFEM has made it strongly. This difference may not significantly affect on stability and error analysis. In fact, it is able to improve imposing boundary conditions strongly for DG. FEniCS also allows us to give strong boundary conditions.

(d) *Error estimates*
According to error estimates theorems and numerical experiments, it is shown that the numerical solutions have second order accuracy in time as well as optimal convergence order in energy norm, respectively. With elliptic regularity, L_2 error estimates become optimal but NIPG also requires super-penalisation.

(e) *Degrees of freedom*
Each degree of freedom in CG is a function value at the corresponding nodal point. On the other hand, degrees of freedom of DG are just coefficients of global basis functions. For the sake of discontinuity for global basis functions, DGFEM has much more degrees of freedom than CGFEM. For example, Table 3.14 describes the number of degrees of freedom with respect to spatial meshes where we consider 2D unit square with linear polynomial basis. As a result, the resulting linear system of DGFEM has much bigger size and it may encounter significant issues on solving linear systems in computational sense such as accumulated round-off and truncation errors while iterative solvers used.

h	1/2	1/4	1/8	1/16	1/32	1/64	1/128
CGFEM	9	25	81	289	1,089	4,225	16,641
DGGEM	24	96	384	1,536	6,144	24,576	98,304

Table 3.14: The number of degrees of freedom on a unit square in 2D for linear polynomials

(f) *Condition numbers*
Table 3.15 indicates that the condition number of CG is of order $O(h^{-2})$ with respect to the spatial mesh, experimentally. It is very similar to standard penalised DG. However, as shown before, in order to ensure optimal L_2 error estimates, DG must be super-penalised for NIPG, which deteriorates solving linear systems for fine meshes in practice.

h \ Δt	1	1/2	1/4	1/8	1/16	1/32
1	2.36	2.29	3.93	5.17	5.64	5.78
1/2	1.19×10	5.44	5.44	7.50	8.97	9.51
1/4	3.94×10	1.24×10	6.83	6.40	9.79	1.18×10
1/8	1.35×10^2	3.76×10	1.27×10	7.40	6.77	1.07×10
1/16	4.89×10^2	1.32×10^2	3.62×10	1.32×10	7.59	6.87
1/32	1.85×10^3	4.96×10^2	1.31×10^2	3.59×10	1.33×10	7.65

Table 3.15: Condition numbers of a global matrix; CGFEM

(g) *etc.*
Overall, it seems that CG is better method in terms of degrees of freedom and solving linear systems. Nevertheless, DG is also good approximation methods to solve PDEs, since it has benefits of hanging nodes issues and local mass conservation. In this thesis, we have not considered hanging nodes and mass conservation in details but we can observe some advantages of DGFEM in [24, 11].

Summary

In Chapter 3, we have formulated DG variational formulations with respect to two forms of internal variables. In a similar way with CGFEM, we have shown stability bounds as well as error bounds without Grönwall's inequality. With sufficiently large penalty parameters, we can derive various properties such as coercivity, continuity, inverse polynomial inequalities, etc. Consequently, we can prove existence and uniqueness of discrete solutions as well as optimal energy error estimates with fixed second order accuracy in time. However, our DG bilinear form, NIPG, requires super-penalisation for optimal L_2 error estimates, which enforces ill conditioned linear systems. In contrast to theory, we can observe L_2 optimality with standard penalisation in the numerical experiments for odd degree of polynomials basis.

Chapter 4

Linear Viscoelastic Problems with Internal Variables

Recall our primal model problem (1.3.17)-(1.3.23) with introducing internal variables (1.3.24)-(1.3.27).

$$
\begin{aligned}
\rho \ddot{\boldsymbol{u}} - \nabla \cdot \underline{\boldsymbol{\sigma}} &= \boldsymbol{f} && \text{in } (0,T] \times \Omega, \\
\boldsymbol{u} &= \boldsymbol{0} && \text{on } [0,T] \times \Gamma_D, \\
\underline{\boldsymbol{\sigma}} \cdot \boldsymbol{n} &= \boldsymbol{g}_N && \text{on } [0,T] \times \Gamma_N, \\
\boldsymbol{u} &= \boldsymbol{u}_0 && \text{on } \{0\} \times \Omega, \\
\dot{\boldsymbol{u}} &= \boldsymbol{w}_0 && \text{on } \{0\} \times \Omega.
\end{aligned}
$$

(Displacement form)

$$
\rho \ddot{\boldsymbol{u}} - \nabla \cdot \left(\underline{\boldsymbol{D}}(0) \underline{\boldsymbol{\varepsilon}}\big(\boldsymbol{u} - \sum_{q=1}^{N_\varphi} \boldsymbol{\psi}_q\big) \right) = \boldsymbol{f} \qquad \text{in } (0,T] \times \Omega,
$$

$$
\tau_q \dot{\boldsymbol{\psi}}_q + \boldsymbol{\psi}_q = \varphi_q \boldsymbol{u} \qquad \text{for } q = 1 \ldots, N_\varphi \text{ in } [0,T] \times \Omega.
$$

and

(Velocity form)

$$
\rho \ddot{\boldsymbol{u}} - \nabla \cdot \left(\varphi_0 \underline{\boldsymbol{D}}(0) \underline{\boldsymbol{\varepsilon}}\big(\boldsymbol{u} + \sum_{q=1}^{N_\varphi} \boldsymbol{\zeta}_q\big) \right) = \boldsymbol{f} + \nabla \cdot \left(\sum_{q=1}^{N_\varphi} \varphi_q e^{-t/\tau_q} \underline{\boldsymbol{D}}(0) \underline{\boldsymbol{\varepsilon}}(\boldsymbol{u}_0) \right)
$$

$$
\text{in } (0,T] \times \Omega,
$$

$$
\tau_q \dot{\boldsymbol{\zeta}}_q + \boldsymbol{\zeta}_q = \tau_q \varphi_q \dot{\boldsymbol{u}} \qquad \text{for } q = 1 \ldots, N_\varphi \text{ in } [0,T] \times \Omega.
$$

In this chapter we will consider finite element approximations to these vector-valued model problems. As shown in Chapter 2 and 3, use of CGFEM and DGFEM allows us to derive numerical approximations and we would consider error estimates and numerical simulations. In other words, we will expand the stability bounds theorems and the error bounds theorems in the scalar problems to vector-valued cases.

We assume that initial conditions, a surface traction, and a body force are sufficiently smooth. Hence we suppose

$$\boldsymbol{g}_N \in C^1(0,T;[L_2(\Gamma_N)]^d), \qquad \boldsymbol{f} \in C(0,T;[L_2(\Omega)]^d),$$

and initial conditions depend on spatial discretisation methods. For the sake of elliptic regularity of solutions, $\boldsymbol{u}_0$ and $\boldsymbol{w}_0$ are also sufficiently smooth.

4.1 CGFEM to Wave Propagation with Viscoelasticity

For a convenient notation, let $\underline{\underline{\boldsymbol{D}}} \leftarrow \underline{\underline{\boldsymbol{D}}}(0)$ for the fourth order tensor which is defined in (1.3.17)-(1.3.27). Consider $\boldsymbol{v} \in \boldsymbol{V}$ where $\boldsymbol{V} = \{\boldsymbol{v} \in [H^1(\Omega)]^d \mid \boldsymbol{v}(\boldsymbol{x}) = 0 \text{ on } \Gamma_D\}$ and then multiplying $\boldsymbol{v}$ by (1.3.17) with integration over the space domain gives

$$(\rho\ddot{\boldsymbol{u}}(t),\boldsymbol{v})_{L_2(\Omega)} - (\nabla\cdot\underline{\boldsymbol{\sigma}}(t),\boldsymbol{v})_{L_2(\Omega)} = (\boldsymbol{f}(t),\boldsymbol{v})_{L_2(\Omega)}. \tag{4.1.1}$$

As seen in the linear elastic problem in Chapter 1, since the stress tensor of the viscoelastic problem is symmetric, integration by parts yields

$$-(\nabla\cdot\underline{\boldsymbol{\sigma}}(t),\boldsymbol{v})_{L_2(\Omega)} = (\underline{\boldsymbol{\sigma}}(t),\underline{\boldsymbol{\varepsilon}}(\boldsymbol{v}))_{L_2(\Omega)} - \int_{\partial\Omega} \underline{\boldsymbol{\sigma}}(t)\cdot\boldsymbol{n}\cdot\boldsymbol{v}\, d\Gamma.$$

Hence imposing the boundary conditions implies

$$(\rho\ddot{\boldsymbol{u}}(t),\boldsymbol{v})_{L_2(\Omega)} + (\underline{\boldsymbol{\sigma}}(t),\underline{\boldsymbol{\varepsilon}}(\boldsymbol{v}))_{L_2(\Omega)} = (\boldsymbol{f}(t),\boldsymbol{v})_{L_2(\Omega)} + (\boldsymbol{g}_N(t),\boldsymbol{v})_{L_2(\Gamma_N)} \tag{4.1.2}$$

where $\boldsymbol{g}_D = \boldsymbol{0}$. Since the stress tensor can be written with internal variables as (1.3.24) and (1.3.26), (4.1.2) is rewritten by

$$\begin{aligned}(\rho\ddot{\boldsymbol{u}}(t),\boldsymbol{v})_{L_2(\Omega)} + (\underline{\underline{\boldsymbol{D}}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{u}(t)),\underline{\boldsymbol{\varepsilon}}(\boldsymbol{v}))_{L_2(\Omega)} - \sum_{q=1}^{N_\varphi}\left(\underline{\underline{\boldsymbol{D}}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{\psi}_q(t)),\underline{\boldsymbol{\varepsilon}}(\boldsymbol{v})\right)_{L_2(\Omega)} \\ = (\boldsymbol{f}(t),\boldsymbol{v})_{L_2(\Omega)} + (\boldsymbol{g}_N(t),\boldsymbol{v})_{L_2(\Gamma_N)}\end{aligned} \tag{4.1.3}$$

and

$$\begin{aligned}(\rho\ddot{\boldsymbol{u}}(t),\boldsymbol{v})_{L_2(\Omega)} + \varphi_0\left(\underline{\underline{\boldsymbol{D}}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{u}(t)),\underline{\boldsymbol{\varepsilon}}(\boldsymbol{v})\right)_{L_2(\Omega)} + \sum_{q=1}^{N_\varphi}\left(\underline{\underline{\boldsymbol{D}}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{\zeta}_q(t)),\underline{\boldsymbol{\varepsilon}}(\boldsymbol{v})\right)_{L_2(\Omega)} \\ = (\boldsymbol{f}(t),\boldsymbol{v})_{L_2(\Omega)} + (\boldsymbol{g}_N(t),\boldsymbol{v})_{L_2(\Gamma_N)} + \sum_{q=1}^{N_\varphi}\varphi_q e^{-t/\tau_q}\left(\underline{\underline{\boldsymbol{D}}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{u}_0),\underline{\boldsymbol{\varepsilon}}(\boldsymbol{v})\right)_{L_2(\Omega)},\end{aligned} \tag{4.1.4}$$

respectively. Let us define a bilinear form $a(\cdot,\cdot)$ by

$$a(\boldsymbol{v},\boldsymbol{w}) = \int_\Omega \underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{v}) : \underline{\boldsymbol{\varepsilon}}(\boldsymbol{w}) d\Omega \tag{4.1.5}$$

where $\boldsymbol{v},\boldsymbol{w} \in [H^1(\Omega)]^d$. Also we have linear forms F_d and F_v such that

$$F_d(t;\boldsymbol{v}) = (\boldsymbol{f}(t),\boldsymbol{v})_{L_2(\Omega)} + (\boldsymbol{g}_N(t),\boldsymbol{v})_{L_2(\Gamma_N)}, \tag{4.1.6}$$

$$F_v(t;\boldsymbol{v}) = (\boldsymbol{f}(t),\boldsymbol{v})_{L_2(\Omega)} + (\boldsymbol{g}_N(t),\boldsymbol{v})_{L_2(\Gamma_N)} + \sum_{q=1}^{N_\varphi} \varphi_q e^{-t/\tau_q} a(\boldsymbol{u}_0,\boldsymbol{v}). \tag{4.1.7}$$

Thus, we can obtain the following variational formulations:
(R1) find $\boldsymbol{u}$ and $\{\boldsymbol{\psi}_q\}_{q=1}^{N_\varphi}$ such that satisfy for all $\boldsymbol{v} \in \boldsymbol{V}$

$$(\rho\ddot{\boldsymbol{u}}(t),\boldsymbol{v})_{L_2(\Omega)} + a(\boldsymbol{u}(t),\boldsymbol{v}) - \sum_{q=1}^{N_\varphi} a\left(\boldsymbol{\psi}_q(t),\boldsymbol{v}\right) = F_d(t;\boldsymbol{v}), \tag{4.1.8}$$

$$a\left(\tau_q\dot{\boldsymbol{\psi}}_q(t) + \boldsymbol{\psi}_q(t),\boldsymbol{v}\right) = a(\varphi_q\boldsymbol{u}(t),\boldsymbol{v}), \tag{4.1.9}$$

for each q, where $\boldsymbol{u}(0) = \boldsymbol{u}_0$, $\dot{\boldsymbol{u}}(0) = \boldsymbol{w}_0$ and $\boldsymbol{\psi}_q(0) = \boldsymbol{0}$. In the same sense,
(R2) find $\boldsymbol{u}$ and $\{\boldsymbol{\zeta}_q\}_{q=1}^{N_\varphi}$ such that satisfy for all $\boldsymbol{v} \in \boldsymbol{V}$

$$(\rho\ddot{\boldsymbol{u}}(t),\boldsymbol{v})_{L_2(\Omega)} + \varphi_0 a(\boldsymbol{u}(t),\boldsymbol{v}) + \sum_{q=1}^{N_\varphi} a\left(\boldsymbol{\zeta}_q(t),\boldsymbol{v}\right) = F_v(t;\boldsymbol{v}), \tag{4.1.10}$$

$$a\left(\tau_q\dot{\boldsymbol{\zeta}}_q(t) + \boldsymbol{\zeta}_q(t),\boldsymbol{v}\right) = a(\tau_q\varphi_q\dot{\boldsymbol{u}}(t),\boldsymbol{v}), \tag{4.1.11}$$

for each q, where $\boldsymbol{u}(0) = \boldsymbol{u}_0$, $\dot{\boldsymbol{u}}(0) = \boldsymbol{w}_0$ and $\boldsymbol{\zeta}_q(0) = \boldsymbol{0}$.

We assume and consider only $d = 2, 3$. Note that according to Korn's inequality (e.g. see [51, 52, 50, 11]),

$$C\|\boldsymbol{v}\|^2_{H^1(\Omega)} \le \lambda_{\min}\|\underline{\boldsymbol{\varepsilon}}(\boldsymbol{v})\|^2_{L_2(\Omega)} \le a(\boldsymbol{v},\boldsymbol{v}) \le \lambda_{\max}\|\underline{\boldsymbol{\varepsilon}}(\boldsymbol{v})\|^2_{L_2(\Omega)}$$

for any $\boldsymbol{v} \in \boldsymbol{V}$ where C is a positive constant independent of $\boldsymbol{v}$, and $\lambda_{\min}$ and $\lambda_{\max}$ are the minimum and maximum eigenvalues of $\underline{\boldsymbol{D}}$. In addition,

$$\begin{aligned}
\|\underline{\boldsymbol{\varepsilon}}(\boldsymbol{v})\|^2_{L_2(\Omega)} =& \frac{1}{4}\sum_{i,j=1}^{d}\int_\Omega (v_{i,j}+v_{j,i})^2 \, d\Omega \\
=& \frac{1}{4}\sum_{i,j=1}^{d}\int_\Omega |v_{i,j}|^2 + 2v_{i,j}v_{j,i} + |v_{j,i}|^2 \, d\Omega \\
=& \frac{1}{2}|\boldsymbol{v}|^2_{H^1(\Omega)} + \frac{1}{2}\sum_{i,j=1}^{d}\int_\Omega v_{i,j}v_{j,i} \, d\Omega
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{2}|\boldsymbol{v}|_{H^1(\Omega)}^2 + \frac{1}{2}\sum_{i,j=1}^{d}\int_\Omega |v_{i,j}v_{j,i}|\, d\Omega \\
&\leq \frac{1}{2}|\boldsymbol{v}|_{H^1(\Omega)}^2 + \frac{1}{4}\sum_{i,j=1}^{d}\int_\Omega |v_{i,j}|^2\, d\Omega + \frac{1}{4}\sum_{i,j=1}^{d}\int_\Omega |v_{j,i}|^2\, d\Omega \\
&= \frac{1}{2}|\boldsymbol{v}|_{H^1(\Omega)}^2 + \frac{1}{2}|\boldsymbol{v}|_{H^1(\Omega)}^2 = |\boldsymbol{v}|_{H^1(\Omega)}^2 \\
&\leq \|\boldsymbol{v}\|_{H^1(\Omega)}^2
\end{aligned}
$$

by Cauchy-Schwarz inequality and Young's inequality. Thus it holds for any $\boldsymbol{v} \in \boldsymbol{V}$

$$
C_0 \|\boldsymbol{v}\|_{H^1(\Omega)}^2 \leq a(\boldsymbol{v}, \boldsymbol{v}) \leq C_1 \|\boldsymbol{v}\|_{H^1(\Omega)}^2 \tag{4.1.12}
$$

for some positive C_0 and C_1. Furthermore, we also have for any $\boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{V}$

$$
\begin{aligned}
|a(\boldsymbol{v}, \boldsymbol{w})| &\leq \lambda_{\max} \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{L_2(\Omega)} \|\boldsymbol{\varepsilon}(\boldsymbol{w})\|_{L_2(\Omega)} \\
&\leq \lambda_{\max} \|\boldsymbol{v}\|_{H^1(\Omega)} \|\boldsymbol{w}\|_{H^1(\Omega)} .
\end{aligned} \tag{4.1.13}
$$

Whence we define the energy norm $\|\cdot\|_V$ on $\boldsymbol{V}$ by

$$
\|\boldsymbol{v}\|_V = \sqrt{a(\boldsymbol{v}, \boldsymbol{v})},
$$

we can observe norm equivalence between H^1 norm and the energy norm on $\boldsymbol{V}$ by (4.1.12).

Consequently, as shown in the above, the bilinear form is coercive and continuous and the linear forms are continuous when data terms are given well.

4.1.1 Fully Discrete Formulation

While we approximate a solution with using CGFEM, we consider also time discretisation with Crank-Nicolson finite difference method. Hence our numerical solutions are defined by using the same notations and suppositions in the scalar problem but using bold symbols with the relation

$$
\frac{\boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n}{2} = \frac{\boldsymbol{U}_h^{n+1} - \boldsymbol{U}_h^n}{\Delta t}. \tag{4.1.14}
$$

Note that our finite dimensional test space $\boldsymbol{V}^h$ is a vector-valued analogue of V^h in Chapter 2 such that

$$
\boldsymbol{V}^h = \{\boldsymbol{v} \in \boldsymbol{V} \mid \boldsymbol{v} \in [V^h]^d\}.
$$

Hence $\boldsymbol{V}^h$ is the finite dimensional subspace of polynomials of degree k in $\boldsymbol{V}$. Consequently, we can derive the following fully discrete formulations with respect to internal variables:

(R1) find $\boldsymbol{W}_h^n$, $\boldsymbol{U}_h^n$ and $\boldsymbol{\Psi}_{hq}^n$ in $\boldsymbol{V}^h$ for $n = 0, \dots, N$ and $q = 1, \dots, N_\varphi$ such that satisfy for all $\boldsymbol{v} \in \boldsymbol{V}^h$

$$\left(\rho \frac{\boldsymbol{W}_h^{n+1} - \boldsymbol{W}_h^n}{\Delta t}, \boldsymbol{v}\right)_{L_2(\Omega)} + a\left(\frac{\boldsymbol{U}_h^{n+1} + \boldsymbol{U}_h^n}{2}, \boldsymbol{v}\right) - \sum_{q=1}^{N_\varphi} a\left(\frac{\boldsymbol{\Psi}_{hq}^{n+1} + \boldsymbol{\Psi}_{hq}^n}{2}, \boldsymbol{v}\right)$$

$$= \bar{F}_d^n(\boldsymbol{v}), \qquad \text{for } n = 0, \dots, N-1, \tag{4.1.15}$$

$$a\left(\tau_q \frac{\boldsymbol{\Psi}_{hq}^{n+1} - \boldsymbol{\Psi}_{hq}^n}{\Delta t} + \frac{\boldsymbol{\Psi}_{hq}^{n+1} + \boldsymbol{\Psi}_{hq}^n}{2}, \boldsymbol{v}\right) = a\left(\varphi_q \frac{\boldsymbol{U}_h^{n+1} + \boldsymbol{U}_h^n}{2}, \boldsymbol{v}\right),$$

$$\text{for } n = 0, \dots, N-1, \text{ and for } q = 1, \dots, N_\varphi, \tag{4.1.16}$$

$$a\left(\boldsymbol{U}_h^0, \boldsymbol{v}\right) = a\left(\boldsymbol{u}_0, \boldsymbol{v}, \right) \tag{4.1.17}$$

$$\left(\boldsymbol{W}_h^0, \boldsymbol{v}\right)_{L_2(\Omega)} = \left(\boldsymbol{w}_0, \boldsymbol{v}\right)_{L_2(\Omega)}, \tag{4.1.18}$$

for each q, $\boldsymbol{\Psi}_{hq}^0 = \boldsymbol{0}$.

(R2) find $\boldsymbol{W}_h^n$, $\boldsymbol{U}_h^n$ and $\boldsymbol{\mathcal{S}}_{hq}^n$ in $\boldsymbol{V}^h$ for $n = 0, \dots, N$ and $q = 1, \dots, N_\varphi$ such that satisfy for all $\boldsymbol{v} \in \boldsymbol{V}^h$

$$\left(\rho \frac{\boldsymbol{W}_h^{n+1} - \boldsymbol{W}_h^n}{\Delta t}, \boldsymbol{v}\right)_{L_2(\Omega)} + \varphi_0 a\left(\frac{\boldsymbol{U}_h^{n+1} + \boldsymbol{U}_h^n}{2}, \boldsymbol{v}\right) + \sum_{q=1}^{N_\varphi} a\left(\frac{\boldsymbol{\mathcal{S}}_{hq}^{n+1} + \boldsymbol{\mathcal{S}}_{hq}^n}{2}, \boldsymbol{v}\right)$$

$$= \bar{F}_v^n(\boldsymbol{v}), \qquad \text{for } n = 0, \dots, N-1, \tag{4.1.19}$$

$$a\left(\tau_q \frac{\boldsymbol{\mathcal{S}}_{hq}^{n+1} - \boldsymbol{\mathcal{S}}_{hq}^n}{\Delta t} + \frac{\boldsymbol{\mathcal{S}}_{hq}^{n+1} + \boldsymbol{\mathcal{S}}_{hq}^n}{2}, \boldsymbol{v}\right) = a\left(\tau_q \varphi_q \frac{\boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n}{2}, \boldsymbol{v}\right),$$

$$\text{for } n = 0, \dots, N-1, \text{ and for } q = 1, \dots, N_\varphi, \tag{4.1.20}$$

$$a\left(\boldsymbol{U}_h^0, \boldsymbol{v}\right) = a\left(\boldsymbol{u}_0, \boldsymbol{v}\right), \tag{4.1.21}$$

$$\left(\boldsymbol{W}_h^0, \boldsymbol{v}\right)_{L_2(\Omega)} = \left(\boldsymbol{w}_0, \boldsymbol{v}\right)_{L_2(\Omega)}, \tag{4.1.22}$$

for each q, $\boldsymbol{\mathcal{S}}_{hq}^0 = \boldsymbol{0}$.

(4.1.17), (4.1.18), (4.1.21) and (4.1.22) yield

$$\left\|\boldsymbol{U}_h^0\right\|_V \leq \left\|\boldsymbol{u}_0\right\|_V, \qquad \left\|\boldsymbol{W}_h^0\right\|_{L_2(\Omega)} \leq \left\|\boldsymbol{w}_0\right\|_{L_2(\Omega)},$$

by Cauchy-Schwarz inequality. Due to trace inequalities, we can analyse boundary terms in the stability bounds of the scalar-valued problem. In the same sense, it is observed that for any $\boldsymbol{v} \in \boldsymbol{V}$

$$\left\|v_i\right\|_{L_2(\partial\Omega)} \leq C \left\|v_i\right\|_{H^1(\Omega)} \text{ for each } i = 1, \dots, d$$

by (2.1.13), and hence

$$\left\|\boldsymbol{v}\right\|_{L_2(\partial\Omega)}^2 = \sum_{i=1}^{d} \left\|v_i\right\|_{L_2(\partial\Omega)}^2 \leq C \sum_{i=1}^{d} \left\|v_i\right\|_{H^1(\Omega)}^2 = C \left\|\boldsymbol{v}\right\|_{H^1(\Omega)}^2$$

$$\Rightarrow \|\boldsymbol{v}\|_{L_2(\partial\Omega)} \le C\,\|\boldsymbol{v}\|_{H^1(\Omega)} \le C\,\|\boldsymbol{v}\|_V \tag{4.1.23}$$

for some positive C by (4.1.12) and trace inequalities. As shown in the stability bounds for a scalar analogue, the existence and uniqueness of the fully discrete solutions can be obtained, once we use the same techniques but in vector-valued cases.

Theorem 4.1. *Consider the fully discrete solution of* **(R1)**. *There exists a positive constant C such that*

$$\max_{0\le n\le N} \|\boldsymbol{W}_h^n\|_{L_2(\Omega)}^2 + \max_{0\le n\le N} \|\boldsymbol{U}_h^n\|_V^2 + \sum_{q=1}^{N_\varphi} \left\|\boldsymbol{\Psi}_{hq}^m\right\|_V^2 + \sum_{q=1}^{N_\varphi}\sum_{n=0}^{m-1} \frac{1}{\Delta t}\left\|\boldsymbol{\Psi}_{hq}^{n+1} - \boldsymbol{\Psi}_{hq}^n\right\|_V^2$$
$$\le C\Bigg(\|\boldsymbol{w}_0\|_{L_2(\Omega)}^2 + \|\boldsymbol{u}_0\|_V^2 + \|\boldsymbol{f}\|_{L_\infty(0,T;L_2(\Omega))}^2 + \|\dot{\boldsymbol{g}}_N\|_{L_2(0,T;L_2(\Gamma_N))}^2$$
$$+ \|\boldsymbol{g}_N\|_{L_\infty(0,T;L_2(\Gamma_N))}^2\Bigg),$$

for any $m = 1,\dots,N$. Here, C is independent of h, Δt and numerical solutions but depends on the final time T.

Proof. Let $m\in\mathbb{N}$ such that $1\le m\le N$. By taking $\boldsymbol{v} = \boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n$ into (4.1.15) for $n = 0,\dots,m-1$ and adding the m equations together with (4.1.14), we have

$$\rho\,\|\boldsymbol{W}_h^m\|_{L_2(\Omega)}^2 + \|\boldsymbol{U}_h^m\|_V^2 = \rho\left\|\boldsymbol{W}_h^0\right\|_{L_2(\Omega)}^2 + \left\|\boldsymbol{U}_h^0\right\|_V^2 + \Delta t\sum_{n=1}^{m-1} \bar{F}_d^n\left(\boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n\right)$$
$$+ \sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi} a(\boldsymbol{\Psi}_{hq}^{n+1} + \boldsymbol{\Psi}_{hq}^n, \boldsymbol{U}_h^{n+1} - \boldsymbol{U}_h^n). \tag{4.1.24}$$

For each $q\in\{1,\dots,N_\varphi\}$, and $n = 0,\dots,m-1$, put $\boldsymbol{v} = \boldsymbol{\Psi}_{hq}^{n+1} - \boldsymbol{\Psi}_{hq}^n$ into (4.1.16) and take summation all with using summation by parts then we can obtain

$$\sum_{n=0}^{m-1} a(\boldsymbol{U}_h^{n+1} - \boldsymbol{U}_h^n, \boldsymbol{\Psi}_{hq}^{n+1} + \boldsymbol{\Psi}_{hq}^n)$$
$$= 2a(\boldsymbol{U}_h^m, \boldsymbol{\Psi}_{hq}^m) - \frac{2\tau_q}{\Delta t\varphi_q}\sum_{n=0}^{m-1}\left\|\boldsymbol{\Psi}_{hq}^{n+1} - \boldsymbol{\Psi}_{hq}^n\right\|_V^2 - \frac{1}{\varphi_q}\left\|\boldsymbol{\Psi}_{hq}^m\right\|_V^2 \tag{4.1.25}$$

since $\boldsymbol{\Psi}_{hq}^0 = 0$. Hence, combining (4.1.24) and (4.1.25) gives

$$\rho\,\|\boldsymbol{W}_h^m\|_{L_2(\Omega)}^2 + \|\boldsymbol{U}_h^m\|_V^2 + \sum_{q=1}^{N_\varphi}\frac{1}{\varphi_q}\left\|\boldsymbol{\Psi}_{hq}^m\right\|_V^2 + \sum_{q=1}^{N_\varphi}\sum_{n=0}^{m-1}\frac{2\tau_q}{\Delta t\varphi_q}\left\|\boldsymbol{\Psi}_{hq}^{n+1} - \boldsymbol{\Psi}_{hq}^n\right\|_V^2$$
$$= \rho\left\|\boldsymbol{W}_h^0\right\|_{L_2(\Omega)}^2 + \left\|\boldsymbol{U}_h^0\right\|_V^2 + \Delta t\sum_{n=0}^{m-1}\bar{F}_d^n\left(\boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n\right) + \sum_{q=1}^{N_\varphi} 2a(\boldsymbol{U}_h^m, \boldsymbol{\Psi}_{hq}^m).$$

Now, we shall consider bounds for the right hand side. As seen in the previous proofs, using triangular inequalities, Cauchy-Schwarz inequalities, Young's inequalities and integration by parts(also summation by parts), the following bound can be given.

$$\begin{aligned}
&\Delta t \sum_{n=0}^{m-1} \bar{F}_d^n \left(\boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n\right) \\
\leq & \frac{\Delta t}{\epsilon_a} \sum_{n=0}^{m-1} \left\|\bar{\boldsymbol{f}}^n\right\|_{L_2(\Omega)}^2 + \Delta t \epsilon_a \sum_{n=1}^{m-1} \left\|\boldsymbol{W}_h^n\right\|_{L_2(\Omega)}^2 + \frac{C}{\epsilon_b} \int_0^{t_m} \left\|\dot{\boldsymbol{g}}_N(t')\right\|_{L_2(\Gamma_N)}^2 dt' \\
& + C \Delta t \epsilon_b \sum_{n=0}^{m-1} \left\|\boldsymbol{U}_h^n\right\|_V^2 + \frac{\Delta t \epsilon_a}{2} \left\|\boldsymbol{W}_h^m\right\|_{L_2(\Omega)}^2 + C \epsilon_b \left\|\boldsymbol{U}_h^m\right\|_V^2 + \frac{\Delta t \epsilon_a}{2} \left\|\boldsymbol{W}_h^0\right\|_{L_2(\Omega)}^2 \\
& + C \left\|\boldsymbol{U}_h^0\right\|_V^2 + \frac{C}{\epsilon_b} \left\|\bar{\boldsymbol{g}}_N^m\right\|_{L_2(\Gamma_N)}^2 + C \left\|\bar{\boldsymbol{g}}_N^0\right\|_{L_2(\Gamma_N)}^2,
\end{aligned}$$

for any positive ϵ_a and ϵ_b, and some positive C. Details are shown in the proof of fully discrete stability bounds for **(P1)**. Here, to estimate L_2 norm on Γ_N, (4.1.23) is used. Moreover, taking into account the property of maximum yields

$$\begin{aligned}
&\Delta t \sum_{n=0}^{m-1} \bar{F}_d^n \left(\boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n\right) \\
\leq & \frac{\Delta t}{\epsilon_a} \sum_{n=0}^{N-1} \left\|\bar{\boldsymbol{f}}^n\right\|_{L_2(\Omega)}^2 + T \epsilon_a \max_{0 \leq n \leq N} \left\|\boldsymbol{W}_h^n\right\|_{L_2(\Omega)}^2 + \frac{C}{\epsilon_b} \left\|\dot{\boldsymbol{g}}_N\right\|_{L_2(0,T;L_2(\Gamma_N))}^2 \\
& + C \epsilon_b (T+1) \max_{0 \leq n \leq N} \left\|\boldsymbol{U}_h^n\right\|_V^2 + \frac{\Delta t \epsilon_a}{2} \left\|\boldsymbol{W}_h^0\right\|_{L_2(\Omega)}^2 + C \left\|\boldsymbol{U}_h^0\right\|_V^2 \\
& + \left(\frac{C}{\epsilon_b} + C\right) \max_{0 \leq n \leq N-1} \left\|\bar{\boldsymbol{g}}_N^n\right\|_{L_2(\Gamma_N)}^2.
\end{aligned}$$

Also, Cauchy-Schwarz inequality implies

$$\sum_{q=1}^{N_\varphi} 2a(\boldsymbol{U}_h^m, \boldsymbol{\Psi}_{hq}^m) \leq \sum_{q=1}^{N_\varphi} \epsilon_q \left\|\boldsymbol{U}_h^m\right\|_V^2 + \sum_{q=1}^{N_\varphi} \frac{1}{\epsilon_q} \left\|\boldsymbol{\Psi}_{hq}^m\right\|_V^2,$$

with $\epsilon_q = \varphi_q + \varphi_0/(2N_\varphi) > 0$ for each q. As a consequence, these two bounds give

$$\begin{aligned}
&\rho \left\|\boldsymbol{W}_h^m\right\|_{L_2(\Omega)}^2 + \frac{\varphi_0}{2} \left\|\boldsymbol{U}_h^m\right\|_V^2 + \sum_{q=1}^{N_\varphi} \frac{\varphi_0}{2\varphi_q^2 N_\varphi + \varphi_q \varphi_0} \left\|\boldsymbol{\Psi}_{hq}^m\right\|_V^2 \\
& + \sum_{q=1}^{N_\varphi} \sum_{n=0}^{m-1} \frac{2\tau_q}{\Delta t \varphi_q} \left\|\boldsymbol{\Psi}_{hq}^{n+1} - \boldsymbol{\Psi}_{hq}^n\right\|_V^2 \\
\leq & \left(1 + \frac{\Delta t \epsilon_a}{2}\right) \left\|\boldsymbol{W}_h^0\right\|_{L_2(\Omega)}^2 + (1+C) \left\|\boldsymbol{U}_h^0\right\|_V^2 + \frac{\Delta t}{\epsilon_a} \sum_{n=0}^{N-1} \left\|\bar{\boldsymbol{f}}^n\right\|_{L_2(\Omega)}^2
\end{aligned}$$

$$+\frac{C}{\epsilon_b}\left\|\dot{\boldsymbol{g}}_N\right\|^2_{L_2(0,T;L_2(\Gamma_N))}+\left(\frac{C}{\epsilon_b}+C\right)\max_{0\le n\le N-1}\left\|\bar{\boldsymbol{g}}_N^n\right\|^2_{L_2(\Gamma_N)}$$
$$+T\epsilon_a\max_{0\le n\le N}\left\|\boldsymbol{W}_h^n\right\|^2_{L_2(\Omega)}+C\epsilon_b(T+1)\max_{0\le n\le N}\left\|\boldsymbol{U}_h^n\right\|^2_V,$$

for any m. Therefore, by setting $\epsilon_a=\rho/6T$ and $\epsilon_b=\varphi_0/12C(T+1)$, and on account of maximum, we can conclude that

$$\frac{\rho}{2}\max_{0\le n\le N}\left\|\boldsymbol{W}_h^n\right\|^2_{L_2(\Omega)}+\frac{\varphi_0}{4}\max_{0\le n\le N}\left\|\boldsymbol{U}_h^n\right\|^2_V+\sum_{q=1}^{N_\varphi}\frac{\varphi_0}{2\varphi_q^2N_\varphi+\varphi_q\varphi_0}\left\|\boldsymbol{\Psi}_{hq}^m\right\|^2_V$$
$$+\sum_{q=1}^{N_\varphi}\sum_{n=0}^{m-1}\frac{2\tau_q}{\Delta t\varphi_q}\left\|\boldsymbol{\Psi}_{hq}^{n+1}-\boldsymbol{\Psi}_{hq}^n\right\|^2_V$$
$$\le C\Bigg(\left\|\boldsymbol{W}_h^0\right\|^2_{L_2(\Omega)}+\left\|\boldsymbol{U}_h^0\right\|^2_V+\Delta t\sum_{n=0}^{N-1}\left\|\bar{\boldsymbol{f}}^n\right\|^2_{L_2(\Omega)}+\left\|\dot{\boldsymbol{g}}_N\right\|^2_{L_2(0,T;L_2(\Gamma_N))}$$
$$+\max_{0\le n\le N-1}\left\|\bar{\boldsymbol{g}}_N^n\right\|^2_{L_2(\Gamma_N)}\Bigg),$$
$$\le C\Bigg(\left\|\boldsymbol{w}_0\right\|^2_{L_2(\Omega)}+\left\|\boldsymbol{u}_0\right\|^2_V+\left\|\boldsymbol{f}\right\|^2_{L_\infty(0,T;L_2(\Omega))}+\left\|\dot{\boldsymbol{g}}_N\right\|^2_{L_2(0,T;L_2(\Gamma_N))}$$
$$+\left\|\boldsymbol{g}_N\right\|^2_{L_\infty(0,T;L_2(\Gamma_N))}\Bigg),$$

for some positive constant C. □

In Theorem 4.1, Grönwall's inequality are never used so that the constant C does not increase exponentially but depending on the final time T. In the same way, a stability bound for **(R2)** can be also derived.

Theorem 4.2. *Suppose we have a fully discrete solution of* **(R2)**. *For any $m\in\mathbb{N}$ such that $1\le m\le N$, there exists a positive constant C such that*

$$\max_{0\le n\le N}\left\|\boldsymbol{W}_h^n\right\|^2_{L_2(\Omega)}+\max_{0\le n\le N}\left\|\boldsymbol{U}_h^n\right\|^2_V+\sum_{q=1}^{N_\varphi}\left\|\boldsymbol{\mathcal{S}}_{hq}^m\right\|^2_V+\sum_{q=1}^{N_\varphi}\sum_{n=0}^{m-1}\Delta t\left\|\boldsymbol{\mathcal{S}}_{hq}^{n+1}+\boldsymbol{\mathcal{S}}_{hq}^n\right\|^2_V$$
$$\le C\Bigg(\left\|\boldsymbol{w}_0\right\|^2_{L_2(\Omega)}+\left\|\boldsymbol{u}_0\right\|^2_V+\left\|\boldsymbol{f}\right\|^2_{L_\infty(0,T;L_2(\Omega))}+\left\|\boldsymbol{g}_N\right\|^2_{L_\infty(0,T;L_2(\Gamma_N))}$$
$$+\left\|\dot{\boldsymbol{g}}_N\right\|^2_{L_2(0,T;L_2(\Gamma_N))}\Bigg).$$

Proof. Let us consider (4.1.19) with $\boldsymbol{v}=\boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n$ for $0\le n\le m-1$, $m=1,\dots,N$. Summation over m gives

$$\rho\left\|\boldsymbol{W}_h^m\right\|^2_{L_2(\Omega)}+\varphi_0\left\|\boldsymbol{U}_h^m\right\|^2_V+\frac{\Delta t}{2}\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi}a(\boldsymbol{\mathcal{S}}_{hq}^{n+1}+\boldsymbol{\mathcal{S}}_{hq}^n,\boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n)$$

$$=\rho \left\|\boldsymbol{W}_h^0\right\|_{L_2(\Omega)}^2 + \varphi_0 \left\|\boldsymbol{U}_h^0\right\|_V^2 + \Delta t \sum_{n=0}^{m-1} \bar{F}_v^n \left(\boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n\right) \tag{4.1.26}$$

by (4.1.14). By taking $\boldsymbol{v} = \boldsymbol{S}_{hq}^{n+1} + \boldsymbol{S}_{hq}^n$ into (4.1.20) for $n = 0, \ldots, m-1$, adding the m equations implies

$$\sum_{n=0}^{m-1} a(\boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n, \boldsymbol{S}_{hq}^{n+1} + \boldsymbol{S}_{hq}^n) = \frac{2}{\varphi_q \Delta t} \left\|\boldsymbol{S}_{hq}^m\right\|_V^2 + \sum_{n=0}^{m-1} \frac{1}{\tau_q \varphi_q} \left\|\boldsymbol{S}_{hq}^{n+1} + \boldsymbol{S}_{hq}^n\right\|_V^2, \tag{4.1.27}$$

since $\boldsymbol{S}_{hq}^0 = 0$ for any q. By substitution of (4.1.27) into (4.1.26), we have

$$\begin{aligned} &\rho \left\|\boldsymbol{W}_h^m\right\|_{L_2(\Omega)}^2 + \varphi_0 \left\|\boldsymbol{U}_h^m\right\|_V^2 + \sum_{q=1}^{N_\varphi} \frac{1}{\varphi_q} \left\|\boldsymbol{S}_{hq}^m\right\|_V^2 + \sum_{q=1}^{N_\varphi} \sum_{n=0}^{m-1} \frac{\Delta t}{2\tau_q \varphi_q} \left\|\boldsymbol{S}_{hq}^{n+1} + \boldsymbol{S}_{hq}^n\right\|_V^2 \\ =&\rho \left\|\boldsymbol{W}_h^0\right\|_{L_2(\Omega)}^2 + \varphi_0 \left\|\boldsymbol{U}_h^0\right\|_V^2 + \Delta t \sum_{n=0}^{m-1} \bar{F}_v^n \left(\boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n\right). \end{aligned} \tag{4.1.28}$$

With based on the same knowledge before such as using Cauchy-Schwarz inequalities, Young's inequalities, maximum properties and (4.1.23), we can estimate the following bound for $\Delta t \sum_{n=0}^{m-1} \bar{F}_v^n \left(\boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n\right)$ by

$$\begin{aligned} &\Delta t \sum_{n=0}^{m-1} \bar{F}_v^n \left(\boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n\right) \\ \leq&\rho \left\|\boldsymbol{W}_h^0\right\|_{L_2(\Omega)}^2 + (C+2) \left\|\boldsymbol{U}_h^0\right\|_V^2 + \frac{\Delta t}{\epsilon_a} \sum_{n=0}^{N-1} \left\|\bar{\boldsymbol{f}}^n\right\|_{L_2(\Omega)}^2 \\ &+ \left(\frac{C}{\epsilon_b} + C\right) \max_{0 \leq n \leq N-1} \left\|\bar{\boldsymbol{g}}_N^n\right\|_{L_2(\Gamma_N)}^2 + C\epsilon_d \left\|\dot{\boldsymbol{g}}_N\right\|_{L_2(0,T;L_2(\Gamma_N))}^2 \\ &+ \left(\frac{1}{\epsilon_c} + \sum_{q=1}^{N_\varphi} \frac{1}{4\tau_q \epsilon_e}\right) \left\|\boldsymbol{u}_0\right\|_V^2 + \Delta t \epsilon_a \left(N + \frac{1}{2}\right) \max_{0 \leq n \leq N} \left\|\boldsymbol{W}_h^n\right\|_{L_2(\Omega)}^2 \\ &+ \left(C\epsilon_b + \epsilon_c + C\epsilon_d T + \frac{\epsilon_e T}{2}\right) \max_{0 \leq n \leq N} \left\|\boldsymbol{U}_h^n\right\|_V^2 \end{aligned} \tag{4.1.29}$$

for positive ϵ_a, ϵ_b, ϵ_c, ϵ_d and ϵ_e. Hence, substituting (4.1.29) into (4.1.28) and applying the property of maximum yield

$$\begin{aligned} &\rho \max_{0 \leq n \leq N} \left\|\boldsymbol{W}_h^n\right\|_{L_2(\Omega)}^2 + \varphi_0 \max_{0 \leq n \leq N} \left\|\boldsymbol{U}_h^n\right\|_V^2 + \sum_{q=1}^{N_\varphi} \frac{1}{\varphi_q} \left\|\boldsymbol{S}_{hq}^m\right\|_V^2 \\ &+ \sum_{q=1}^{N_\varphi} \sum_{n=0}^{m-1} \frac{\Delta t}{2\tau_q \varphi_q} \left\|\boldsymbol{S}_{hq}^{n+1} + \boldsymbol{S}_{hq}^n\right\|_V^2 \end{aligned}$$

$$
\begin{aligned}
\leq& 3\Bigg(\rho \left\| \boldsymbol{W}_h^0 \right\|_{L_2(\Omega)}^2 + (C+2) \left\| \boldsymbol{U}_h^0 \right\|_V^2 + \frac{\Delta t}{\epsilon_a} \sum_{n=0}^{N-1} \left\| \bar{\boldsymbol{f}}^n \right\|_{L_2(\Omega)}^2 \\
&+ \left(\frac{C}{\epsilon_b} + C \right) \max_{0\leq n\leq N-1} \|\bar{\boldsymbol{g}}_N^n\|_{L_2(\Gamma_N)}^2 + C\epsilon_d \|\dot{\boldsymbol{g}}_N\|_{L_2(0,T;L_2(\Gamma_N))}^2 \\
&+ \left(\frac{1}{\epsilon_c} + \sum_{q=1}^{N_\varphi} \frac{1}{4\tau_q \epsilon_e} \right) \|\boldsymbol{u}_0\|_V^2 + \Delta t \epsilon_a \left(N + \frac{1}{2} \right) \max_{0\leq n\leq N} \|\boldsymbol{W}_h^n\|_{L_2(\Omega)}^2 \\
&+ \left(C\epsilon_b + \epsilon_c + C\epsilon_d T + \frac{\epsilon_e T}{2} \right) \max_{0\leq n\leq N} \|\boldsymbol{U}_h^n\|_V^2 \Bigg).
\end{aligned}
$$

Whence we set

$$
\epsilon_a = \frac{\rho}{\Delta t(6N+3)} = \frac{\rho}{6T + 3\Delta t} > 0,
$$

$$
\epsilon_b = \frac{\varphi_0}{24C} > 0,\ \epsilon_c = \frac{\varphi_0}{24} > 0,\ \epsilon_d = \frac{\varphi_0}{24CT} > 0,\ \text{and } \epsilon_e = \frac{\varphi_0}{12T} > 0,
$$

we have a positive constant C such that

$$
\begin{aligned}
&\frac{\rho}{2} \max_{0\leq n\leq N} \|\boldsymbol{W}_h^n\|_{L_2(\Omega)}^2 + \frac{\varphi_0}{2} \max_{0\leq n\leq N} \|\boldsymbol{U}_h^n\|_V^2 + \sum_{q=1}^{N_\varphi} \frac{1}{\varphi_q} \left\| \boldsymbol{S}_{hq}^m \right\|_V^2 \\
&+ \sum_{q=1}^{N_\varphi} \sum_{n=0}^{m-1} \frac{\Delta t}{2\tau_q \varphi_q} \left\| \boldsymbol{S}_{hq}^{n+1} + \boldsymbol{S}_{hq}^n \right\|_V^2 \\
\leq& C\Bigg(\left\| \boldsymbol{W}_h^0 \right\|_{L_2(\Omega)}^2 + \left\| \boldsymbol{U}_h^0 \right\|_V^2 + \Delta t \sum_{n=0}^{N-1} \left\| \bar{\boldsymbol{f}}^n \right\|_{L_2(\Omega)}^2 + \max_{0\leq n\leq N-1} \|\bar{\boldsymbol{g}}_N^n\|_{L_2(\Gamma_N)}^2 \\
&+ \|\dot{\boldsymbol{g}}_N\|_{L_2(0,T;L_2(\Gamma_N))}^2 + \|\boldsymbol{u}_0\|_V^2 \Bigg) \\
\leq& C\Bigg(\|\boldsymbol{w}_0\|_{L_2(\Omega)}^2 + \|\boldsymbol{u}_0\|_V^2 + \|\boldsymbol{f}\|_{L_\infty(0,T;L_2(\Omega))}^2 + \|\boldsymbol{g}_N\|_{L_\infty(0,T;L_2(\Gamma_N))}^2 \\
&+ \|\dot{\boldsymbol{g}}_N\|_{L_2(0,T;L_2(\Gamma_N))}^2 \Bigg).
\end{aligned}
$$

□

Theorem 4.1 and Theorem 4.2 are sufficient to show the existence and uniqueness of the fully discrete solutions for the displacement form and the velocity form, respectively. More precisely, the discrete solutions are bounded by only data terms such as source, traction and initial conditions. It means, zero data imply a trivial solution so that the linear system can be solved uniquely.

From now on, we shall introduce a new elliptic projection operator $\boldsymbol{R}$ defined by

$$
\boldsymbol{R} : \boldsymbol{V} \mapsto \boldsymbol{V}^h \text{ such that for } \boldsymbol{w} \in \boldsymbol{V} \subset [H^1(\Omega)]^d,\ a(\boldsymbol{w}, \boldsymbol{v}) = a(\boldsymbol{R}\boldsymbol{w}, \boldsymbol{v}),\ \forall \boldsymbol{v} \in \boldsymbol{V}^h. \tag{4.1.30}
$$

Obviously, we can observe Galerkin orthogonality such that $a\left(\boldsymbol{w}-\boldsymbol{R}\boldsymbol{w},\boldsymbol{v}\right)=0,\ \forall\boldsymbol{v}\in\boldsymbol{V}^h$ and use of interpolation estimates gives the following H^1 estimates

$$\|\boldsymbol{w}-\boldsymbol{R}\boldsymbol{w}\|_{H^1(\Omega)}\leq C|\boldsymbol{w}|_{H^r(\Omega)}h^{r-1} \tag{4.1.31}$$

for $\boldsymbol{w}\in[H^s(\Omega)]^d$ where $r=\min(k+1,s)$. Furthermore, elliptic regularity estimation provides an optimal L_2 estimates such that

$$\|\boldsymbol{w}-\boldsymbol{R}\boldsymbol{w}\|_{L_2(\Omega)}\leq C|\boldsymbol{w}|_{H^r(\Omega)}h^{r} \tag{4.1.32}$$

if elliptic regularity is given. See e.g. [14, 11, 66]. To consider error bounds for our problems let us define

$$\begin{aligned}
&\boldsymbol{\theta}:=\boldsymbol{u}-\boldsymbol{R}\boldsymbol{u}, &&\boldsymbol{\chi}^n:=\boldsymbol{U}_h^n-\boldsymbol{R}\boldsymbol{u}^n, &&\boldsymbol{\varpi}^n:=\boldsymbol{W}_h^n-\boldsymbol{R}\dot{\boldsymbol{u}}^n,\\
&\boldsymbol{\vartheta}_q:=\boldsymbol{\psi}_q-\boldsymbol{R}\boldsymbol{\psi}_q, &&\boldsymbol{\varsigma}_q^n:=\boldsymbol{\Psi}_{hq}^n-\boldsymbol{R}\boldsymbol{\psi}_q^n,\ \forall q\in\{1,\ldots,N_\varphi\},\\
&\boldsymbol{\nu}_q:=\boldsymbol{\zeta}_q-\boldsymbol{R}\boldsymbol{\zeta}_q, &&\boldsymbol{\Upsilon}_q^n:=\boldsymbol{S}_{hq}^n-\boldsymbol{R}\boldsymbol{\zeta}_q^n,\ \forall q\in\{1,\ldots,N_\varphi\},
\end{aligned}$$

where $\boldsymbol{u}^n=\boldsymbol{u}(t_n)$. We follows the same argument in scalar cases to see error bounds using the properties of elliptic projections such as Galerkin orthogonality, certain error estimates (4.1.31), (4.1.32) and other techniques.

Lemma 4.1. *Suppose* $\boldsymbol{u}\in H^4(0,T;[H^s(\Omega)]^d)\cap W^1_\infty(0,T;[H^s(\Omega)]^d)$. *If the fully discrete solution satisfies* **(R1)**, *then*

$$\max_{0\leq n\leq N}\|\boldsymbol{\varpi}^n\|_{L_2(\Omega)}+\max_{0\leq n\leq N}\|\boldsymbol{\chi}^n\|_V\leq C(h^{\min(k+1,s)-1}+\Delta t^2).$$

If we also assume elliptic regularity,

$$\max_{0\leq n\leq N}\|\boldsymbol{\varpi}^n\|_{L_2(\Omega)}+\max_{0\leq n\leq N}\|\boldsymbol{\chi}^n\|_V\leq C(h^{\min(k+1,s)}+\Delta t^2).$$

Proof. Consider (4.1.15) and (4.1.8) for average between $t=t_{n+1}$ and $t=t_n$, and subtract each other. Then we have

$$\begin{aligned}
&\left(\frac{\rho}{2}(\ddot{\boldsymbol{u}}^{n+1}+\ddot{\boldsymbol{u}}^n)-\frac{\rho}{\Delta t}(\boldsymbol{W}_h^{n+1}-\boldsymbol{W}_h^n),\boldsymbol{v}\right)_{L_2(\Omega)}+\frac{1}{2}a\left((\boldsymbol{u}^{n+1}+\boldsymbol{u}^n)-(\boldsymbol{U}_h^{n+1}+\boldsymbol{U}_h^n),\boldsymbol{v}\right)\\
&-\frac{1}{2}\sum_{q=1}^{N_\varphi}a\left((\boldsymbol{\psi}_q^{n+1}+\boldsymbol{\psi}_q^n)-(\boldsymbol{\Psi}_{hq}^{n+1}+\boldsymbol{\Psi}_{hq}^n),\boldsymbol{v}\right)=0,
\end{aligned}$$

for any $\boldsymbol{v}\in\boldsymbol{V}^h$. When we define

$$\begin{aligned}
\boldsymbol{\mathcal{E}}_1(t)&:=\frac{\ddot{\boldsymbol{u}}(t+\Delta t)+\ddot{\boldsymbol{u}}(t)}{2}-\frac{\dot{\boldsymbol{u}}(t+\Delta t)-\dot{\boldsymbol{u}}(t)}{\Delta t},\\
\boldsymbol{\mathcal{E}}_2(t)&:=\frac{\dot{\boldsymbol{\theta}}(t+\Delta t)+\dot{\boldsymbol{\theta}}(t)}{2}-\frac{\boldsymbol{\theta}(t+\Delta t)-\boldsymbol{\theta}(t)}{\Delta t},\\
\boldsymbol{\mathcal{E}}_3(t)&:=\frac{\boldsymbol{u}(t+\Delta t)-\boldsymbol{u}(t)}{\Delta t}-\frac{\dot{\boldsymbol{u}}(t+\Delta t)+\dot{\boldsymbol{u}}(t)}{2},
\end{aligned}$$

Galerkin orthogonality gives

$$\frac{\rho}{\Delta t}\left(\boldsymbol{\varpi}^{n+1}-\boldsymbol{\varpi}^{n},\boldsymbol{v}\right)_{L_2(\Omega)}+\frac{1}{2}a\left(\boldsymbol{\chi}^{n+1}+\boldsymbol{\chi}^{n},\boldsymbol{v}\right)-\frac{1}{2}\sum_{q=1}^{N_\varphi}a\left(\boldsymbol{\varsigma}_q^{n+1}+\boldsymbol{\varsigma}_q^{n},\boldsymbol{v}\right)$$
$$=\frac{\rho}{\Delta t}\left(\dot{\boldsymbol{\theta}}^{n+1}-\dot{\boldsymbol{\theta}}^{n},\boldsymbol{v}\right)_{L_2(\Omega)}+\rho\left(\boldsymbol{\mathcal{E}}_1^n,\boldsymbol{v}\right)_{L_2(\Omega)}.$$

Taking $\boldsymbol{v}=\frac{\boldsymbol{\chi}^{n+1}-\boldsymbol{\chi}^n}{\Delta t}$ into the above and using the fact,

$$\frac{\boldsymbol{\chi}^{n+1}-\boldsymbol{\chi}^n}{\Delta t}=\frac{\boldsymbol{\varpi}^{n+1}+\boldsymbol{\varpi}^n}{2}-\boldsymbol{\mathcal{E}}_2^n-\boldsymbol{\mathcal{E}}_3^n,$$

yield

$$\frac{\rho}{2\Delta t}\left(\|\boldsymbol{\varpi}^{n+1}\|^2_{L_2(\Omega)}-\|\boldsymbol{\varpi}^{n}\|^2_{L_2(\Omega)}\right)+\frac{1}{2\Delta t}\left(\|\boldsymbol{\chi}^{n+1}\|^2_V-\|\boldsymbol{\chi}^{n}\|^2_V\right)$$
$$-\frac{1}{2}\sum_{q=1}^{N_\varphi}a\left(\boldsymbol{\varsigma}_q^{n+1}+\boldsymbol{\varsigma}_q^{n},\frac{\boldsymbol{\chi}^{n+1}-\boldsymbol{\chi}^n}{\Delta t}\right),$$
$$=\frac{\rho}{2\Delta t}\left(\dot{\boldsymbol{\theta}}^{n+1}-\dot{\boldsymbol{\theta}}^{n},\boldsymbol{\varpi}^{n+1}+\boldsymbol{\varpi}^{n}\right)_{L_2(\Omega)}-\frac{\rho}{\Delta t}\left(\dot{\boldsymbol{\theta}}^{n+1}-\dot{\boldsymbol{\theta}}^{n},\boldsymbol{\mathcal{E}}_2^n\right)_{L_2(\Omega)}$$
$$-\frac{\rho}{\Delta t}\left(\dot{\boldsymbol{\theta}}^{n+1}-\dot{\boldsymbol{\theta}}^{n},\boldsymbol{\mathcal{E}}_3^n\right)_{L_2(\Omega)}+\frac{\rho}{2}\left(\boldsymbol{\mathcal{E}}_1^n,\boldsymbol{\varpi}^{n+1}+\boldsymbol{\varpi}^{n}\right)_{L_2(\Omega)}-\rho\left(\boldsymbol{\mathcal{E}}_1^n,\boldsymbol{\mathcal{E}}_2^n\right)_{L_2(\Omega)}$$
$$-\rho\left(\boldsymbol{\mathcal{E}}_1^n,\boldsymbol{\mathcal{E}}_3^n\right)_{L_2(\Omega)}+\frac{\rho}{\Delta t}\left(\boldsymbol{\varpi}^{n+1}-\boldsymbol{\varpi}^{n},\boldsymbol{\mathcal{E}}_2^n\right)_{L_2(\Omega)}+\frac{\rho}{\Delta t}\left(\boldsymbol{\varpi}^{n+1}-\boldsymbol{\varpi}^{n},\boldsymbol{\mathcal{E}}_3^n\right)_{L_2(\Omega)},$$

summation from $n=0$ to $m-1$ for $m\le N$, implies

$$\frac{\rho}{2\Delta t}\|\boldsymbol{\varpi}^{m}\|^2_{L_2(\Omega)}+\frac{1}{2\Delta t}\|\boldsymbol{\chi}^{m}\|^2_V-\frac{1}{2\Delta t}\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi}a\left(\boldsymbol{\varsigma}_q^{n+1}+\boldsymbol{\varsigma}_q^{n},\boldsymbol{\chi}^{n+1}-\boldsymbol{\chi}^n\right),$$
$$=\frac{\rho}{2\Delta t}\|\boldsymbol{\varpi}^{0}\|^2_{L_2(\Omega)}+\frac{1}{2\Delta t}\|\boldsymbol{\chi}^{0}\|^2_V+\frac{\rho}{2\Delta t}\sum_{n=0}^{m-1}\left(\dot{\boldsymbol{\theta}}^{n+1}-\dot{\boldsymbol{\theta}}^{n},\boldsymbol{\varpi}^{n+1}+\boldsymbol{\varpi}^{n}\right)_{L_2(\Omega)}$$
$$-\frac{\rho}{\Delta t}\sum_{n=0}^{m-1}\left(\dot{\boldsymbol{\theta}}^{n+1}-\dot{\boldsymbol{\theta}}^{n},\boldsymbol{\mathcal{E}}_2^n\right)_{L_2(\Omega)}-\frac{\rho}{\Delta t}\sum_{n=0}^{m-1}\left(\dot{\boldsymbol{\theta}}^{n+1}-\dot{\boldsymbol{\theta}}^{n},\boldsymbol{\mathcal{E}}_3^n\right)_{L_2(\Omega)}$$
$$+\frac{\rho}{2}\sum_{n=0}^{m-1}\left(\boldsymbol{\mathcal{E}}_1^n,\boldsymbol{\varpi}^{n+1}+\boldsymbol{\varpi}^{n}\right)_{L_2(\Omega)}-\rho\sum_{n=0}^{m-1}\left(\boldsymbol{\mathcal{E}}_1^n,\boldsymbol{\mathcal{E}}_2^n\right)_{L_2(\Omega)}-\rho\sum_{n=0}^{m-1}\left(\boldsymbol{\mathcal{E}}_1^n,\boldsymbol{\mathcal{E}}_3^n\right)_{L_2(\Omega)}$$
$$+\frac{\rho}{\Delta t}\sum_{n=0}^{m-1}\left(\boldsymbol{\varpi}^{n+1}-\boldsymbol{\varpi}^{n},\boldsymbol{\mathcal{E}}_2^n\right)_{L_2(\Omega)}+\frac{\rho}{\Delta t}\sum_{n=0}^{m-1}\left(\boldsymbol{\varpi}^{n+1}-\boldsymbol{\varpi}^{n},\boldsymbol{\mathcal{E}}_3^n\right)_{L_2(\Omega)}. \qquad (4.1.33)$$

To rewrite $a\left(\boldsymbol{\varsigma}_q^{n+1}+\boldsymbol{\varsigma}_q^{n},\boldsymbol{\chi}^{n+1}-\boldsymbol{\chi}^n\right)$, let us consider the difference of (4.1.16) and (4.1.9) such that for each q

$$\tau_q a\left(\frac{\dot{\boldsymbol{\psi}}_q^{\,n+1}+\dot{\boldsymbol{\psi}}_q^{\,n}}{2}-\frac{\boldsymbol{\Psi}_{hq}^{n+1}-\boldsymbol{\Psi}_{hq}^{n}}{\Delta t},\boldsymbol{v}\right)+\frac{1}{2}a\left((\boldsymbol{\psi}_q^{n+1}+\boldsymbol{\psi}_q^{n})-(\boldsymbol{\Psi}_{hq}^{n+1}+\boldsymbol{\Psi}_{hq}^{n}),\boldsymbol{v}\right)$$

$$
=\frac{\varphi_q}{2} a\left((\boldsymbol{u}^{n+1}+\boldsymbol{u}^n)-(\boldsymbol{U}_h^{n+1}+\boldsymbol{U}_h^n),\boldsymbol{v}\right).
$$

By setting $\boldsymbol{v}=\frac{\boldsymbol{\varsigma}_q^{n+1}-\boldsymbol{\varsigma}_q^n}{\Delta t}$ and taking summation for $n=0,\ldots,m-1$ with summation by parts, Galerkin orthogonality gives

$$
\begin{aligned}
&\frac{\varphi_q}{2\Delta t}\sum_{n=0}^{m-1} a\left(\boldsymbol{\chi}^{n+1}-\boldsymbol{\chi}^n,\boldsymbol{\varsigma}_q^{n+1}+\boldsymbol{\varsigma}_q^n\right)\\
&\quad=\frac{\varphi_q}{\Delta t} a\left(\boldsymbol{\chi}^m,\boldsymbol{\varsigma}_q^m\right)-\frac{\tau_q}{\Delta t^2}\sum_{n=0}^{m-1}\left\|\boldsymbol{\varsigma}_q^{n+1}-\boldsymbol{\varsigma}_q^n\right\|_V^2-\frac{1}{2\Delta t}\left\|\boldsymbol{\varsigma}_q^m\right\|_V^2+\frac{\tau_q}{\Delta t}a\left(\boldsymbol{E}_q^{m-1},\boldsymbol{\varsigma}_q^m\right)\\
&\qquad-\frac{\tau_q}{\Delta t}\sum_{n=0}^{m-2} a\left(\boldsymbol{E}_q^{n+1}-\boldsymbol{E}_q^n,\boldsymbol{\varsigma}_q^{n+1}\right).
\end{aligned}
$$

where

$$
\boldsymbol{E}_q(t)=\frac{\dot{\boldsymbol{\psi}}_q(t+\Delta t)+\dot{\boldsymbol{\psi}}_q(t)}{2}-\frac{\boldsymbol{\psi}_q(t+\Delta t)-\boldsymbol{\psi}_q(t)}{\Delta t},
$$

for each q. Imposing this result into (4.1.33), we can write

$$
\begin{aligned}
&\frac{\rho}{2}\left\|\boldsymbol{\varpi}^m\right\|_{L_2(\Omega)}^2+\frac{1}{2}\left\|\boldsymbol{\chi}^m\right\|_V^2+\frac{1}{2}\sum_{q=1}^{N_\varphi}\frac{1}{\varphi_q}\left\|\boldsymbol{\varsigma}_q^m\right\|_V^2+\Delta t\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi}\frac{\tau_q}{\varphi_q}\left\|\frac{\boldsymbol{\varsigma}_q^{n+1}-\boldsymbol{\varsigma}_q^n}{\Delta t}\right\|_V^2\\
=&\frac{\rho}{2}\left\|\boldsymbol{\varpi}^0\right\|_{L_2(\Omega)}^2+\frac{1}{2}\left\|\boldsymbol{\chi}^0\right\|_V^2+\frac{\rho}{2}\sum_{n=0}^{m-1}\left(\dot{\boldsymbol{\theta}}^{n+1}-\dot{\boldsymbol{\theta}}^n,\boldsymbol{\varpi}^{n+1}+\boldsymbol{\varpi}^n\right)_{L_2(\Omega)}\\
&-\rho\sum_{n=0}^{m-1}\left(\dot{\boldsymbol{\theta}}^{n+1}-\dot{\boldsymbol{\theta}}^n,\boldsymbol{\mathcal{E}}_2^n\right)_{L_2(\Omega)}-\rho\sum_{n=0}^{m-1}\left(\dot{\boldsymbol{\theta}}^{n+1}-\dot{\boldsymbol{\theta}}^n,\boldsymbol{\mathcal{E}}_3^n\right)_{L_2(\Omega)}\\
&+\frac{\rho}{2}\Delta t\sum_{n=0}^{m-1}\left(\boldsymbol{\mathcal{E}}_1^n,\boldsymbol{\varpi}^{n+1}+\boldsymbol{\varpi}^n\right)_{L_2(\Omega)}-\rho\Delta t\sum_{n=0}^{m-1}\left(\boldsymbol{\mathcal{E}}_1^n,\boldsymbol{\mathcal{E}}_2^n\right)_{L_2(\Omega)}-\rho\Delta t\sum_{n=0}^{m-1}\left(\boldsymbol{\mathcal{E}}_1^n,\boldsymbol{\mathcal{E}}_3^n\right)_{L_2(\Omega)}\\
&+\rho\sum_{n=0}^{m-1}\left(\boldsymbol{\varpi}^{n+1}-\boldsymbol{\varpi}^n,\boldsymbol{\mathcal{E}}_2^n\right)_{L_2(\Omega)}+\rho\sum_{n=0}^{m-1}\left(\boldsymbol{\varpi}^{n+1}-\boldsymbol{\varpi}^n,\boldsymbol{\mathcal{E}}_3^n\right)_{L_2(\Omega)}+\sum_{q=1}^{N_\varphi}a\left(\boldsymbol{\chi}^m,\boldsymbol{\varsigma}_q^m\right)\\
&+\sum_{q=1}^{N_\varphi}\frac{\tau_q}{\varphi_q}a\left(\boldsymbol{E}_q^{m-1},\boldsymbol{\varsigma}_q^m\right)-\sum_{n=0}^{m-2}\sum_{q=1}^{N_\varphi}\frac{\tau_q}{\varphi_q}a\left(\boldsymbol{E}_q^{n+1}-\boldsymbol{E}_q^n,\boldsymbol{\varsigma}_q^{n+1}\right).
\end{aligned}
\tag{4.1.34}
$$

Note that we suppose a sufficiently smooth $\boldsymbol{u}$ with respect to time hence Crank-Nicolson finite difference method provides the following bounds such that

$$
|\boldsymbol{\mathcal{E}}_1^n|,|\boldsymbol{\mathcal{E}}_2^n|,|\boldsymbol{\mathcal{E}}_3^n|,|\boldsymbol{E}_q^n|\leq C\Delta t^2,\ \forall q,
$$

for some positive C. Furthermore, use of the same arguments to estimate bounds for (2.3.19) turns out the similar result such that

$$
\frac{\rho}{2}\left\|\boldsymbol{\varpi}^m\right\|_{L_2(\Omega)}^2+\frac{\varphi_0}{4}\left\|\boldsymbol{\chi}^m\right\|_V^2+\sum_{q=1}^{N_\varphi}\frac{\varphi_0}{8\varphi_q^2N_\varphi+4\varphi_0\varphi_q}\left\|\boldsymbol{\varsigma}_q^m\right\|_V^2
$$

$$
\begin{aligned}
&+\Delta t \sum_{n=0}^{m-1} \sum_{q=1}^{N_\varphi} \frac{\tau_q}{\varphi_q} \left\| \frac{\boldsymbol{\varsigma}_q^{n+1} - \boldsymbol{\varsigma}_q^n}{\Delta t} \right\|_V^2 \\
\leq & \frac{\rho}{4(3+N_\varphi)} \max_{0\leq n\leq N} \|\boldsymbol{\varpi}^n\|_{L_2(\Omega)}^2 + \sum_{q=1}^{N_\varphi} \frac{1}{2(3+N_\varphi)} \frac{\varphi_0}{8\varphi_q^2 N_\varphi + 4\varphi_0\varphi_q} \max_{0\leq n\leq N} \|\boldsymbol{\varsigma}_q^n\|_V^2 \\
&+ C(h^{2(\min(k+1,s)-1)} + \Delta t^4),
\end{aligned}
$$

and thus

$$
\begin{aligned}
&\frac{\rho}{4} \max_{0\leq n\leq N} \|\boldsymbol{\varpi}^n\|_{L_2(\Omega)}^2 + \frac{\varphi_0}{4} \max_{0\leq n\leq N} \|\boldsymbol{\chi}^n\|_V^2 + \sum_{q=1}^{N_\varphi} \frac{\varphi_0}{16\varphi_q^2 N_\varphi + 8\varphi_0\varphi_q} \max_{0\leq n\leq N} \|\boldsymbol{\varsigma}_q^n\|_V^2 \\
&+\Delta t \sum_{n=0}^{m-1} \sum_{q=1}^{N_\varphi} \frac{\tau_q}{\varphi_q} \left\| \frac{\boldsymbol{\varsigma}_q^{n+1} - \boldsymbol{\varsigma}_q^n}{\Delta t} \right\|_V^2 \\
\leq & C(h^{2(\min(k+1,s)-1)} + \Delta t^4). \qquad (4.1.35)
\end{aligned}
$$

Consequently, we have

$$
\max_{0\leq n\leq N} \|\boldsymbol{\varpi}^n\|_{L_2(\Omega)} + \max_{0\leq n\leq N} \|\boldsymbol{\chi}^n\|_V \leq C(h^{\min(k+1,s)-1} + \Delta t^2).
$$

If elliptic regularity is equipped with, (4.1.35) could have optimal estimates and hence we can derive

$$
\max_{0\leq n\leq N} \|\boldsymbol{\varpi}^n\|_{L_2(\Omega)} + \max_{0\leq n\leq N} \|\boldsymbol{\chi}^n\|_V \leq C(h^{\min(k+1,s)} + \Delta t^2).
$$

□

In a similar way with Lemma 4.1, we can obtain the following bounds for the velocity form.

Lemma 4.2. *Suppose* $\boldsymbol{u} \in H^4(0,T;[H^s(\Omega)]^d) \cap W_\infty^1(0,T;[H^s(\Omega)]^d)$*. If the fully discrete solution satisfies* **(R2)***, then*

$$
\max_{0\leq n\leq N} \|\boldsymbol{\varpi}^n\|_{L_2(\Omega)} + \max_{0\leq n\leq N} \|\boldsymbol{\chi}^n\|_V \leq C(h^{\min(k+1,s)-1} + \Delta t^2).
$$

If we also assume elliptic regularity,

$$
\max_{0\leq n\leq N} \|\boldsymbol{\varpi}^n\|_{L_2(\Omega)} + \max_{0\leq n\leq N} \|\boldsymbol{\chi}^n\|_V \leq C(h^{\min(k+1,s)} + \Delta t^2).
$$

Proof. A proof will follow the same way in the proof of Lemma 2.13 but in vector-valued cases. Let us consider subtraction of (4.1.19) from (4.1.10). Then we have

$$
\frac{\rho}{\Delta t} \left(\boldsymbol{\varpi}^{n+1} - \boldsymbol{\varpi}^n, \boldsymbol{v}\right)_{L_2(\Omega)} + \frac{\varphi_0}{2} a\left(\boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^n, \boldsymbol{v}\right) + \frac{1}{2} \sum_{q=1}^{N_\varphi} a\left(\boldsymbol{\Upsilon}_q^{n+1} + \boldsymbol{\Upsilon}_q^n, \boldsymbol{v}\right)
$$

$$
=\frac{\rho}{\Delta t}\left(\dot{\boldsymbol{\theta}}^{n+1}-\dot{\boldsymbol{\theta}}^{n},\boldsymbol{v}\right)_{L_2(\Omega)}+\rho\left(\boldsymbol{\mathcal{E}}_1^n,\boldsymbol{v}\right)_{L_2(\Omega)}
$$

for any $\boldsymbol{v}\in\boldsymbol{V}^h$, where $\boldsymbol{\mathcal{E}}_1(t):=\frac{\ddot{\boldsymbol{u}}(t+\Delta t)+\ddot{\boldsymbol{u}}(t)}{2}-\frac{\dot{\boldsymbol{u}}(t+\Delta t)-\dot{\boldsymbol{u}}(t)}{\Delta t}$ by Galerkin orthogonality. Note that (4.1.14) implies

$$
\frac{\boldsymbol{\chi}^{n+1}-\boldsymbol{\chi}^{n}}{\Delta t}=\frac{\boldsymbol{\varpi}^{n+1}+\boldsymbol{\varpi}^{n}}{2}-\boldsymbol{\mathcal{E}}_2^n-\boldsymbol{\mathcal{E}}_3^n
$$

where $\boldsymbol{\mathcal{E}}_2(t):=\frac{\dot{\boldsymbol{\theta}}(t+\Delta t)+\dot{\boldsymbol{\theta}}(t)}{2}-\frac{\boldsymbol{\theta}(t+\Delta t)-\boldsymbol{\theta}(t)}{\Delta t}$, $\boldsymbol{\mathcal{E}}_3(t):=\frac{\boldsymbol{u}(t+\Delta t)-\boldsymbol{u}(t)}{\Delta t}-\frac{\dot{\boldsymbol{u}}(t+\Delta t)+\dot{\boldsymbol{u}}(t)}{2}$. Hence, when we put $\boldsymbol{v}=\frac{\boldsymbol{\chi}^{n+1}-\boldsymbol{\chi}^{n}}{\Delta t}$, we can obtain

$$
\begin{aligned}
&\frac{\rho}{\Delta t}\left(\boldsymbol{\varpi}^{n+1}-\boldsymbol{\varpi}^{n},\frac{\boldsymbol{\varpi}^{n+1}+\boldsymbol{\varpi}^{n}}{2}\right)_{L_2(\Omega)}+\frac{\varphi_0}{2}a\left(\boldsymbol{\chi}^{n+1}+\boldsymbol{\chi}^{n},\frac{\boldsymbol{\chi}^{n+1}-\boldsymbol{\chi}^{n}}{\Delta t}\right)\\
&\quad+\frac{1}{2}\sum_{q=1}^{N_\varphi}a\left(\boldsymbol{\Upsilon}_q^{n+1}+\boldsymbol{\Upsilon}_q^{n},\frac{\boldsymbol{\chi}^{n+1}-\boldsymbol{\chi}^{n}}{\Delta t}\right)\\
&=\frac{\rho}{2\Delta t}\left(\dot{\boldsymbol{\theta}}^{n+1}-\dot{\boldsymbol{\theta}}^{n},\boldsymbol{\varpi}^{n+1}+\boldsymbol{\varpi}^{n}\right)_{L_2(\Omega)}+\frac{\rho}{2}\left(\boldsymbol{\mathcal{E}}_1^n,\boldsymbol{\varpi}^{n+1}+\boldsymbol{\varpi}^{n}\right)_{L_2(\Omega)}\\
&\quad-\frac{\rho}{\Delta t}\left(\dot{\boldsymbol{\theta}}^{n+1}-\dot{\boldsymbol{\theta}}^{n},\boldsymbol{\mathcal{E}}_2^n\right)_{L_2(\Omega)}-\frac{\rho}{\Delta t}\left(\dot{\boldsymbol{\theta}}^{n+1}-\dot{\boldsymbol{\theta}}^{n},\boldsymbol{\mathcal{E}}_3^n\right)_{L_2(\Omega)}\\
&\quad-\rho\left(\boldsymbol{\mathcal{E}}_1^n,\boldsymbol{\mathcal{E}}_2^n\right)_{L_2(\Omega)}-\rho\left(\boldsymbol{\mathcal{E}}_1^n,\boldsymbol{\mathcal{E}}_3^n\right)_{L_2(\Omega)}.
\end{aligned}
\tag{4.1.36}
$$

In this manner, the difference of (4.1.11) and (4.1.20) gives

$$
\begin{aligned}
&\frac{\tau_q}{\Delta t}a\left(\boldsymbol{\Upsilon}_q^{n+1}-\boldsymbol{\Upsilon}_q^{n},\boldsymbol{v}\right)+\frac{1}{2}a\left(\boldsymbol{\Upsilon}_q^{n+1}+\boldsymbol{\Upsilon}_q^{n},\boldsymbol{v}\right)-\frac{\tau_q\varphi_q}{\Delta t}a\left(\boldsymbol{\chi}^{n+1}-\boldsymbol{\chi}^{n},\boldsymbol{v}\right)\\
&=\tau_q a\left(\boldsymbol{E}_q^n,\boldsymbol{v}\right)-\tau_q\varphi_q a\left(\boldsymbol{\mathcal{E}}_3^n,\boldsymbol{v}\right)
\end{aligned}
$$

where

$$
\boldsymbol{E}_q(t):=\frac{\dot{\boldsymbol{\zeta}}_q(t+\Delta t)+\dot{\boldsymbol{\zeta}}_q(t)}{2}-\frac{\boldsymbol{\zeta}_q(t+\Delta t)-\boldsymbol{\zeta}_q(t)}{\Delta t},
$$

for each q, $\forall\boldsymbol{v}\in\boldsymbol{V}^h$. A choice of $\boldsymbol{v}=\frac{\boldsymbol{\Upsilon}_q^{n+1}+\boldsymbol{\Upsilon}_q^{n}}{2}$ yields

$$
\begin{aligned}
&\frac{1}{2\Delta t}a\left(\boldsymbol{\chi}^{n+1}-\boldsymbol{\chi}^{n},\boldsymbol{\Upsilon}_q^{n+1}+\boldsymbol{\Upsilon}_q^{n}\right)\\
&\quad=\frac{1}{2\Delta t\varphi_q}\left(\left\|\boldsymbol{\Upsilon}_q^{n+1}\right\|_V^2-\left\|\boldsymbol{\Upsilon}_q^{n}\right\|_V^2\right)+\frac{1}{2\tau_q\varphi_q}\left\|\boldsymbol{\Upsilon}_q^{n+1}+\boldsymbol{\Upsilon}_q^{n}\right\|_V^2-\frac{1}{2\varphi_q}a\left(\boldsymbol{E}_q^n,\boldsymbol{\Upsilon}_q^{n+1}+\boldsymbol{\Upsilon}_q^{n}\right)\\
&\qquad+\frac{1}{2}a\left(\boldsymbol{\mathcal{E}}_3^n,\boldsymbol{\Upsilon}_q^{n+1}+\boldsymbol{\Upsilon}_q^{n}\right),
\end{aligned}
\tag{4.1.37}
$$

for each q. Now, we can derive

$$
\frac{\rho}{2}\left(\left\|\boldsymbol{\varpi}^{n+1}\right\|_{L_2(\Omega)}^2-\left\|\boldsymbol{\varpi}^{n}\right\|_{L_2(\Omega)}^2\right)+\frac{\varphi_0}{2}\left(\left\|\boldsymbol{\chi}^{n+1}\right\|_V^2-\left\|\boldsymbol{\chi}^{n}\right\|_V^2\right)
$$

$$
\begin{aligned}
&+\frac{1}{2}\sum_{q=1}^{N_\varphi}\frac{1}{\varphi_q}\left(\left\|\boldsymbol{\Upsilon}_q^{n+1}\right\|_V^2-\left\|\boldsymbol{\Upsilon}_q^{n}\right\|_V^2\right)+\Delta t\sum_{q=1}^{N_\varphi}\frac{1}{2\tau_q\varphi_q}\left\|\boldsymbol{\Upsilon}_q^{n+1}+\boldsymbol{\Upsilon}_q^{n}\right\|_V^2\\
=&\frac{\rho}{2}\left(\dot{\boldsymbol{\theta}}^{n+1}-\dot{\boldsymbol{\theta}}^{n},\boldsymbol{\varpi}^{n+1}+\boldsymbol{\varpi}^{n}\right)_{L_2(\Omega)}+\frac{\rho}{2}\Delta t\left(\boldsymbol{\mathcal{E}}_1^n,\boldsymbol{\varpi}^{n+1}+\boldsymbol{\varpi}^{n}\right)_{L_2(\Omega)}\\
&-\rho\left(\dot{\boldsymbol{\theta}}^{n+1}-\dot{\boldsymbol{\theta}}^{n},\boldsymbol{\mathcal{E}}_2^n\right)_{L_2(\Omega)}-\rho\left(\dot{\boldsymbol{\theta}}^{n+1}-\dot{\boldsymbol{\theta}}^{n},\boldsymbol{\mathcal{E}}_3^n\right)_{L_2(\Omega)}-\rho\Delta t\left(\boldsymbol{\mathcal{E}}_1^n,\boldsymbol{\mathcal{E}}_2^n\right)_{L_2(\Omega)}\\
&-\rho\Delta t\left(\boldsymbol{\mathcal{E}}_1^n,\boldsymbol{\mathcal{E}}_3^n\right)_{L_2(\Omega)}+\frac{\Delta t}{2}\sum_{q=1}^{N_\varphi}\frac{1}{\varphi_q}a\left(\boldsymbol{E}_q^n,\boldsymbol{\Upsilon}_q^{n+1}+\boldsymbol{\Upsilon}_q^{n}\right)-\frac{\Delta t}{2}\sum_{q=1}^{N_\varphi}a\left(\boldsymbol{\mathcal{E}}_3^n,\boldsymbol{\Upsilon}_q^{n+1}+\boldsymbol{\Upsilon}_q^{n}\right),
\end{aligned}
$$

by substitution of (4.1.37) into (4.1.36) and multiplication by Δt. Summing this equation from $n=0$ to $n=m-1$ for $m\in\mathbb{N},\ m\le N$, allows us to have

$$
\begin{aligned}
&\frac{\rho}{2}\left\|\boldsymbol{\varpi}^m\right\|_{L_2(\Omega)}^2+\frac{\varphi_0}{2}\left\|\boldsymbol{\chi}^m\right\|_V^2+\frac{1}{2}\sum_{q=1}^{N_\varphi}\frac{1}{\varphi_q}\left\|\boldsymbol{\Upsilon}_q^m\right\|_V^2+\Delta t\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi}\frac{1}{2\tau_q\varphi_q}\left\|\boldsymbol{\Upsilon}_q^{n+1}+\boldsymbol{\Upsilon}_q^{n}\right\|_V^2\\
=&\frac{\rho}{2}\left\|\boldsymbol{\varpi}^0\right\|_{L_2(\Omega)}^2+\frac{\varphi_0}{2}\left\|\boldsymbol{\chi}^0\right\|_V^2+\frac{1}{2}\sum_{q=1}^{N_\varphi}\frac{1}{\varphi_q}\left\|\boldsymbol{\Upsilon}_q^0\right\|_V^2+\frac{\rho}{2}\sum_{n=0}^{m-1}\left(\dot{\boldsymbol{\theta}}^{n+1}-\dot{\boldsymbol{\theta}}^{n},\boldsymbol{\varpi}^{n+1}+\boldsymbol{\varpi}^{n}\right)_{L_2(\Omega)}\\
&+\frac{\rho}{2}\Delta t\sum_{n=0}^{m-1}\left(\boldsymbol{\mathcal{E}}_1^n,\boldsymbol{\varpi}^{n+1}+\boldsymbol{\varpi}^{n}\right)_{L_2(\Omega)}-\rho\sum_{n=0}^{m-1}\left(\dot{\boldsymbol{\theta}}^{n+1}-\dot{\boldsymbol{\theta}}^{n},\boldsymbol{\mathcal{E}}_2^n\right)_{L_2(\Omega)}\\
&-\rho\sum_{n=0}^{m-1}\left(\dot{\boldsymbol{\theta}}^{n+1}-\dot{\boldsymbol{\theta}}^{n},\boldsymbol{\mathcal{E}}_3^n\right)_{L_2(\Omega)}-\rho\Delta t\sum_{n=0}^{m-1}\left(\boldsymbol{\mathcal{E}}_1^n,\boldsymbol{\mathcal{E}}_2^n\right)_{L_2(\Omega)}-\rho\Delta t\sum_{n=0}^{m-1}\left(\boldsymbol{\mathcal{E}}_1^n,\boldsymbol{\mathcal{E}}_3^n\right)_{L_2(\Omega)}\\
&+\frac{\Delta t}{2}\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi}\frac{1}{\varphi_q}a\left(\boldsymbol{E}_q^n,\boldsymbol{\Upsilon}_q^{n+1}+\boldsymbol{\Upsilon}_q^{n}\right)-\frac{\Delta t}{2}\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi}a\left(\boldsymbol{\mathcal{E}}_3^n,\boldsymbol{\Upsilon}_q^{n+1}+\boldsymbol{\Upsilon}_q^{n}\right). \qquad (4.1.38)
\end{aligned}
$$

As seen in the proof of Lemma 2.13, use of Crank-Nicolson finite difference approximations, (4.1.31), (4.1.32), Cauchy-Schwarz inequalities, Young's inequalities and other techniques implies the following bound for (4.1.38) such that

$$
\begin{aligned}
&\frac{\rho}{4}\max_{0\le n\le N}\left\|\boldsymbol{\varpi}^n\right\|_{L_2(\Omega)}^2+\frac{\varphi_0}{2}\max_{0\le n\le N}\left\|\boldsymbol{\chi}^n\right\|_V^2+\sum_{q=1}^{N_\varphi}\frac{1}{2\varphi_q}\left\|\boldsymbol{\Upsilon}_q^m\right\|_V^2\\
&\quad+\Delta t\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi}\frac{1}{4\tau_q\varphi_q}\left\|\boldsymbol{\Upsilon}_q^{n+1}+\boldsymbol{\Upsilon}_q^{n}\right\|_V^2\\
\le&O(h^{2(r-1)}+\Delta t^4),
\end{aligned}
$$

where $r=\min(k+1,s)$. The details follow the bounds for (2.3.34). Consequently, we have

$$
\max_{0\le n\le N}\left\|\boldsymbol{\varpi}^n\right\|_{L_2(\Omega)}+\max_{0\le n\le N}\left\|\boldsymbol{\chi}^n\right\|_V\le C(h^{r-1}+\Delta t^2),
$$

for some positive C. Moreover, if elliptic regularity is given, it holds

$$\max_{0\leq n\leq N} \|\boldsymbol{\varpi}^n\|_{L_2(\Omega)} + \max_{0\leq n\leq N} \|\boldsymbol{\chi}^n\|_V \leq C(h^r + \Delta t^2).$$

□

From Lemma 4.1 and 4.2, error bounds for both fully discrete formulations are observed as we follow.

Theorem 4.3. *Suppose* $\boldsymbol{u} \in H^4(0,T;[H^s(\Omega)]^d) \cap W^1_\infty(0,T;[H^s(\Omega)]^d)$ *and the discrete solutions satisfy either* **(R1)** *or* **(R2)**. *Then we have*

$$\max_{0\leq n\leq N} \|\boldsymbol{u}(t_n) - \boldsymbol{U}_h^n\|_V \leq C(h^{\min(k+1,s)-1} + \Delta t^2)$$

and

$$\max_{0\leq n\leq N} \|\dot{\boldsymbol{u}}(t_n) - \boldsymbol{W}_h^n\|_{L_2(\Omega)} \leq C(h^{\min(k+1,s)-1} + \Delta t^2)$$

for some positive C. With elliptic regularity, it is also observed that

$$\max_{0\leq n\leq N} \|\dot{\boldsymbol{u}}(t_n) - \boldsymbol{W}_h^n\|_{L_2(\Omega)} \leq C(h^{\min(k+1,s)} + \Delta t^2)$$

for some positive C. In addition, we can also see L_2 error estimates of a displacement vector

$$\max_{0\leq n\leq N} \|\boldsymbol{u}(t_n) - \boldsymbol{U}_h^n\|_{L_2(\Omega)} \leq C(h^{\min(k+1,s)-1} + \Delta t^2).$$

If elliptic regularity is given, it shows

$$\max_{0\leq n\leq N} \|\boldsymbol{u}(t_n) - \boldsymbol{U}_h^n\|_{L_2(\Omega)} \leq C(h^{\min(k+1,s)} + \Delta t^2).$$

Proof. A proof is shown in the same way in scalar error estimates theorems by using triangular inequalities, for example,

$$\|\boldsymbol{u}(t_n) - \boldsymbol{U}_h^n\|_V = \|\boldsymbol{u}(t_n) - \boldsymbol{R}\boldsymbol{u}(t_n) - (\boldsymbol{U}_h^n - \boldsymbol{R}\boldsymbol{u}(t_n))\|_V \leq \|\boldsymbol{\theta}(t_n)\|_V + \|\boldsymbol{\chi}^n\|_V .$$

The details follow as seen in Theorems 2.14, 2.17 and Corollaries 2.1, 2.2. □

Due to the norm equivalence between H^1 norm and the energy norm, Theorem 4.3 shows optimal H^1 error estimates. Also, by elliptic regularity estimates, L_2 error estimates could be optimised. Furthermore, stability bounds and error bounds have a constant bound C which is not governed by Grönwall inequalities hence the constant does not increase exponentially by the final time T.

4.1.2 Numerical Experiments

Let a strong solution be a sufficiently smooth with respect to time and spatial domain such that

$$\boldsymbol{u}(x, y, t) = (xye^{1-t}, \cos(t)\sin(xy))$$

on $\Omega = [0,1] \times [0,1]$. Suppose there are two internal variables and their coefficients are given by

$$\varphi_0 = 0.5,\ \varphi_1 = 0.1,\ \varphi_2 = 0.4,\ \tau_1 = 0.5,\ \tau_2 = 1.5.$$

Moreover, we assume an identity fourth order tensor as our $\underline{\boldsymbol{D}}$ so that $\underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}} = \underline{\boldsymbol{\varepsilon}}$. Here, we have a homogeneous Dirichlet boundary $\Gamma_D = \{(x, y) \in \partial\Omega \mid x = 0 \text{ or } y = 0\}$ and then other data such as traction, initial conditions and so forth, can be computed.

Note that the energy norm $\|\cdot\|_V$ is equivalent to $\|\cdot\|_{H^1(\Omega)}$ hence we can obtain H^1 error estimates. We consider the exact errors between the exact solution $\boldsymbol{u}$ and numerical solutions satisfying **(R1)** or **(R2)** with respect to H^1 norm and L_2 norm, respectively. Hence let us define first

$$\boldsymbol{e}_h^n := \boldsymbol{u}(t_n) - \boldsymbol{U}_h^n, \qquad \tilde{\boldsymbol{e}}_h^n := \dot{\boldsymbol{u}}(t_n) - \boldsymbol{W}_h^n,$$

where $\boldsymbol{U}_h^n$ and $\boldsymbol{W}_h^n$ are the numerical solutions to **(R1)** or **(R2)**, for $n = 0, \ldots, N$.

Code implementation has been constructed with based on FEniCS as similar as scalar CG. Due to Theorem 4.1 and Theorem 4.2, approximate solutions exist uniquely and so we could compute the exact error for the final time. In other words, we will consider $\|\boldsymbol{e}_h^N\|_{H^1(\Omega)}$, $\|\tilde{\boldsymbol{e}}_h^N\|_{L_2(\Omega)}$ and $\|\boldsymbol{e}_h^N\|_{L_2(\Omega)}$. Since our spatial domain Ω is convex, elliptic regularity is given and hence we would expect optimal L_2 estimates as well as optimal H_1 estimates by Theorem 4.3.

As seen in Tables 4.1 and 4.2, it is observed that

$$\|\boldsymbol{e}_h^N\|_{H^1(\Omega)} = O(h + \Delta t^2), \qquad \|\tilde{\boldsymbol{e}}_h^N\|_{L_2(\Omega)},\ \|\boldsymbol{e}_h^N\|_{L_2(\Omega)} = O(h^2 + \Delta t^2)$$

when we take into account linear Lagrange finite element. On the other hand, Tables 4.3 and 4.4 indicate higher order of accuracy with quadratic polynomial basis. Thus, it shows that

$$\|\boldsymbol{e}_h^N\|_{H^1(\Omega)} = O(h^2 + \Delta t^2), \qquad \|\tilde{\boldsymbol{e}}_h^N\|_{L_2(\Omega)},\ \|\boldsymbol{e}_h^N\|_{L_2(\Omega)} = O(h^3 + \Delta t^2).$$

$\|\boldsymbol{e}_h^N\|_{H^1(\Omega)}$

h \ Δt	1/4	1/8	1/16	1/32	1/64	1/128	1/256	1/512
1/4	1.6489×10^{-1}	1.6355×10^{-1}	1.6311×10^{-1}	1.6299×10^{-1}	1.6295×10^{-1}	1.6295×10^{-1}	1.6294×10^{-1}	1.6294×10^{-1}
1/8	8.2304×10^{-2}	8.1498×10^{-2}	8.1376×10^{-2}	8.1353×10^{-2}	8.1349×10^{-2}	8.1348×10^{-2}	8.1347×10^{-2}	8.1347×10^{-2}
1/16	4.1094×10^{-2}	4.0269×10^{-2}	4.0171×10^{-2}	4.0151×10^{-2}	4.0146×10^{-2}	4.0144×10^{-2}	4.0144×10^{-2}	4.0144×10^{-2}
1/32	2.1299×10^{-2}	2.0055×10^{-2}	1.9949×10^{-2}	1.9936×10^{-2}	1.9934×10^{-2}	1.9933×10^{-2}	1.9933×10^{-2}	1.9933×10^{-2}
1/64	1.2249×10^{-2}	1.0117×10^{-2}	9.9522×10^{-3}	9.9389×10^{-3}	9.9373×10^{-3}	9.9370×10^{-3}	9.9369×10^{-3}	9.9369×10^{-3}
1/128	8.6420×10^{-3}	5.2849×10^{-3}	4.9856×10^{-3}	4.9648×10^{-3}	4.9631×10^{-3}	4.9629×10^{-3}	4.9629×10^{-3}	4.9629×10^{-3}
1/256	7.4763×10^{-3}	3.0618×10^{-3}	2.5218×10^{-3}	2.4833×10^{-3}	2.4807×10^{-3}	2.4805×10^{-3}	2.4805×10^{-3}	2.4805×10^{-3}
1/512	7.1556×10^{-3}	2.1773×10^{-3}	1.3190×10^{-3}	1.2453×10^{-3}	1.2404×10^{-3}	1.2401×10^{-3}	1.2401×10^{-3}	1.2401×10^{-3}

$\|\tilde{\boldsymbol{e}}_h^N\|_{L_2(\Omega)}$

h \ Δt	1/4	1/8	1/16	1/32	1/64	1/128	1/256	1/512
1/4	3.0930×10^{-2}	2.7442×10^{-2}	2.6732×10^{-2}	2.6553×10^{-2}	2.6508×10^{-2}	2.6497×10^{-2}	2.6494×10^{-2}	2.6493×10^{-2}
1/8	1.3233×10^{-2}	8.7675×10^{-3}	8.0053×10^{-3}	7.8566×10^{-3}	7.8221×10^{-3}	7.8140×10^{-3}	7.8120×10^{-3}	7.8115×10^{-3}
1/16	9.1264×10^{-3}	3.3978×10^{-3}	2.3217×10^{-3}	2.1459×10^{-3}	2.1115×10^{-3}	2.1033×10^{-3}	2.1013×10^{-3}	2.1008×10^{-3}
1/32	8.4072×10^{-3}	2.2946×10^{-3}	8.5664×10^{-4}	5.9292×10^{-4}	5.5055×10^{-4}	5.4226×10^{-4}	5.4032×10^{-4}	5.3984×10^{-4}
1/64	8.2646×10^{-3}	2.1132×10^{-3}	5.7435×10^{-4}	2.1473×10^{-4}	1.4924×10^{-4}	1.3879×10^{-4}	1.3674×10^{-4}	1.3627×10^{-4}
1/128	8.2317×10^{-3}	2.0778×10^{-3}	5.2920×10^{-4}	1.4364×10^{-4}	5.3720×10^{-5}	3.7383×10^{-5}	3.4782×10^{-5}	3.4272×10^{-5}
1/256	8.2236×10^{-3}	2.0697×10^{-3}	5.2045×10^{-4}	1.3237×10^{-4}	3.5912×10^{-5}	1.3432×10^{-5}	9.3506×10^{-6}	8.7015×10^{-6}
1/512	8.2216×10^{-3}	2.0677×10^{-3}	5.1844×10^{-4}	1.3018×10^{-4}	3.3096×10^{-5}	8.9781×10^{-6}	3.3580×10^{-6}	2.3382×10^{-6}

$\|\boldsymbol{e}_h^N\|_{L_2(\Omega)}$

h \ Δt	1/4	1/8	1/16	1/32	1/64	1/128	1/256	1/512
1/4	1.6293×10^{-2}	1.6013×10^{-2}	1.5988×10^{-2}	1.5986×10^{-2}	1.5986×10^{-2}	1.5986×10^{-2}	1.5986×10^{-2}	1.5986×10^{-2}
1/8	5.3705×10^{-3}	4.8472×10^{-3}	4.8246×10^{-3}	4.8272×10^{-3}	4.8284×10^{-3}	4.8287×10^{-3}	4.8287×10^{-3}	4.8288×10^{-3}
1/16	2.7020×10^{-3}	1.4108×10^{-3}	1.2961×10^{-3}	1.2933×10^{-3}	1.2943×10^{-3}	1.2947×10^{-3}	1.2948×10^{-3}	1.2948×10^{-3}
1/32	2.4211×10^{-3}	6.9237×10^{-4}	3.5896×10^{-4}	3.3150×10^{-4}	3.3098×10^{-4}	3.3126×10^{-4}	3.3136×10^{-4}	3.3138×10^{-4}
1/64	2.4081×10^{-3}	6.2640×10^{-4}	1.7417×10^{-4}	9.0259×10^{-5}	8.3482×10^{-5}	8.3363×10^{-5}	8.3437×10^{-5}	8.3462×10^{-5}
1/128	2.4089×10^{-3}	6.2446×10^{-4}	1.5787×10^{-4}	4.3612×10^{-5}	2.2605×10^{-5}	2.0917×10^{-5}	2.0888×10^{-5}	2.0907×10^{-5}
1/256	2.4094×10^{-3}	6.2496×10^{-4}	1.5746×10^{-4}	3.9544×10^{-5}	1.0908×10^{-5}	5.6544×10^{-6}	5.2327×10^{-6}	5.2254×10^{-6}
1/512	2.4095×10^{-3}	6.2514×10^{-4}	1.5760×10^{-4}	3.9447×10^{-5}	9.8909×10^{-6}	2.7272×10^{-6}	1.4138×10^{-6}	1.3083×10^{-6}

Table 4.1: Errors of **(R1)** for linear polynomial basis

$\|\boldsymbol{e}_h^N\|_{H^1(\Omega)}$

h \ Δt	1/4	1/8	1/16	1/32	1/64	1/128	1/256	1/512
1/4	1.6469×10^{-1}	1.6352×10^{-1}	1.6310×10^{-1}	1.6298×10^{-1}	1.6295×10^{-1}	1.6295×10^{-1}	1.6294×10^{-1}	1.6294×10^{-1}
1/8	8.2096×10^{-2}	8.1477×10^{-2}	8.1373×10^{-2}	8.1353×10^{-2}	8.1349×10^{-2}	8.1348×10^{-2}	8.1347×10^{-2}	8.1347×10^{-2}
1/16	4.0740×10^{-2}	4.0244×10^{-2}	4.0168×10^{-2}	4.0151×10^{-2}	4.0145×10^{-2}	4.0144×10^{-2}	4.0144×10^{-2}	4.0144×10^{-2}
1/32	2.0625×10^{-2}	2.0009×10^{-2}	1.9946×10^{-2}	1.9936×10^{-2}	1.9934×10^{-2}	1.9933×10^{-2}	1.9933×10^{-2}	1.9933×10^{-2}
1/64	1.1039×10^{-2}	1.0027×10^{-2}	9.9465×10^{-3}	9.9385×10^{-3}	9.9373×10^{-3}	9.9370×10^{-3}	9.9369×10^{-3}	9.9369×10^{-3}
1/128	6.8191×10^{-3}	5.1107×10^{-3}	4.9742×10^{-3}	4.9641×10^{-3}	4.9631×10^{-3}	4.9629×10^{-3}	4.9629×10^{-3}	4.9629×10^{-3}
1/256	5.2642×10^{-3}	2.7504×10^{-3}	2.4992×10^{-3}	2.4819×10^{-3}	2.4806×10^{-3}	2.4805×10^{-3}	2.4805×10^{-3}	2.4805×10^{-3}
1/512	4.7977×10^{-3}	1.7119×10^{-3}	1.2753×10^{-3}	1.2424×10^{-3}	1.2403×10^{-3}	1.2401×10^{-3}	1.2401×10^{-3}	1.2401×10^{-3}

$\|\tilde{\boldsymbol{e}}_h^N\|_{L_2(\Omega)}$

h \ Δt	1/4	1/8	1/16	1/32	1/64	1/128	1/256	1/512
1/4	3.0131×10^{-2}	2.7299×10^{-2}	2.6700×10^{-2}	2.6546×10^{-2}	2.6506×10^{-2}	2.6496×10^{-2}	2.6494×10^{-2}	2.6493×10^{-2}
1/8	1.2023×10^{-2}	8.5887×10^{-3}	7.9746×10^{-3}	7.8499×10^{-3}	7.8205×10^{-3}	7.8136×10^{-3}	7.8119×10^{-3}	7.8115×10^{-3}
1/16	7.5915×10^{-3}	3.1027×10^{-3}	2.2797×10^{-3}	2.1389×10^{-3}	2.1100×10^{-3}	2.1029×10^{-3}	2.1012×10^{-3}	2.1008×10^{-3}
1/32	6.8013×10^{-3}	1.9092×10^{-3}	7.8348×10^{-4}	5.8267×10^{-4}	5.4883×10^{-4}	5.4189×10^{-4}	5.4023×10^{-4}	5.3982×10^{-4}
1/64	6.6477×10^{-3}	1.7090×10^{-3}	4.7779×10^{-4}	1.9651×10^{-4}	1.4669×10^{-4}	1.3836×10^{-4}	1.3665×10^{-4}	1.3625×10^{-4}
1/128	6.6126×10^{-3}	1.6707×10^{-3}	4.2784×10^{-4}	1.1948×10^{-4}	4.9167×10^{-5}	3.6749×10^{-5}	3.4676×10^{-5}	3.4249×10^{-5}
1/256	6.6041×10^{-3}	1.6620×10^{-3}	4.1834×10^{-4}	1.0700×10^{-4}	2.9872×10^{-5}	1.2294×10^{-5}	9.1921×10^{-6}	8.6748×10^{-6}
1/512	6.6019×10^{-3}	1.6599×10^{-3}	4.1618×10^{-4}	1.0463×10^{-4}	2.6753×10^{-5}	7.4678×10^{-6}	3.0734×10^{-6}	2.2983×10^{-6}

$\|\boldsymbol{e}_h^N\|_{L_2(\Omega)}$

h \ Δt	1/4	1/8	1/16	1/32	1/64	1/128	1/256	1/512
1/4	1.6593×10^{-2}	1.6112×10^{-2}	1.6014×10^{-2}	1.5993×10^{-2}	1.5987×10^{-2}	1.5986×10^{-2}	1.5986×10^{-2}	1.5986×10^{-2}
1/8	5.3872×10^{-3}	4.9188×10^{-3}	4.8473×10^{-3}	4.8332×10^{-3}	4.8299×10^{-3}	4.8290×10^{-3}	4.8288×10^{-3}	4.8288×10^{-3}
1/16	2.1847×10^{-3}	1.4124×10^{-3}	1.3134×10^{-3}	1.2987×10^{-3}	1.2957×10^{-3}	1.2950×10^{-3}	1.2949×10^{-3}	1.2948×10^{-3}
1/32	1.6247×10^{-3}	5.5319×10^{-4}	3.5921×10^{-4}	3.3574×10^{-4}	3.3230×10^{-4}	3.3161×10^{-4}	3.3145×10^{-4}	3.3141×10^{-4}
1/64	1.5501×10^{-3}	4.1750×10^{-4}	1.3880×10^{-4}	9.0311×10^{-5}	8.4534×10^{-5}	8.3693×10^{-5}	8.3524×10^{-5}	8.3484×10^{-5}
1/128	1.5373×10^{-3}	4.0086×10^{-4}	1.0506×10^{-4}	3.4735×10^{-5}	2.2617×10^{-5}	2.1180×10^{-5}	2.0971×10^{-5}	2.0929×10^{-5}
1/256	1.5345×10^{-3}	3.9814×10^{-4}	1.0102×10^{-4}	2.6307×10^{-5}	8.6862×10^{-6}	5.6573×10^{-6}	5.2983×10^{-6}	5.2462×10^{-6}
1/512	1.5338×10^{-3}	3.9755×10^{-4}	1.0037×10^{-4}	2.5305×10^{-5}	6.5793×10^{-6}	2.1717×10^{-6}	1.4146×10^{-6}	1.3249×10^{-6}

Table 4.2: Errors of **(R2)** for linear polynomial basis

$\|\boldsymbol{e}_h^N\|_{H^1(\Omega)}$

h \ Δt	1/4	1/8	1/16	1/32	1/64	1/128	1/256	1/512
1/4	7.8614×10^{-3}	3.8924×10^{-3}	3.4808×10^{-3}	3.4521×10^{-3}	3.4500×10^{-3}	3.4498×10^{-3}	3.4498×10^{-3}	3.4498×10^{-3}
1/8	7.1090×10^{-3}	2.0021×10^{-3}	1.0057×10^{-3}	9.0693×10^{-4}	9.0024×10^{-4}	8.9979×10^{-4}	8.9975×10^{-4}	8.9975×10^{-4}
1/16	7.0500×10^{-3}	1.8024×10^{-3}	5.0317×10^{-4}	2.5576×10^{-4}	2.3164×10^{-4}	2.3004×10^{-4}	2.2993×10^{-4}	2.2993×10^{-4}
1/32	7.0457×10^{-3}	1.7888×10^{-3}	4.5121×10^{-4}	1.2609×10^{-4}	6.4553×10^{-5}	5.8596×10^{-5}	5.8202×10^{-5}	5.8178×10^{-5}
1/64	7.0455×10^{-3}	1.7879×10^{-3}	4.4768×10^{-4}	1.1281×10^{-4}	3.1563×10^{-5}	1.6222×10^{-5}	1.4741×10^{-5}	1.4644×10^{-5}
1/128	7.0454×10^{-3}	1.7878×10^{-3}	4.4746×10^{-4}	1.1192×10^{-4}	2.8203×10^{-5}	7.8962×10^{-6}	4.0663×10^{-6}	3.6975×10^{-6}
1/256	7.0454×10^{-3}	1.7878×10^{-3}	4.4744×10^{-4}	1.1186×10^{-4}	2.7978×10^{-5}	7.0509×10^{-6}	1.9748×10^{-6}	1.0181×10^{-6}

$\|\tilde{\boldsymbol{e}}_h^N\|_{L_2(\Omega)}$

h \ Δt	1/4	1/8	1/16	1/32	1/64	1/128	1/256	1/512
1/4	8.2211×10^{-3}	2.0684×10^{-3}	5.3806×10^{-4}	2.1098×10^{-4}	1.7458×10^{-4}	1.7296×10^{-4}	1.7309×10^{-4}	1.7316×10^{-4}
1/8	8.2209×10^{-3}	2.0668×10^{-3}	5.1776×10^{-4}	1.3084×10^{-4}	3.8747×10^{-5}	2.3366×10^{-5}	2.2174×10^{-5}	2.2127×10^{-5}
1/16	8.2210×10^{-3}	2.0670×10^{-3}	5.1777×10^{-4}	1.2951×10^{-4}	3.2466×10^{-5}	8.5268×10^{-6}	3.4232×10^{-6}	2.8274×10^{-6}
1/32	8.2210×10^{-3}	2.0670×10^{-3}	5.1778×10^{-4}	1.2952×10^{-4}	3.2384×10^{-5}	8.1016×10^{-6}	2.0519×10^{-6}	6.1388×10^{-7}
1/64	8.2209×10^{-3}	2.0670×10^{-3}	5.1778×10^{-4}	1.2952×10^{-4}	3.2385×10^{-5}	8.0965×10^{-6}	2.0245×10^{-6}	5.0801×10^{-7}
1/128	8.2209×10^{-3}	2.0670×10^{-3}	5.1778×10^{-4}	1.2952×10^{-4}	3.2385×10^{-5}	8.0965×10^{-6}	2.0242×10^{-6}	5.0627×10^{-7}
1/256	8.2209×10^{-3}	2.0670×10^{-3}	5.1778×10^{-4}	1.2952×10^{-4}	3.2385×10^{-5}	8.0964×10^{-6}	2.0241×10^{-6}	5.0620×10^{-7}

$\|\boldsymbol{e}_h^N\|_{L_2(\Omega)}$

h \ Δt	1/4	1/8	1/16	1/32	1/64	1/128	1/256	1/512
1/4	2.4088×10^{-3}	6.3134×10^{-4}	1.9133×10^{-4}	1.2016×10^{-4}	1.1517×10^{-4}	1.1509×10^{-4}	1.1514×10^{-4}	1.1516×10^{-4}
1/8	2.4098×10^{-3}	6.2517×10^{-4}	1.5805×10^{-4}	4.1759×10^{-5}	1.7276×10^{-5}	1.4539×10^{-5}	1.4379×10^{-5}	1.4376×10^{-5}
1/16	2.4097×10^{-3}	6.2521×10^{-4}	1.5767×10^{-4}	3.9525×10^{-5}	1.0025×10^{-5}	3.0402×10^{-6}	1.8942×10^{-6}	1.8021×10^{-6}
1/32	2.4096×10^{-3}	6.2521×10^{-4}	1.5767×10^{-4}	3.9501×10^{-5}	9.8821×10^{-6}	2.4797×10^{-6}	6.5655×10^{-7}	2.7280×10^{-7}
1/64	2.4096×10^{-3}	6.2521×10^{-4}	1.5767×10^{-4}	3.9501×10^{-5}	9.8805×10^{-6}	2.4706×10^{-6}	6.1827×10^{-7}	1.5710×10^{-7}
1/128	2.4096×10^{-3}	6.2521×10^{-4}	1.5767×10^{-4}	3.9501×10^{-5}	9.8805×10^{-6}	2.4705×10^{-6}	6.1769×10^{-7}	1.5464×10^{-7}
1/256	2.4096×10^{-3}	6.2521×10^{-4}	1.5767×10^{-4}	3.9501×10^{-5}	9.8805×10^{-6}	2.4705×10^{-6}	6.1768×10^{-7}	1.5461×10^{-7}

Table 4.3: Errors of **(R1)** for quadratic polynomial basis

$\left\|\boldsymbol{e}_h^N\right\|_{H^1(\Omega)}$

h \ Δt	1/4	1/8	1/16	1/32	1/64	1/128	1/256	1/512
1/4	5.8188×10^{-3}	3.6590×10^{-3}	3.4658×10^{-3}	3.4514×10^{-3}	3.4500×10^{-3}	3.4498×10^{-3}	3.4498×10^{-3}	3.4498×10^{-3}
1/8	4.7255×10^{-3}	1.4850×10^{-3}	9.4830×10^{-4}	9.0316×10^{-4}	9.0003×10^{-4}	8.9978×10^{-4}	8.9975×10^{-4}	8.9975×10^{-4}
1/16	4.6383×10^{-3}	1.2000×10^{-3}	3.7423×10^{-4}	2.4158×10^{-4}	2.3070×10^{-4}	2.2998×10^{-4}	2.2993×10^{-4}	2.2993×10^{-4}
1/32	4.6323×10^{-3}	1.1789×10^{-3}	3.0058×10^{-4}	9.3944×10^{-5}	6.1037×10^{-5}	5.8361×10^{-5}	5.8188×10^{-5}	5.8177×10^{-5}
1/64	4.6319×10^{-3}	1.1775×10^{-3}	2.9523×10^{-4}	7.5169×10^{-5}	2.3539×10^{-5}	1.5346×10^{-5}	1.4683×10^{-5}	1.4640×10^{-5}
1/128	4.6319×10^{-3}	1.1774×10^{-3}	2.9489×10^{-4}	7.3819×10^{-5}	1.8794×10^{-5}	5.8920×10^{-6}	3.8481×10^{-6}	3.6828×10^{-6}
1/256	4.6319×10^{-3}	1.1774×10^{-3}	2.9487×10^{-4}	7.3733×10^{-5}	1.8455×10^{-5}	4.6989×10^{-6}	1.4740×10^{-6}	9.6359×10^{-7}

$\left\|\tilde{\boldsymbol{e}}_h^N\right\|_{L_2(\Omega)}$

h \ Δt	1/4	1/8	1/16	1/32	1/64	1/128	1/256	1/512
1/4	6.5989×10^{-3}	1.6605×10^{-3}	4.4170×10^{-4}	1.9717×10^{-4}	1.7376×10^{-4}	1.7296×10^{-4}	1.7310×10^{-4}	1.7316×10^{-4}
1/8	6.6009×10^{-3}	1.6588×10^{-3}	4.1548×10^{-4}	1.0564×10^{-4}	3.3656×10^{-5}	2.2890×10^{-5}	2.2148×10^{-5}	2.2127×10^{-5}
1/16	6.6012×10^{-3}	1.6592×10^{-3}	4.1546×10^{-4}	1.0392×10^{-4}	2.6090×10^{-5}	7.0309×10^{-6}	3.2062×10^{-6}	2.8122×10^{-6}
1/32	6.6012×10^{-3}	1.6592×10^{-3}	4.1548×10^{-4}	1.0392×10^{-4}	2.5982×10^{-5}	6.5026×10^{-6}	1.6589×10^{-6}	5.3475×10^{-7}
1/64	6.6012×10^{-3}	1.6592×10^{-3}	4.1548×10^{-4}	1.0392×10^{-4}	2.5983×10^{-5}	6.4959×10^{-6}	1.6245×10^{-6}	4.0849×10^{-7}
1/128	6.6012×10^{-3}	1.6592×10^{-3}	4.1548×10^{-4}	1.0392×10^{-4}	2.5983×10^{-5}	6.4959×10^{-6}	1.6240×10^{-6}	4.0630×10^{-7}
1/256	6.6012×10^{-3}	1.6592×10^{-3}	4.1548×10^{-4}	1.0392×10^{-4}	2.5983×10^{-5}	6.4959×10^{-6}	1.6240×10^{-6}	4.0627×10^{-7}

$\left\|\boldsymbol{e}_h^N\right\|_{L_2(\Omega)}$

h \ Δt	1/4	1/8	1/16	1/32	1/64	1/128	1/256	1/512
1/4	1.5276×10^{-3}	4.0384×10^{-4}	1.4576×10^{-4}	1.1567×10^{-4}	1.1478×10^{-4}	1.1504×10^{-4}	1.1513×10^{-4}	1.1516×10^{-4}
1/8	1.5331×10^{-3}	3.9695×10^{-4}	1.0050×10^{-4}	2.8290×10^{-5}	1.5396×10^{-5}	1.4385×10^{-5}	1.4364×10^{-5}	1.4374×10^{-5}
1/16	1.5336×10^{-3}	3.9733×10^{-4}	1.0016×10^{-4}	2.5116×10^{-5}	6.4840×10^{-6}	2.3544×10^{-6}	1.8291×10^{-6}	1.7971×10^{-6}
1/32	1.5336×10^{-3}	3.9736×10^{-4}	1.0018×10^{-4}	2.5097×10^{-5}	6.2789×10^{-6}	1.5828×10^{-6}	4.5004×10^{-7}	2.4480×10^{-7}
1/64	1.5336×10^{-3}	3.9736×10^{-4}	1.0019×10^{-4}	2.5098×10^{-5}	6.2777×10^{-6}	1.5697×10^{-6}	3.9331×10^{-7}	1.0213×10^{-7}
1/128	1.5336×10^{-3}	3.9736×10^{-4}	1.0019×10^{-4}	2.5098×10^{-5}	6.2778×10^{-6}	1.5697×10^{-6}	3.9249×10^{-7}	9.8375×10^{-8}
1/256	1.5336×10^{-3}	3.9736×10^{-4}	1.0019×10^{-4}	2.5098×10^{-5}	6.2779×10^{-6}	1.5697×10^{-6}	3.9251×10^{-7}	9.8356×10^{-8}

Table 4.4: Errors of **(R2)** for quadratic polynomial basis

4.2 DGFEM to Wave Propagation with Viscoelasticity

Recall Chapter 3 for DG frameworks. As seen in the previous section, numerical approximation to vector-valued problem could be implemented by using DGFEM as well as CGFEM. In particular, the scalar DG approximation would be elevated to vector field spaces. However, we have to consider Korn's inequalities in broken Sobolev space to see coercivity. In addition, we could gain fully discrete formulations with respect to internal variable forms. As following scalar problems, stability bounds and error bounds would be dealt with by the same process and similar proofs.

Remark DGFEM for elasticity problems was introduced in [15, 16, 17]. The extension of DG formulation for the elasticity models will be used but we want to consider mixed DG for viscoelasticity. Either SIPG or NIPG for viscoelasticity was given by Rivére, Shaw and Whiteman [17, 26].

Let us recall a broken Sobolev space $H^s(\mathcal{E}_h)$ and a finite dimensional space $\mathcal{D}_k(\mathcal{E}_h)$ with the subdivision $\mathcal{E}_h$ as we defined before for $s > 3/2$. Then we can define a piecewise Sobolev vector field $[H^s(\mathcal{E}_h)]^d$ and finite dimensional vector field $[\mathcal{D}_k(\mathcal{E}_h)]^d$. Define DG bilinear forms $a_{\pm1} : [H^s(\mathcal{E}_h)]^d \times [H^s(\mathcal{E}_h)]^d \mapsto \mathbb{R}$ by for any $\boldsymbol{v}, \boldsymbol{w} \in [H^s(\mathcal{E}_h)]^d$

$$a_{\pm1}(\boldsymbol{v}, \boldsymbol{w}) = \sum_{E\in\mathcal{E}_h} \int_E \underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{v}) : \underline{\boldsymbol{\varepsilon}}(\boldsymbol{w})\, dE - \sum_{e\subset\Gamma_h\cup\Gamma_D} \int_e \{\underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{v}) \cdot \boldsymbol{n}_e\} \cdot [\boldsymbol{w}] de$$
$$\pm \sum_{e\subset\Gamma_h\cup\Gamma_D} \int_e \{\underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{w}) \cdot \boldsymbol{n}_e\} \cdot [\boldsymbol{v}] de + \boldsymbol{J}_0^{\alpha_0,\beta_0}(\boldsymbol{v}, \boldsymbol{w}),$$

where the jump and average operator are given in Section 1.4.2 and

$$J_0^{\alpha_0,\beta_0}(\boldsymbol{v}, \boldsymbol{w}) = \sum_{e\subset\Gamma_h\cup\Gamma_D} \frac{\alpha_0}{|e|^{\beta_0}} \int_e [\boldsymbol{v}] \cdot [\boldsymbol{w}] de.$$

$a_1(\cdot,\cdot)$ is a DG bilinear form of NIPG whereas $a_{-1}(\cdot,\cdot)$ is for SIPG. Also we have linear forms given by

$$F_d(t; \boldsymbol{v}) = \int_\Omega \boldsymbol{f}(t) \cdot \boldsymbol{v}\, d\Omega + \sum_{e\subset\Gamma_N} \int_e \boldsymbol{g}_N(t) \cdot \boldsymbol{v}\, de$$

and

$$F_v(t; \boldsymbol{v}) = \int_\Omega \boldsymbol{f}(t) \cdot \boldsymbol{v}\, d\Omega + \sum_{e\subset\Gamma_N} \int_e \boldsymbol{g}_N(t) \cdot \boldsymbol{v}\, de - \sum_{q=1}^{N_\varphi} \varphi_q e^{-t/\tau_q} a_1(\boldsymbol{u}_0, \boldsymbol{v}),$$

for any $\boldsymbol{v} \in [H^s(\mathcal{E}_h)]^d$, $s > 3/2$. As a consequence, we now formulate the following variational forms of (1.3.17)-(1.3.27) with respect to internal variable forms.
(S1) Find $\boldsymbol{u}$ and $\{\boldsymbol{\psi}_q\}_{q=1}^{N_\varphi}$ such that satisfy for all $\boldsymbol{v} \in [H^s(\mathcal{E}_h)]^d$

$$(\rho\ddot{\boldsymbol{u}}(t), \boldsymbol{v})_{L_2(\Omega)} + a_1(\boldsymbol{u}(t), \boldsymbol{v}) - \sum_{q=1}^{N_\varphi} a_{-1}(\boldsymbol{\psi}_q(t), \boldsymbol{v}) + J_0^{\alpha_0,\beta_0}(\dot{\boldsymbol{u}}(t), \boldsymbol{v}) = F_d(t; \boldsymbol{v}), \quad (4.2.1)$$

$$a_{-1}\left(\tau_q\dot{\boldsymbol{\psi}}_q(t)+\boldsymbol{\psi}_q(t),\boldsymbol{v}\right)=a_{-1}\left(\varphi_q\boldsymbol{u}(t),\boldsymbol{v}\right), \tag{4.2.2}$$

for each q, where $\boldsymbol{u}(0)=\boldsymbol{u}_0$, $\dot{\boldsymbol{u}}(0)=\boldsymbol{w}_0$ and $\boldsymbol{\psi}_q(0)=\mathbf{0}$. In the same sense,
(S2) find $\boldsymbol{u}$ and $\{\boldsymbol{\zeta}_q\}_{q=1}^{N_\varphi}$ such that satisfy for all $\boldsymbol{v}\in[H^s(\mathcal{E}_h)]^d$

$$(\rho\ddot{\boldsymbol{u}}(t),\boldsymbol{v})_{L_2(\Omega)}+\varphi_0a_1\left(\boldsymbol{u}(t),\boldsymbol{v}\right)+\sum_{q=1}^{N_\varphi}a_{-1}\left(\boldsymbol{\zeta}_q(t),\boldsymbol{v}\right)+J_0^{\alpha_0,\beta_0}(\dot{\boldsymbol{u}}(t),\boldsymbol{v})=F_v(t;\boldsymbol{v}), \tag{4.2.3}$$

$$a_{-1}\left(\tau_q\dot{\boldsymbol{\zeta}}_q(t)+\boldsymbol{\zeta}_q(t),\boldsymbol{v}\right)=a_{-1}\left(\tau_q\varphi_q\dot{\boldsymbol{u}}(t),\boldsymbol{v}\right), \tag{4.2.4}$$

for each q, where $\boldsymbol{u}(0)=\boldsymbol{u}_0$, $\dot{\boldsymbol{u}}(0)=\boldsymbol{w}_0$ and $\boldsymbol{\zeta}_q(0)=\mathbf{0}$.

Remark As we concerned before in a scalar analogue, we consider non-symmetric variational problems as in (4.2.1) and (4.2.3). Even though we could choose symmetric bilinear forms and solve strong forms of internal variables, we have restricted the weak forms for challenges. In order to manage the difficulty of non-symmetric problems and discontinuity of the velocity $\dot{\boldsymbol{u}}$ over the edges, we introduce the jump penalty for the velocity.

Integration by parts with respect to the space domain yields the weak formulations with introducing interior penalty and jump penalty.

Theorem 4.4. *If the solution $\boldsymbol{u}(t)$ and $\{\boldsymbol{\psi}_q(t)\}_{q=1}^{N_\varphi}$ to (1.3.24), (1.3.25), (1.3.17)-(1.3.21), (1.3.13) and (1.3.14) belong to $[H^s(\mathcal{E}_h)]^d$ for all $t\in[0,T]$ and $3/2<s\in\mathbb{N}$, then $\boldsymbol{u}$ and $\{\boldsymbol{\psi}_q\}_{q=1}^{N_\varphi}$ satisfy* **(S1)**.

Proof. For any $E\in\mathcal{E}_h$, $\forall\boldsymbol{v}\in[H^s(\mathcal{E}_h)]^d$, integration by parts gives

$$\begin{aligned}&\left(-\nabla\cdot\left(\underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}\big(\boldsymbol{u}(t)-\sum_{q=1}^{N_\varphi}\boldsymbol{\psi}_q(t)\big)\right),\boldsymbol{v}\right)_{L_2(E)}\\ &=\left(\underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}\big(\boldsymbol{u}(t)-\sum_{q=1}^{N_\varphi}\boldsymbol{\psi}_q(t)\big),\underline{\boldsymbol{\varepsilon}}(\boldsymbol{v})\right)_{L_2(E)}-\sum_{e\subset\partial E}\left(\underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}\big(\boldsymbol{u}(t)-\sum_{q=1}^{N_\varphi}\boldsymbol{\psi}_q(t)\big)\cdot\boldsymbol{n}_e,\boldsymbol{v}\right)_{L_2(e)}\end{aligned}$$

since $\underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}$ is symmetric. Note that continuity by embedding theorem with respect to space and homogeneous Dirichlet boundary condition imply

$$[\boldsymbol{u}(t)]=\mathbf{0},\qquad[\boldsymbol{\psi}_q(t)]=\mathbf{0},\forall q\qquad\text{on }\Gamma_h\cup\Gamma_D.$$

Hence, use of the same arguments in the proof of 3.1 allows us to claim that $\boldsymbol{u}(t)$ and $\{\boldsymbol{\psi}_q(t)\}_{q=1}^{N_\varphi}$ fulfil (4.2.1). Moreover, (1.3.25) yields (4.2.2) straightforwardly, since the bilinear form is well-defined. □

In this manner, the strong solution with internal variables of the velocity form belonging to $[H^s(\mathcal{E}_h)]^d$ satisfies **(S2)**.

4.2.1 Fully Discrete Formulation

Before obtaining fully discrete formulations, let us consider coercivity and continuity on the finite dimensional space $[\mathcal{D}_k(\mathcal{E}_h)]^d$. Define a DG energy norm by

$$\|\boldsymbol{v}\|_{\mathcal{V}} = \left(\sum_{E\in\mathcal{E}_h} \int_E \underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{v}) : \underline{\boldsymbol{\varepsilon}}(\boldsymbol{v})dE + J_0^{\alpha_0,\beta_0}(\boldsymbol{v},\boldsymbol{v}) \right)^{1/2}, \qquad \text{for } \boldsymbol{v} \in [H^s(\mathcal{E}_h)]^d.$$

Remark Due to Cauchy-Schwarz inequalities, we have for any $\boldsymbol{v} \in [H^1(\Omega)]^d$

$$\sum_{i,j=1}^{d} \varepsilon_{ij}(\boldsymbol{v})\varepsilon_{ij}(\boldsymbol{v}) \leq \sum_{i,j=1}^{d} |v_{i,j}|^2. \tag{4.2.5}$$

Lemma 4.3. *Let $\boldsymbol{v} \in [H^s(\mathcal{E}_h)]^d$. If we assume $\beta_0(d-1) \geq 1$,*

$$\sum_{E\in\mathcal{E}_h} |\boldsymbol{v}|^2_{H^1(E)} \leq C\,\|\boldsymbol{v}\|^2_{\mathcal{V}}$$

for some positive C independent of $\boldsymbol{v}$.

In [36], Korn's inequalities for piecewise H^1 vector fields have been introduced. Use of the Korn's inequalities allows us to obtain Lemma 4.3, since $\underline{\boldsymbol{D}}$ is symmetric positive definite and the jump penalty is defined on not only interior edges but also positive measured Dirichlet boundary.

Theorem 4.5. *Both NIPG and SIPG bilinear forms are coercive on $[\mathcal{D}_k(\mathcal{E}_h)]^d$ with large penalty parameters α_0 and β_0. Thus there exists a positive constant κ such that*

$$a_{-1}(\boldsymbol{v},\boldsymbol{v}),\ a_1(\boldsymbol{v},\boldsymbol{v}) \geq \kappa\,\|\boldsymbol{v}\|^2_{\mathcal{V}}, \qquad \forall \boldsymbol{v} \in [\mathcal{D}_k(\mathcal{E}_h)]^d.$$

Proof. Let $\boldsymbol{v} \in [\mathcal{D}_k(\mathcal{E}_h)]^d$ and $\beta_0(d-1) \geq 1$. It is true that $a_1(\boldsymbol{v},\boldsymbol{v}) = \|\boldsymbol{v}\|^2_{\mathcal{V}}$. So we shall consider SIPG only. By the definition, we have

$$a_{-1}(\boldsymbol{v},\boldsymbol{v}) = \|\boldsymbol{v}\|^2_{\mathcal{V}} - 2\sum_{e\subset\Gamma_h\cup\Gamma_D} \int_e \{\underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{v})\cdot\boldsymbol{n}_e\}\cdot[\boldsymbol{v}]de.$$

We will show that

$$\sum_{e\subset\Gamma_h\cup\Gamma_D} \int_e \{\underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{v})\cdot\boldsymbol{n}_e\}\cdot[\boldsymbol{v}]de \leq \frac{C}{\sqrt{\alpha_0}}\,\|\boldsymbol{v}\|^2_{\mathcal{V}}.$$

If this is true, then the proof is completed by taking sufficiently large α_0.

Taking into account a bound of $\sum_{e\subset\Gamma_h\cup\Gamma_D} \int_e \{\underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{v})\cdot\boldsymbol{n}_e\}\cdot[\boldsymbol{v}]de$, we can obtain

$$\sum_{e\subset\Gamma_h\cup\Gamma_D} \int_e \{\underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{v})\cdot\boldsymbol{n}_e\}\cdot[\boldsymbol{v}]de \leq \sum_{e\subset\Gamma_h\cup\Gamma_D} \|\{\underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{v})\cdot\boldsymbol{n}_e\}\|_{L_2(e)}\,\|[\boldsymbol{v}]\|_{L_2(e)}$$

by Cauchy-Schwarz inequality. Note that $\underline{\boldsymbol{D}}$ is symmetric positive definite and bounded. As following a similar argument in Theorem 3.3, inverse polynomial trace theorem gives that if $\beta_0(d-1) \geq 1$,

$$\sum_{e \subset \Gamma_h \cup \Gamma_D} \int_e \{\underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{v}) \cdot \boldsymbol{n}_e\} \cdot [\boldsymbol{v}] de \leq \frac{C}{\sqrt{\alpha_0}} \|\boldsymbol{v}\|_{\mathcal{V}}^2 .$$

Thus, there exists κ such that

$$a_{-1}(\boldsymbol{v}, \boldsymbol{v}) \geq \big(1 - \frac{C}{\sqrt{\alpha_0}}\big) \|\boldsymbol{v}\|_{\mathcal{V}}^2 = \kappa \|\boldsymbol{v}\|_{\mathcal{V}}^2$$

with sufficiently large α_0 as $1 - \frac{C}{\sqrt{\alpha_0}} > 0$. □

As shown the above, we can observe that the following bound such that

$$\sum_{e \subset \Gamma_h \cup \Gamma_D} \int_e \{\underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{v}) \cdot \boldsymbol{n}_e\} \cdot [\boldsymbol{w}] de \leq \frac{C}{\sqrt{\alpha_0}} \Big(\|\boldsymbol{v}\|_{\mathcal{V}}^2 + J_0^{\alpha_0,\beta_0}(\boldsymbol{w}, \boldsymbol{w}) \Big) . \tag{4.2.6}$$

By (4.2.6), we can show continuity of DG bilinear forms.

Theorem 4.6. *Let $\alpha_0 > 0$ and $\beta_0(d-1) \geq 1$. For any $\boldsymbol{v}, \boldsymbol{w} \in [\mathcal{D}_k(\mathcal{E}_h)]^d$, there exists a positive constant K such that*

$$|a_{-1}(\boldsymbol{v}, \boldsymbol{w})|, |a_1(\boldsymbol{v}, \boldsymbol{w})| \leq K \|\boldsymbol{v}\|_{\mathcal{V}} \|\boldsymbol{w}\|_{\mathcal{V}} .$$

Proof. Let $\boldsymbol{v}, \boldsymbol{w} \in [\mathcal{D}_k(\mathcal{E}_h)]^d$. By the definition of $a_{\pm 1}(\cdot, \cdot)$, Cauchy-Schwarz inequality yields

$$\begin{aligned}
|a_{\pm 1}(\boldsymbol{v}, \boldsymbol{w})| = & \Bigg| \sum_{E \in \mathcal{E}_h} \int_E \underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{v}) : \underline{\boldsymbol{\varepsilon}}(\boldsymbol{w}) \, dE - \sum_{e \subset \Gamma_h \cup \Gamma_D} \int_e \{\underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{v}) \cdot \boldsymbol{n}_e\} \cdot [\boldsymbol{w}] de \\
& \pm \sum_{e \subset \Gamma_h \cup \Gamma_D} \int_e \{\underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{w}) \cdot \boldsymbol{n}_e\} \cdot [\boldsymbol{v}] de + \boldsymbol{J}_0^{\alpha_0,\beta_0}(\boldsymbol{v}, \boldsymbol{w}) \Bigg| \\
\leq & \sum_{E \in \mathcal{E}_h} \left\| \underline{\boldsymbol{D}}^{1/2} \underline{\boldsymbol{\varepsilon}}(\boldsymbol{v}) \right\|_{L_2(E)} \left\| \underline{\boldsymbol{D}}^{1/2} \underline{\boldsymbol{\varepsilon}}(\boldsymbol{w}) \right\|_{L_2(E)} \\
& + \sum_{e \subset \Gamma_h \cup \Gamma_D} \|\{\underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{v}) \cdot \boldsymbol{n}_e\}\|_{L_2(e)} \|[\boldsymbol{w}]\|_{L_2(e)} \\
& + \sum_{e \subset \Gamma_h \cup \Gamma_D} \|\{\underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{w}) \cdot \boldsymbol{n}_e\}\|_{L_2(e)} \|[\boldsymbol{v}]\|_{L_2(e)} \\
& + \sum_{e \subset \Gamma_h \cup \Gamma_D} \frac{\alpha_0}{|e|^{\beta_0}} \|[\boldsymbol{v}]\|_{L_2(e)} \|[\boldsymbol{w}]\|_{L_2(e)}
\end{aligned}$$

where $\underline{\boldsymbol{D}}^{1/2}$ is the symmetric positive definite fourth order tensor such that satisfies $\underline{\boldsymbol{D}} = \underline{\boldsymbol{D}}^{1/2}\underline{\boldsymbol{D}}^{1/2}$. In a similar manner in Theorem 3.4, we can conclude that there exists a positive constant K such that

$$|a_{\pm 1}(\boldsymbol{v}, \boldsymbol{w})| \leq K \|\boldsymbol{v}\|_{\mathcal{V}} \|\boldsymbol{w}\|_{\mathcal{V}} .$$

□

Remark Let us define a skew symmetric bilinear form in NIPG such that for $\boldsymbol{v}, \boldsymbol{w} \in [\mathcal{D}_k(\mathcal{E}_h)]^d$

$$B(\boldsymbol{v}, \boldsymbol{w}) = a_1(\boldsymbol{v}, \boldsymbol{w}) - a_1(\boldsymbol{w}, \boldsymbol{v}).$$

Then (4.2.6) implies that

$$|B(\boldsymbol{v}, \boldsymbol{w})| \leq \frac{C}{\sqrt{\alpha_0}}\Big(\|\boldsymbol{v}\|^2_{\mathcal{V}} + \|\boldsymbol{w}\|^2_{\mathcal{V}} \Big) \tag{4.2.7}$$

if $\beta_0(d-1) \geq 1$.

With applying Crank-Nicolson finite difference method to time discretisation, we can formulate the fully discrete numerical schemes with respect to two internal variable forms with (4.1.14).

(S1) find $\boldsymbol{W}^n_h$, $\boldsymbol{U}^n_h$ and $\boldsymbol{\Psi}^n_{hq}$ in $[\mathcal{D}_k(\mathcal{E}_h)]^d$ for $n = 0, \dots, N$ and $q = 1, \dots, N_\varphi$ such that satisfy for all $\boldsymbol{v} \in [\mathcal{D}_k(\mathcal{E}_h)]^d$

$$\left(\rho \frac{\boldsymbol{W}^{n+1}_h - \boldsymbol{W}^n_h}{\Delta t}, \boldsymbol{v}\right)_{L_2(\Omega)} + a_1\left(\frac{\boldsymbol{U}^{n+1}_h + \boldsymbol{U}^n_h}{2}, \boldsymbol{v}\right) - \sum_{q=1}^{N_\varphi} a_{-1}\left(\frac{\boldsymbol{\Psi}^{n+1}_{hq} + \boldsymbol{\Psi}^n_{hq}}{2}, \boldsymbol{v}\right)$$

$$+ J_0^{\alpha_0,\beta_0}\left(\frac{\boldsymbol{W}^{n+1}_h + \boldsymbol{W}^n_h}{2}, \boldsymbol{v}\right) = \bar{F}^n_d(\boldsymbol{v}), \text{for } n = 0, \dots, N-1, \tag{4.2.8}$$

$$a_{-1}\left(\tau_q \frac{\boldsymbol{\Psi}^{n+1}_{hq} - \boldsymbol{\Psi}^n_{hq}}{\Delta t} + \frac{\boldsymbol{\Psi}^{n+1}_{hq} + \boldsymbol{\Psi}^n_{hq}}{2}, \boldsymbol{v}\right) = a_{-1}\left(\varphi_q \frac{\boldsymbol{U}^{n+1}_h + \boldsymbol{U}^n_h}{2}, \boldsymbol{v}\right),$$

$$\text{for } n = 0, \dots, N-1, \text{ and for } q = 1, \dots, N_\varphi, \tag{4.2.9}$$

$$a_1\left(\boldsymbol{U}^0_h, \boldsymbol{v}\right) = a_1(\boldsymbol{u}_0, \boldsymbol{v},) \tag{4.2.10}$$

$$\left(\boldsymbol{W}^0_h, \boldsymbol{v}\right)_{L_2(\Omega)} = (\boldsymbol{w}_0, \boldsymbol{v})_{L_2(\Omega)}, \tag{4.2.11}$$

for each q, $\boldsymbol{\Psi}^0_{hq} = \boldsymbol{0}$.

In this manner, we can formulate a fully discrete problem of the velocity form as following:

(S2) find $\boldsymbol{W}^n_h$, $\boldsymbol{U}^n_h$ and $\boldsymbol{\mathcal{S}}^n_{hq}$ in $[\mathcal{D}_k(\mathcal{E}_h)]^d$ for $n = 0, \dots, N$ and $q = 1, \dots, N_\varphi$ such that satisfy for all $\boldsymbol{v} \in [\mathcal{D}_k(\mathcal{E}_h)]^d$

$$\left(\rho \frac{\boldsymbol{W}^{n+1}_h - \boldsymbol{W}^n_h}{\Delta t}, \boldsymbol{v}\right)_{L_2(\Omega)} + \varphi_0 a_1\left(\frac{\boldsymbol{U}^{n+1}_h + \boldsymbol{U}^n_h}{2}, \boldsymbol{v}\right) + \sum_{q=1}^{N_\varphi} a_{-1}\left(\frac{\boldsymbol{\mathcal{S}}^{n+1}_{hq} + \boldsymbol{\mathcal{S}}^n_{hq}}{2}, \boldsymbol{v}\right)$$

$$+ J_0^{\alpha_0,\beta_0}\left(\frac{\boldsymbol{W}^{n+1}_h + \boldsymbol{W}^n_h}{2}, \boldsymbol{v}\right) = \bar{F}^n_v(\boldsymbol{v}), \qquad \text{for } n = 0, \dots, N-1, \tag{4.2.12}$$

$$a_{-1}\left(\tau_q \frac{\boldsymbol{\mathcal{S}}^{n+1}_{hq} - \boldsymbol{\mathcal{S}}^n_{hq}}{\Delta t} + \frac{\boldsymbol{\mathcal{S}}^{n+1}_{hq} + \boldsymbol{\mathcal{S}}^n_{hq}}{2}, \boldsymbol{v}\right) = a_{-1}\left(\tau_q \varphi_q \frac{\boldsymbol{W}^{n+1}_h + \boldsymbol{W}^n_h}{2}, \boldsymbol{v}\right),$$

$$\text{for } n = 0, \dots, N-1, \text{ and for } q = 1, \dots, N_\varphi, \tag{4.2.13}$$

$$a_1\left(\boldsymbol{U}_h^0,\boldsymbol{v}\right)=a_1\left(\boldsymbol{u}_0,\boldsymbol{v}\right), \tag{4.2.14}$$

$$\left(\boldsymbol{W}_h^0,\boldsymbol{v}\right)_{L_2(\Omega)}=\left(\boldsymbol{w}_0,\boldsymbol{v}\right)_{L_2(\Omega)}, \tag{4.2.15}$$

and $\boldsymbol{\mathcal{S}}_{hq}^0=\mathbf{0}$ for each q.

Note that the continuity of the NIPG bilinear form allows us to have

$$\left\|\boldsymbol{U}_h^0\right\|_{\mathcal{V}}^2=a_1\left(\boldsymbol{U}_h^0,\boldsymbol{U}_h^0\right)=a_1\left(\boldsymbol{u}_0,\boldsymbol{U}_h^0\right)\leq K\left\|\boldsymbol{U}_h^0\right\|_{\mathcal{V}}\left\|\boldsymbol{u}_0\right\|_{\mathcal{V}}\Rightarrow\left\|\boldsymbol{U}_h^0\right\|_{\mathcal{V}}\leq K\left\|\boldsymbol{u}_0\right\|_{\mathcal{V}}$$

from (4.2.10) and (4.2.14), and applying Cauchy-Schwarz inequality to (4.2.11) and (4.2.15) gives

$$\left(\boldsymbol{W}_h^0,\boldsymbol{W}_h^0\right)_{L_2(\Omega)}=\left(\boldsymbol{w}_0,\boldsymbol{W}_h^0\right)_{L_2(\Omega)}\leq\left\|\boldsymbol{W}_h^0\right\|_{L_2(\Omega)}\left\|\boldsymbol{w}_0\right\|_{L_2(\Omega)}\Rightarrow\left\|\boldsymbol{W}_h^0\right\|_{L_2(\Omega)}\leq\left\|\boldsymbol{w}_0\right\|_{L_2(\Omega)}$$

for both **(S1)** and **(S2)**.

In Chapter 3, stability bounds and error bounds are proved for the scalar problems with DGFEM and we can also elevate these to vector-valued cases. More precisely, use of the certain techniques such as integration by parts, summation by parts, using a variety of inequalities e.g. inverse polynomial trace inequalities, Young's inequality, Cauchy-Schwarz inequality, etc, as in Chapter 3 leads us to have the following discrete stability theorems and error estimates theorems.

Theorem 4.7. *Suppose $\beta_0(d-1)\geq 1$ and α_0 is large enough. Assume $\boldsymbol{W}_h^n$, $\boldsymbol{U}_h^n$ and $\{\boldsymbol{\Psi}_{hq}^n\}_{q=1}^{N_\varphi}$ in $[\mathcal{D}_k(\mathcal{E}_h)]^d$ for $n=0,\ldots,N$ satisfy the fully discrete formulation of* **(S1)**. *There exists a positive constant C such that*

$$\begin{aligned}
&\max_{0\leq n\leq N}\left\|\rho^{1/2}\boldsymbol{W}_h^n\right\|_{L_2(\Omega)}^2+\max_{0\leq n\leq N}\left\|\boldsymbol{U}_h^n\right\|_{\mathcal{V}}^2+\sum_{q=1}^{N_\varphi}\max_{0\leq n\leq N}\left\|\boldsymbol{\Psi}_{hq}^n\right\|_{\mathcal{V}}^2\\
&+\sum_{n=0}^{N-1}\sum_{q=1}^{N_\varphi}\frac{1}{\Delta t}\left\|\boldsymbol{\Psi}_{hq}^{n+1}-\boldsymbol{\Psi}_{hq}^n\right\|_{\mathcal{V}}^2+\Delta t\sum_{n=0}^{N-1}J_0^{\alpha_0,\beta_0}(\boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n,\boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n)\\
\leq&C\Bigg(\left\|\rho^{1/2}\boldsymbol{w}_0\right\|_{L_2(\Omega)}^2+\left\|\boldsymbol{u}_0\right\|_{\mathcal{V}}^2+\left\|\boldsymbol{f}\right\|_{L_\infty(0,T;L_2(\Omega))}^2+h^{-1}\left\|\boldsymbol{g}_N\right\|_{L_\infty(0,T;L_2(\Gamma_N))}^2\\
&+h^{-1}\left\|\dot{\boldsymbol{g}}_N\right\|_{L_2(0,T;L_2(\Gamma_N))}^2\Bigg).
\end{aligned}$$

Proof. Let $m\in\mathbb{N}$ such that $m\leq N$. By substitution of $\boldsymbol{v}=\boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n$ into (4.2.8), we have

$$\begin{aligned}
&\frac{\rho}{\Delta t}\left(\left\|\boldsymbol{W}_h^{n+1}\right\|_{L_2(\Omega)}^2-\left\|\boldsymbol{W}_h^n\right\|_{L_2(\Omega)}^2\right)+\frac{1}{\Delta t}\left(\left\|\boldsymbol{U}_h^{n+1}\right\|_{\mathcal{V}}^2-\left\|\boldsymbol{U}_h^n\right\|_{\mathcal{V}}^2\right)\\
&-\frac{1}{\Delta t}B\left(\boldsymbol{U}_h^{n+1},\boldsymbol{U}_h^n\right)-\frac{1}{\Delta t}\sum_{q=1}^{N_\varphi}a_{-1}\left(\boldsymbol{\Psi}_{hq}^{n+1}+\boldsymbol{\Psi}_{hq}^n,\boldsymbol{U}_h^{n+1}-\boldsymbol{U}_h^n\right)\\
&+\frac{1}{2}J_0^{\alpha_0,\beta_0}(\boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n,\boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n)
\end{aligned}$$

$$=\bar{F}_d^n(\boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n) \tag{4.2.16}$$

for $n=0,\ldots,m-1$. On the other hand, a choice of $\boldsymbol{v}=2(\boldsymbol{\Psi}_{hq}^{n+1}-\boldsymbol{\Psi}_{hq}^n)$ in (4.2.9) for each q and symmetry of SIPG give

$$\begin{aligned}a_{-1}\left(\boldsymbol{\Psi}_{hq}^{n+1}+\boldsymbol{\Psi}_{hq}^n,\boldsymbol{U}_h^{n+1}-\boldsymbol{U}_h^n\right)=&2a_{-1}\left(\boldsymbol{\Psi}_{hq}^{n+1},\boldsymbol{U}_h^{n+1}\right)-2a_{-1}\left(\boldsymbol{\Psi}_{hq}^n,\boldsymbol{U}_h^n\right)\\&-\frac{2\tau_q}{\Delta t\varphi_q}a_{-1}\left(\boldsymbol{\Psi}_{hq}^{n+1}-\boldsymbol{\Psi}_{hq}^n,\boldsymbol{\Psi}_{hq}^{n+1}-\boldsymbol{\Psi}_{hq}^n\right)\\&-\frac{1}{\varphi_q}\left(a_{-1}\left(\boldsymbol{\Psi}_{hq}^{n+1},\boldsymbol{\Psi}_{hq}^{n+1}\right)-a_{-1}\left(\boldsymbol{\Psi}_{hq}^n,\boldsymbol{\Psi}_{hq}^n\right)\right),\end{aligned} \tag{4.2.17}$$

since

$$\begin{aligned}a_{-1}\left(\boldsymbol{\Psi}_{hq}^{n+1}-\boldsymbol{\Psi}_{hq}^n,\boldsymbol{U}_h^{n+1}+\boldsymbol{U}_h^n\right)=&2a_{-1}\left(\boldsymbol{\Psi}_{hq}^{n+1},\boldsymbol{U}_h^{n+1}\right)-2a_{-1}\left(\boldsymbol{\Psi}_{hq}^n,\boldsymbol{U}_h^n\right)\\&-a_{-1}\left(\boldsymbol{\Psi}_{hq}^{n+1}+\boldsymbol{\Psi}_{hq}^n,\boldsymbol{U}_h^{n+1}-\boldsymbol{U}_h^n\right).\end{aligned}$$

Combining (4.2.16) and (4.2.17), multiplying Δt and summing from $n=0$ to $n=m-1$ yield

$$\begin{aligned}&\rho\left\|\boldsymbol{W}_h^m\right\|_{L_2(\Omega)}^2+\left\|\boldsymbol{U}_h^m\right\|_{\mathcal{V}}^2+\sum_{q=1}^{N_\varphi}\frac{1}{\varphi_q}a_{-1}\left(\boldsymbol{\Psi}_{hq}^m,\boldsymbol{\Psi}_{hq}^m\right)\\&+\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi}\frac{2\tau_q}{\Delta t\varphi_q}a_{-1}\left(\boldsymbol{\Psi}_{hq}^{n+1}-\boldsymbol{\Psi}_{hq}^n,\boldsymbol{\Psi}_{hq}^{n+1}-\boldsymbol{\Psi}_{hq}^n\right)\\&+\frac{\Delta t}{2}\sum_{n=0}^{m-1}J_0^{\alpha_0,\beta_0}(\boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n,\boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n)\\=&\rho\left\|\boldsymbol{W}_h^0\right\|_{L_2(\Omega)}^2+\left\|\boldsymbol{U}_h^0\right\|_{\mathcal{V}}^2+\sum_{n=0}^{m-1}\Delta t\bar{F}_d^n(\boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n)+2\sum_{q=1}^{N_\varphi}a_{-1}\left(\boldsymbol{\Psi}_{hq}^m,\boldsymbol{U}_h^m\right)\\&+\sum_{n=0}^{m-1}B\left(\boldsymbol{U}_h^{n+1},\boldsymbol{U}_h^n\right)\end{aligned} \tag{4.2.18}$$

since $\boldsymbol{\Psi}_{hq}^0=0,\ \forall q\in\{1,\ldots,N_\varphi\}$. Use of coercivity of SIPG and expansion of SIPG reduces (4.2.18) to

$$\begin{aligned}&\rho\left\|\boldsymbol{W}_h^m\right\|_{L_2(\Omega)}^2+\left\|\boldsymbol{U}_h^m\right\|_{\mathcal{V}}^2+\sum_{q=1}^{N_\varphi}\frac{1}{\varphi_q}\left\|\boldsymbol{\Psi}_{hq}^m\right\|_{\mathcal{V}}^2+\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi}\frac{2\kappa\tau_q}{\Delta t\varphi_q}\left\|\boldsymbol{\Psi}_{hq}^{n+1}-\boldsymbol{\Psi}_{hq}^n\right\|_{\mathcal{V}}^2\\&+\frac{\Delta t}{2}\sum_{n=0}^{m-1}J_0^{\alpha_0,\beta_0}(\boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n,\boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n)\end{aligned}$$

$$\leq \Bigg| \rho \left\| \boldsymbol{w}_0 \right\|^2_{L_2(\Omega)} + K \left\| \boldsymbol{u}_0 \right\|^2_{\mathcal{V}} + \sum_{n=0}^{m-1} \Delta t \bar{F}^n_d (\boldsymbol{W}^{n+1}_h + \boldsymbol{W}^n_h) + 2 \sum_{q=1}^{N_\varphi} a_{-1} \left(\boldsymbol{\Psi}^m_{hq}, \boldsymbol{U}^m_h \right)$$

$$+ \sum_{n=0}^{m-1} B \left(\boldsymbol{U}^{n+1}_h, \boldsymbol{U}^n_h \right) + \sum_{q=1}^{N_\varphi} \frac{2}{\varphi_q} \sum_{e \subset \Gamma_h \cup \Gamma_D} \int_e \{ \underline{\underline{\boldsymbol{D}}} \boldsymbol{\varepsilon}(\boldsymbol{\Psi}^m_{hq}) \cdot \boldsymbol{n}_e \} \cdot [\boldsymbol{\Psi}^m_{hq}] de \Bigg| \qquad (4.2.19)$$

since

$$\left\| \boldsymbol{W}^0_h \right\|_{L_2(\Omega)} \leq \left\| \boldsymbol{w}_0 \right\|_{L_2(\Omega)} \text{ and } \left\| \boldsymbol{U}^0_h \right\|_{\mathcal{V}} \leq K \left\| \boldsymbol{u}_0 \right\|_{\mathcal{V}}.$$

In a similar way with the proof of Theorem 3.11, we can observe the bound for the right hand side of (4.2.19). To be specific, we would use the same arguments in the scalar DG problem but we should introduce (4.2.5), (4.2.6), (4.2.7) and Lemma 4.3 more. In the end, it is able to conclude that

$$\max_{0\leq n\leq N} \left\| \rho^{1/2} \boldsymbol{W}^n_h \right\|^2_{L_2(\Omega)} + \max_{0\leq n\leq N} \left\| \boldsymbol{U}^n_h \right\|^2_{\mathcal{V}} + \sum_{q=1}^{N_\varphi} \left\| \boldsymbol{\Psi}^m_{hq} \right\|^2_{\mathcal{V}} + \sum_{n=0}^{m-1} \sum_{q=1}^{N_\varphi} \frac{1}{\Delta t} \left\| \boldsymbol{\Psi}^{n+1}_{hq} - \boldsymbol{\Psi}^n_{hq} \right\|^2_{\mathcal{V}}$$

$$+ \Delta t \sum_{n=0}^{m-1} J^{\alpha_0,\beta_0}_0 (\boldsymbol{W}^{n+1}_h + \boldsymbol{W}^n_h, \boldsymbol{W}^{n+1}_h + \boldsymbol{W}^n_h)$$

$$\leq C \Bigg(\left\| \rho^{1/2} \boldsymbol{w}_0 \right\|^2_{L_2(\Omega)} + \left\| \boldsymbol{u}_0 \right\|^2_{\mathcal{V}} + \Delta t \sum_{n=0}^{N-1} \left\| \bar{\boldsymbol{f}}^n \right\|^2_{L_2(\Omega)} + h^{-1} \max_{0\leq n\leq N-1} \left\| \bar{\boldsymbol{g}}^n_N \right\|^2_{L_2(\Gamma_N)}$$

$$+ h^{-1} \left\| \dot{\boldsymbol{g}}_N \right\|^2_{L_2(0,T;L_2(\Gamma_N))} \Bigg)$$

$$\leq C \Bigg(\left\| \rho^{1/2} \boldsymbol{w}_0 \right\|^2_{L_2(\Omega)} + \left\| \boldsymbol{u}_0 \right\|^2_{\mathcal{V}} + \left\| \boldsymbol{f} \right\|^2_{L_\infty(0,T;L_2(\Omega))} + h^{-1} \left\| \boldsymbol{g}_N \right\|^2_{L_\infty(0,T;L_2(\Gamma_N))}$$

$$+ h^{-1} \left\| \dot{\boldsymbol{g}}_N \right\|^2_{L_2(0,T;L_2(\Gamma_N))} \Bigg),$$

for a sufficiently large α_0 and

$$\max_{0\leq n\leq N} \left\| \rho^{1/2} \boldsymbol{W}^n_h \right\|^2_{L_2(\Omega)} + \max_{0\leq n\leq N} \left\| \boldsymbol{U}^n_h \right\|^2_{\mathcal{V}} + \sum_{q=1}^{N_\varphi} \max_{0\leq n\leq N} \left\| \boldsymbol{\Psi}^n_{hq} \right\|^2_{\mathcal{V}}$$

$$+ \sum_{n=0}^{N-1} \sum_{q=1}^{N_\varphi} \frac{1}{\Delta t} \left\| \boldsymbol{\Psi}^{n+1}_{hq} - \boldsymbol{\Psi}^n_{hq} \right\|^2_{\mathcal{V}} + \Delta t \sum_{n=0}^{N-1} J^{\alpha_0,\beta_0}_0 (\boldsymbol{W}^{n+1}_h + \boldsymbol{W}^n_h, \boldsymbol{W}^{n+1}_h + \boldsymbol{W}^n_h)$$

$$\leq C \Bigg(\left\| \rho^{1/2} \boldsymbol{w}_0 \right\|^2_{L_2(\Omega)} + \left\| \boldsymbol{u}_0 \right\|^2_{\mathcal{V}} + \left\| \boldsymbol{f} \right\|^2_{L_\infty(0,T;L_2(\Omega))} + h^{-1} \left\| \boldsymbol{g}_N \right\|^2_{L_\infty(0,T;L_2(\Gamma_N))}$$

$$+ h^{-1} \left\| \dot{\boldsymbol{g}}_N \right\|^2_{L_2(0,T;L_2(\Gamma_N))} \Bigg),$$

since m is arbitrary. □

In this same manner, a stability bound for the velocity form is given.

Theorem 4.8. *Suppose $\beta_0(d-1) \geq 1$ and α_0 is large enough. Assume $\boldsymbol{W}_h^n$, $\boldsymbol{U}_h^n$ and $\{\boldsymbol{\mathcal{S}}_{hq}^n\}_{q=1}^{N_\varphi}$ in $[\mathcal{D}_k(\mathcal{E}_h)]^d$ for $n = 0, \dots, N$ satisfy the fully discrete formulation of* **(S2)**. *There exists a positive constant C such that*

$$\begin{aligned}
&\max_{0\leq n\leq N} \left\|\rho^{1/2}\boldsymbol{W}_h^n\right\|_{L_2(\Omega)}^2 + \max_{0\leq n\leq N} \|\boldsymbol{U}_h^n\|_{\mathcal{V}}^2 + \sum_{q=1}^{N_\varphi} \max_{0\leq n\leq N} \left\|\boldsymbol{\mathcal{S}}_{hq}^n\right\|_{\mathcal{V}}^2 \\
&+ \Delta t \sum_{n=0}^{N-1}\sum_{q=1}^{N_\varphi} \left\|\boldsymbol{\mathcal{S}}_{hq}^{n+1} + \boldsymbol{\mathcal{S}}_{hq}^n\right\|_{\mathcal{V}}^2 + \Delta t \sum_{n=0}^{N-1} J_0^{\alpha_0,\beta_0}(\boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n, \boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n) \\
\leq& C\Bigg(\left\|\rho^{1/2}\boldsymbol{w}_0\right\|_{L_2(\Omega)}^2 + \|\boldsymbol{u}_0\|_{\mathcal{V}}^2 + \|\boldsymbol{f}\|_{L_\infty(0,T;L_2(\Omega))}^2 + h^{-1}\|\boldsymbol{g}_N\|_{L_\infty(0,T;L_2(\Gamma_N))}^2 \\
&+ h^{-1}\|\dot{\boldsymbol{g}}_N\|_{L_2(0,T;L_2(\Gamma_N))}^2 \Bigg).
\end{aligned}$$

Proof. One can show a stability bound for the fully discrete formulation of **(S2)** as following the same way as in Theorem 3.13. Taking $\boldsymbol{v} = \boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n$ for $n = 0, \dots, m-1$ where $m \in \{1, \dots, N\}$ in (4.2.12), adding from $n = 0$ to $n = m-1$ and multiplying Δt, we can obtain

$$\begin{aligned}
&\rho\|\boldsymbol{W}_h^m\|_{L_2(\Omega)}^2 + \varphi_0\|\boldsymbol{U}_h^m\|_{\mathcal{V}}^2 + \frac{\Delta t}{2}\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi} a_{-1}\left(\boldsymbol{S}_{hq}^{n+1} + \boldsymbol{S}_{hq}^n, \boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n\right) \\
&+ \frac{\Delta t}{2}\sum_{n=0}^{m-1} J_0^{\alpha_0,\beta_0}(\boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n, \boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n) \\
=&\rho\left\|\boldsymbol{W}_h^0\right\|_{L_2(\Omega)}^2 + \varphi_0\left\|\boldsymbol{U}_h^0\right\|_{\mathcal{V}}^2 + \Delta t\sum_{n=0}^{m-1}\bar{F}_v^n\left(\boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n\right) + \varphi_0\sum_{n=0}^{m-1} B(\boldsymbol{U}_h^{n+1}, \boldsymbol{U}_h^n).
\end{aligned} \tag{4.2.20}$$

Once we consider putting $\boldsymbol{v} = \boldsymbol{S}_{hq}^{n+1} + \boldsymbol{S}_{hq}^n$ for $n = 0, \dots, m-1$ into (4.2.13) and summing it from $n = 0$ to $n = m-1$, we can have the following equation such that

$$\begin{aligned}
\sum_{n=0}^{m-1} a_{-1}(\boldsymbol{S}_{hq}^{n+1} + \boldsymbol{S}_{hq}^n, \boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n) =& \frac{2}{\varphi_q\Delta t} a_{-1}\left(\boldsymbol{S}_{hq}^m, \boldsymbol{S}_{hq}^m\right) \\
&+ \sum_{n=0}^{m-1}\frac{1}{\tau_q\varphi_q} a_{-1}\left(\boldsymbol{S}_{hq}^{n+1} + \boldsymbol{S}_{hq}^n, \boldsymbol{S}_{hq}^{n+1} + \boldsymbol{S}_{hq}^n\right),
\end{aligned}$$

so that substitution of the resulting equation to (4.2.20) yields

$$\rho\|\boldsymbol{W}_h^m\|_{L_2(\Omega)}^2 + \varphi_0\|\boldsymbol{U}_h^m\|_{\mathcal{V}}^2 + \sum_{q=1}^{N_\varphi}\frac{1}{\varphi_q} a_{-1}\left(\boldsymbol{S}_{hq}^m, \boldsymbol{S}_{hq}^m\right)$$

$$+\frac{\Delta t}{2}\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi}\frac{1}{\tau_q\varphi_q}a_{-1}\left(\boldsymbol{S}_{hq}^{n+1}+\boldsymbol{S}_{hq}^{n},\boldsymbol{S}_{hq}^{n+1}+\boldsymbol{S}_{hq}^{n}\right)$$

$$+\frac{\Delta t}{2}\sum_{n=0}^{m-1}J_0^{\alpha_0,\beta_0}(\boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n,\boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n)$$

$$=\rho\left\|\boldsymbol{W}_h^0\right\|_{L_2(\Omega)}^2+\varphi_0\left\|\boldsymbol{U}_h^0\right\|_{\mathcal{V}}^2+\Delta t\sum_{n=0}^{m-1}\bar{F}_v^n\left(\boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n\right)+\varphi_0\sum_{n=0}^{m-1}B(\boldsymbol{U}_h^{n+1},\boldsymbol{U}_h^n)$$

Coercivity implies

$$\rho\left\|\boldsymbol{W}_h^m\right\|_{L_2(\Omega)}^2+\varphi_0\left\|\boldsymbol{U}_h^m\right\|_{\mathcal{V}}^2+\sum_{q=1}^{N_\varphi}\frac{\kappa}{\varphi_q}\left\|\boldsymbol{S}_{hq}^m\right\|_{\mathcal{V}}^2+\frac{\Delta t}{2}\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi}\frac{\kappa}{\tau_q\varphi_q}\left\|\boldsymbol{S}_{hq}^{n+1}+\boldsymbol{S}_{hq}^n\right\|_{\mathcal{V}}^2$$

$$+\frac{\Delta t}{2}\sum_{n=0}^{m-1}J_0^{\alpha_0,\beta_0}(\boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n,\boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n)$$

$$\leq\Bigg|\rho\left\|\boldsymbol{W}_h^0\right\|_{L_2(\Omega)}^2+\varphi_0\left\|\boldsymbol{U}_h^0\right\|_{\mathcal{V}}^2+\Delta t\sum_{n=0}^{m-1}\bar{F}_v^n\left(\boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n\right)$$

$$+\varphi_0\sum_{n=0}^{m-1}B(\boldsymbol{U}_h^{n+1},\boldsymbol{U}_h^n)\Bigg|. \tag{4.2.21}$$

Note that we have $\left\|\boldsymbol{W}_h^0\right\|_{L_2(\Omega)}\leq\left\|\boldsymbol{w}_0\right\|_{L_2(\Omega)}$ and $\left\|\boldsymbol{U}_h^0\right\|_{\mathcal{V}}\leq K\left\|\boldsymbol{u}_0\right\|_{\mathcal{V}}$. As seen in Theorem 3.13, we can derive bounds of $\Delta t\sum_{n=0}^{m-1}\bar{F}_v^n\left(\boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n\right)$ and $\sum_{n=0}^{m-1}B(\boldsymbol{U}_h^{n+1},\boldsymbol{U}_h^n)$ with using the same arguments in the scalar case but introducing (4.2.5), (4.2.6), (4.2.7), and Lemma 4.3 for estimates of traces with respect to the strain tensor.

Consequently, we can obtain

$$\max_{0\leq n\leq N}\left\|\boldsymbol{W}_h^n\right\|_{L_2(\Omega)}^2+\max_{0\leq n\leq N}\left\|\boldsymbol{U}_h^n\right\|_{\mathcal{V}}^2+\sum_{q=1}^{N_\varphi}\left\|\boldsymbol{S}_{hq}^m\right\|_{\mathcal{V}}^2+\Delta t\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi}\left\|\boldsymbol{S}_{hq}^{n+1}+\boldsymbol{S}_{hq}^n\right\|_{\mathcal{V}}^2$$

$$+\Delta t\sum_{n=0}^{m-1}J_0^{\alpha_0,\beta_0}(\boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n,\boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n)$$

$$\leq C\big(\left\|\boldsymbol{w}_0\right\|_{L_2(\Omega)}^2+\left\|\boldsymbol{u}_0\right\|_{\mathcal{V}}^2+\Delta t\sum_{n=0}^{N-1}\left\|\bar{\boldsymbol{f}}^n\right\|_{L_2(\Omega)}^2+h^{-1}\max_{0\leq n\leq N-1}\left\|\bar{\boldsymbol{g}}_N^n\right\|_{L_2(\Gamma_N)}^2$$

$$+h^{-1}\left\|\dot{\boldsymbol{g}}_N\right\|_{L_2(0,T;L_2(\Gamma_N))}^2\big)$$

$$\leq C\big(\left\|\boldsymbol{w}_0\right\|_{L_2(\Omega)}^2+\left\|\boldsymbol{u}_0\right\|_{\mathcal{V}}^2+\left\|\boldsymbol{f}\right\|_{L_\infty(0,T;L_2(\Omega))}^2+h^{-1}\left\|\boldsymbol{g}_N\right\|_{L_\infty(0,T;L_2(\Gamma_N))}^2$$

$$+h^{-1}\left\|\dot{\boldsymbol{g}}_N\right\|_{L_2(0,T;L_2(\Gamma_N))}^2\big)$$

with large enough α_0 for some C, furthermore, since m is arbitrary, it is concluded that

$$
\begin{aligned}
&\max_{0\le n\le N}\|\boldsymbol{W}_h^n\|_{L_2(\Omega)}^2+\max_{0\le n\le N}\|\boldsymbol{U}_h^n\|_{\mathcal{V}}^2+\sum_{q=1}^{N_\varphi}\max_{0\le n\le N}\left\|\boldsymbol{S}_{hq}^n\right\|_{\mathcal{V}}^2+\Delta t\sum_{n=0}^{N-1}\sum_{q=1}^{N_\varphi}\left\|\boldsymbol{S}_{hq}^{n+1}+\boldsymbol{S}_{hq}^n\right\|_{\mathcal{V}}^2\\
&\quad+\Delta t\sum_{n=0}^{N-1}J_0^{\alpha_0,\beta_0}(\boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n,\boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n)\\
&\le C\big(\|\boldsymbol{w}_0\|_{L_2(\Omega)}^2+\|\boldsymbol{u}_0\|_{\mathcal{V}}^2+\|\boldsymbol{f}\|_{L_\infty(0,T;L_2(\Omega))}^2+h^{-1}\|\boldsymbol{g}_N\|_{L_\infty(0,T;L_2(\Gamma_N))}^2\\
&\quad+h^{-1}\|\dot{\boldsymbol{g}}_N\|_{L_2(0,T;L_2(\Gamma_N))}^2\big)
\end{aligned}
$$

for a sufficiently large α_0. □

The discrete stability bounds in Theorem 4.7 and 4.8 indicate that our numerical solutions exist uniquely. Also, the bound constant C independent of mesh sizes is increasing in the final time T but not exponentially since we do not use Grönwall inequalities. As seen in Chapter 3, use of maximum gives us $C \propto T$ and so is the vector value problems. In a similar sense with scalar DG problems, since DGFEM has imposed boundary condition weakly, h^{-1} terms exist but it is not observed in numerical experiments and error estimations.

Now we shall consider DG elliptic projection in order to use elliptic error estimates. Let us define DG elliptic projectors $\boldsymbol{R}_{-1}$ and $\boldsymbol{R}_1$ by for $\boldsymbol{u} \in [H^s(\mathcal{E}_h)]^d$

$$\boldsymbol{R}_{-1}:[H^s(\mathcal{E}_h)]^d\mapsto[\mathcal{D}_k(\mathcal{E}_h)]^d \text{ such that } a_{-1}(\boldsymbol{u},\boldsymbol{v})=a_{-1}(\boldsymbol{R}_{-1}\boldsymbol{u},\boldsymbol{v}),\ \forall\boldsymbol{v}\in[\mathcal{D}_k(\mathcal{E}_h)]^d,$$

$$\boldsymbol{R}_{1}:[H^s(\mathcal{E}_h)]^d\mapsto[\mathcal{D}_k(\mathcal{E}_h)]^d \text{ such that } a_{1}(\boldsymbol{u},\boldsymbol{v})=a_{1}(\boldsymbol{R}_{1}\boldsymbol{u},\boldsymbol{v}),\ \forall\boldsymbol{v}\in[\mathcal{D}_k(\mathcal{E}_h)]^d.$$

According to [24, 17], Theorem 3.7 can be extended in vector-valued functions. Hence if $\boldsymbol{u} \in [H^s(\mathcal{E}_h)]^d$ for $s \in \mathbb{N}$ such that $s > 3/2$ and sufficiently large penalty parameters, α_0 and $\beta_0 \ge (d-1)^{-1}$, it satisfies

$$\|\boldsymbol{u}-\boldsymbol{R}_{-1}\boldsymbol{u}\|_{\mathcal{V}}\le Ch^{\min(k+1,s)-1}|||\boldsymbol{u}|||_{H^s(\mathcal{E}_h)},\tag{4.2.22}$$

$$\|\boldsymbol{u}-\boldsymbol{R}_{-1}\boldsymbol{u}\|_{L_2(\Omega)}\le Ch^{\min(k+1,s)-1}|||\boldsymbol{u}|||_{H^s(\mathcal{E}_h)}.\tag{4.2.23}$$

Moreover, with the convex domain Ω

$$\|\boldsymbol{u}-\boldsymbol{R}_{-1}\boldsymbol{u}\|_{L_2(\Omega)}\le Ch^{\min(k+1,s)}|||\boldsymbol{u}|||_{H^s(\mathcal{E}_h)}.\tag{4.2.24}$$

(4.2.22)-(4.2.24) hold for NIPG elliptic operator $\boldsymbol{R}_1$ too but the super-penalisation is needed for optimal L_2 norm. Thus, we could derive error estimates by using (4.2.22)-(4.2.24) and a similar argument in the scalar-valued problem. Define

$$
\begin{aligned}
&\boldsymbol{\theta}:=\boldsymbol{u}-\boldsymbol{R}_1\boldsymbol{u}, &&\boldsymbol{\chi}^n:=\boldsymbol{U}_h^n-\boldsymbol{R}_1\boldsymbol{u}^n, \qquad \boldsymbol{\varpi}^n:=\boldsymbol{W}_h^n-\boldsymbol{R}_1\dot{\boldsymbol{u}}^n,\\
&\boldsymbol{\vartheta}_q:=\boldsymbol{\psi}_q-\boldsymbol{R}_{-1}\boldsymbol{\psi}_q, &&\boldsymbol{\varsigma}_q^n:=\boldsymbol{\Psi}_{hq}^n-\boldsymbol{R}_{-1}\boldsymbol{\psi}_q^n,\ \forall q\in\{1,\dots,N_\varphi\},\\
&\boldsymbol{\nu}_q:=\boldsymbol{\zeta}_q-\boldsymbol{R}_{-1}\boldsymbol{\zeta}_q, &&\boldsymbol{\Upsilon}_q^n:=\boldsymbol{S}_{hq}^n-\boldsymbol{R}_{-1}\boldsymbol{\zeta}_q^n,\ \forall q\in\{1,\dots,N_\varphi\},
\end{aligned}
$$

for $t \in [0,T]$ and $n = 0,\dots,N$. Then (4.1.14) implies that equation

$$\frac{\boldsymbol{\chi}^{n+1}-\boldsymbol{\chi}^{n}}{\Delta t} = \frac{\boldsymbol{\varpi}^{n+1}+\boldsymbol{\varpi}^{n}}{2} - \boldsymbol{\mathcal{E}}_2^n - \boldsymbol{\mathcal{E}}_3^n, \tag{4.2.25}$$

for $n = 0,\dots,N-1$ where

$$\boldsymbol{\mathcal{E}}_2(t) := \frac{\dot{\boldsymbol{\theta}}(t+\Delta t)+\dot{\boldsymbol{\theta}}(t)}{2} - \frac{\boldsymbol{\theta}(t+\Delta t)-\boldsymbol{\theta}(t)}{\Delta t},$$
$$\boldsymbol{\mathcal{E}}_3(t) := \frac{\boldsymbol{u}(t+\Delta t)-\boldsymbol{u}(t)}{\Delta t} - \frac{\dot{\boldsymbol{u}}(t+\Delta t)+\dot{\boldsymbol{u}}(t)}{2}.$$

Note that Galerkin orthogonality helps us to have error estimates theorems, and continuity of the strong solution and homogeneous Dirichlet boundary condition imply

$$[\boldsymbol{\theta}(t)],\ [\dot{\boldsymbol{\theta}}(t)],\ [\boldsymbol{\vartheta}_q(t)],\ [\dot{\boldsymbol{\vartheta}}_q(t)],\ [\boldsymbol{\nu}_q(t)],\ [\dot{\boldsymbol{\nu}}_q(t)] = \mathbf{0} \tag{4.2.26}$$

for any t, $\forall q \in \{1,\dots,N_\varphi\}$ on $\Gamma_h \cup \Gamma_D$. Hence (4.2.26) gives

$$[\boldsymbol{\mathcal{E}}_2(t)], [\boldsymbol{\mathcal{E}}_3(t)] = \mathbf{0} \tag{4.2.27}$$

for $t \in [0, T-\Delta t]$. In addition, (4.2.26) yields

$$a_{-1}\left(\boldsymbol{\theta}(t),\boldsymbol{v}\right), a_{-1}\left(\dot{\boldsymbol{\theta}}(t),\boldsymbol{v}\right) = 0,\ \forall \boldsymbol{v} \in [\mathcal{D}_k(\mathcal{E}_h)]^d.$$

Lemma 4.4. *Suppose $\boldsymbol{u} \in H^4(0,T;[C^2(\Omega)]^d) \cap W^1_\infty(0,T;[H^s(\mathcal{E}_h)]^d)$ and $\beta_0(d-1) \geq 1$ for $s > 3/2$. Let $\boldsymbol{U}_h^n$,$\boldsymbol{W}_h^n$ and $\{\boldsymbol{\Psi}_{hq}^n\}_{q=1}^{N_\varphi}$ be the numerical solution to* **(S1)** *for $n = 0,\dots,N$. For large enough α_0, there exists a positive constant C such that*

$$\max_{0\leq n\leq N} \|\boldsymbol{\varpi}^n\|_{L_2(\Omega)} + \max_{0\leq n\leq N} \|\boldsymbol{\chi}^n\|_{\mathcal{V}} \leq C(h^{\min{(k+1,s)}-1} + \Delta t^2).$$

Furthermore, if Ω is convex or elliptic regularity is given with the super-penalisation, $\beta_0(d-1) \geq 3$, we have

$$\max_{0\leq n\leq N} \|\boldsymbol{\varpi}^n\|_{L_2(\Omega)} + \max_{0\leq n\leq N} \|\boldsymbol{\chi}^n\|_{\mathcal{V}} \leq C(h^{\min{(k+1,s)}} + \Delta t^2).$$

Proof. A proof parallels to that of Lemma 3.6. A difference of (4.2.8) and (4.2.1) for average between $t = t_{n+1}$ and $t = t_n$ gives

$$\frac{\rho}{\Delta t}\left(\boldsymbol{\varpi}^{n+1}-\boldsymbol{\varpi}^{n},\boldsymbol{v}\right)_{L_2(\Omega)} + \frac{1}{2}a_1\left(\boldsymbol{\chi}^{n+1}+\boldsymbol{\chi}^{n},\boldsymbol{v}\right) - \frac{1}{2}\sum_{q=1}^{N_\varphi} a_{-1}\left(\boldsymbol{\varsigma}_q^{n+1}+\boldsymbol{\varsigma}_q^{n},\boldsymbol{v}\right)$$
$$+\frac{1}{2}J_0^{\alpha_0,\beta_0}(\boldsymbol{\varpi}^{n+1}+\boldsymbol{\varpi}^{n},\boldsymbol{v}) = \frac{\rho}{\Delta t}\left(\dot{\boldsymbol{\theta}}^{n+1}-\dot{\boldsymbol{\theta}}^{n},\boldsymbol{v}\right)_{L_2(\Omega)} + \rho\left(\boldsymbol{\mathcal{E}}_1^n,\boldsymbol{v}\right)_{L_2(\Omega)}, \tag{4.2.28}$$

$\forall \boldsymbol{v} \in [\mathcal{D}_k(\mathcal{E}_h)]^d$ for $0 \leq n \leq N-1$, where

$$\boldsymbol{\mathcal{E}}_1(t) = \frac{\ddot{\boldsymbol{u}}(t+\Delta t)+\ddot{\boldsymbol{u}}(t)}{2} - \frac{\dot{\boldsymbol{u}}(t+\Delta t)-\dot{\boldsymbol{u}}(t)}{\Delta t},$$

since Galerkin orthogonality and (4.2.26) hold. A choice of $\boldsymbol{v} = (\boldsymbol{\chi}^{n+1} - \boldsymbol{\chi}^n)/\Delta t$ in (4.2.28) implies

$$
\begin{aligned}
&\frac{\rho}{2\Delta t}\left(\|\boldsymbol{\varpi}^{n+1}\|^2_{L_2(\Omega)} - \|\boldsymbol{\varpi}^n\|^2_{L_2(\Omega)}\right) - \frac{\rho}{\Delta t}\left(\boldsymbol{\varpi}^{n+1} - \boldsymbol{\varpi}^n, \boldsymbol{\mathcal{E}}_2^n\right)_{L_2(\Omega)} \\
&- \frac{\rho}{\Delta t}\left(\boldsymbol{\varpi}^{n+1} - \boldsymbol{\varpi}^n, \boldsymbol{\mathcal{E}}_3^n\right)_{L_2(\Omega)} + \frac{1}{2\Delta t}\left(\|\boldsymbol{\chi}^{n+1}\|^2_{\mathcal{V}} - \|\boldsymbol{\chi}^n\|^2_{\mathcal{V}}\right) - \frac{1}{2\Delta t}B(\boldsymbol{\chi}^{n+1}, \boldsymbol{\chi}^n) \\
&- \frac{1}{2\Delta t}\sum_{q=1}^{N_\varphi} a_{-1}\left(\boldsymbol{\varsigma}_q^{n+1} + \boldsymbol{\varsigma}_q^n, \boldsymbol{\chi}^{n+1} - \boldsymbol{\chi}^n\right) + \frac{1}{4}J_0^{\alpha_0,\beta_0}\left(\boldsymbol{\varpi}^{n+1} + \boldsymbol{\varpi}^n, \boldsymbol{\varpi}^{n+1} + \boldsymbol{\varpi}^n\right) \\
=&\frac{\rho}{2\Delta t}\left(\dot{\boldsymbol{\theta}}^{n+1} - \dot{\boldsymbol{\theta}}^n, \boldsymbol{\varpi}^{n+1} + \boldsymbol{\varpi}^n\right)_{L_2(\Omega)} - \frac{\rho}{\Delta t}\left(\dot{\boldsymbol{\theta}}^{n+1} - \dot{\boldsymbol{\theta}}^n, \boldsymbol{\mathcal{E}}_2^n\right)_{L_2(\Omega)} \\
&- \frac{\rho}{\Delta t}\left(\dot{\boldsymbol{\theta}}^{n+1} - \dot{\boldsymbol{\theta}}^n, \boldsymbol{\mathcal{E}}_3^n\right)_{L_2(\Omega)} + \frac{\rho}{2}\left(\boldsymbol{\mathcal{E}}_1^n, \boldsymbol{\varpi}^{n+1} + \boldsymbol{\varpi}^n\right)_{L_2(\Omega)} - \rho\left(\boldsymbol{\mathcal{E}}_1^n, \boldsymbol{\mathcal{E}}_2^n\right)_{L_2(\Omega)} \\
&- \rho\left(\boldsymbol{\mathcal{E}}_1^n, \boldsymbol{\mathcal{E}}_3^n\right)_{L_2(\Omega)}
\end{aligned}
$$

by using skew symmetric $B(\cdot,\cdot)$, (4.2.25) and (4.2.27). Taking into account summation from $n = 0$ to $n = m - 1$ for $0 < m \leq N$, we have

$$
\begin{aligned}
&\frac{\rho}{2\Delta t}\|\boldsymbol{\varpi}^m\|^2_{L_2(\Omega)} + \frac{1}{2\Delta t}\|\boldsymbol{\chi}^m\|^2_{\mathcal{V}} - \frac{1}{2\Delta t}\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi} a_{-1}\left(\boldsymbol{\varsigma}_q^{n+1} + \boldsymbol{\varsigma}_q^n, \boldsymbol{\chi}^{n+1} - \boldsymbol{\chi}^n\right) \\
&\quad + \frac{1}{4}\sum_{n=0}^{m-1} J_0^{\alpha_0,\beta_0}\left(\boldsymbol{\varpi}^{n+1} + \boldsymbol{\varpi}^n, \boldsymbol{\varpi}^{n+1} + \boldsymbol{\varpi}^n\right) \\
&=\frac{\rho}{2\Delta t}\|\boldsymbol{\varpi}^0\|^2_{L_2(\Omega)} + \frac{1}{2\Delta t}\|\boldsymbol{\chi}^0\|^2_{\mathcal{V}} + \frac{\rho}{2\Delta t}\sum_{n=0}^{m-1}\left(\dot{\boldsymbol{\theta}}^{n+1} - \dot{\boldsymbol{\theta}}^n, \boldsymbol{\varpi}^{n+1} + \boldsymbol{\varpi}^n\right)_{L_2(\Omega)} \\
&\quad - \frac{\rho}{\Delta t}\sum_{n=0}^{m-1}\left(\dot{\boldsymbol{\theta}}^{n+1} - \dot{\boldsymbol{\theta}}^n, \boldsymbol{\mathcal{E}}_2^n\right)_{L_2(\Omega)} - \frac{\rho}{\Delta t}\sum_{n=0}^{m-1}\left(\dot{\boldsymbol{\theta}}^{n+1} - \dot{\boldsymbol{\theta}}^n, \mathcal{E}_3^n\right)_{L_2(\Omega)} \\
&\quad + \frac{\rho}{2}\sum_{n=0}^{m-1}\left(\boldsymbol{\mathcal{E}}_1^n, \boldsymbol{\varpi}^{n+1} + \boldsymbol{\varpi}^n\right)_{L_2(\Omega)} - \rho\sum_{n=0}^{m-1}\left(\boldsymbol{\mathcal{E}}_1^n, \boldsymbol{\mathcal{E}}_2^n\right)_{L_2(\Omega)} - \rho\sum_{n=0}^{m-1}\left(\boldsymbol{\mathcal{E}}_1^n, \boldsymbol{\mathcal{E}}_3^n\right)_{L_2(\Omega)} \\
&\quad + \frac{\rho}{\Delta t}\sum_{n=0}^{m-1}\left(\boldsymbol{\varpi}^{n+1} - \boldsymbol{\varpi}^n, \boldsymbol{\mathcal{E}}_2^n\right)_{L_2(\Omega)} + \frac{\rho}{\Delta t}\sum_{n=0}^{m-1}\left(\boldsymbol{\varpi}^{n+1} - \boldsymbol{\varpi}^n, \boldsymbol{\mathcal{E}}_3^n\right)_{L_2(\Omega)} \\
&\quad + \frac{1}{2\Delta t}\sum_{n=0}^{m-1} B(\boldsymbol{\chi}^{n+1}, \boldsymbol{\chi}^n). \qquad (4.2.29)
\end{aligned}
$$

However, a subtraction of (4.2.9) from (4.2.2) gives

$$
\begin{aligned}
&\frac{\tau_q}{\Delta t}a_{-1}\left(\boldsymbol{\varsigma}_q^{n+1} - \boldsymbol{\varsigma}_q^n, \boldsymbol{v}\right) + \frac{1}{2}a_{-1}\left(\boldsymbol{\varsigma}_q^{n+1} + \boldsymbol{\varsigma}_q^n, \boldsymbol{v}\right) - \frac{\varphi_q}{2}a_{-1}\left(\boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^n, \boldsymbol{v}\right) \\
&=\tau_q a_{-1}\left(\boldsymbol{E}_q^n, \boldsymbol{v}\right)
\end{aligned}
$$

by Galerkin orthogonality, for any $\boldsymbol{v} \in [\mathcal{D}_k(\mathcal{E}_h)]^d$, where

$$\boldsymbol{E}_q(t) = \frac{\dot{\boldsymbol{\psi}}_q(t+\Delta t) + \dot{\boldsymbol{\psi}}_q(t)}{2} - \frac{\boldsymbol{\psi}_q(t+\Delta t) - \boldsymbol{\psi}_q(t)}{\Delta t} \text{ for each } q.$$

Whence inserting $\boldsymbol{v} = \boldsymbol{\varsigma}_q^{n+1} - \boldsymbol{\varsigma}_q^n/\Delta t$ and summing for $n = 0, \ldots, m-1$, we can derive

$$\begin{aligned}
&\frac{\varphi_q}{2\Delta t} \sum_{n=0}^{m-1} a_{-1}\left(\boldsymbol{\chi}^{n+1} - \boldsymbol{\chi}^n, \boldsymbol{\varsigma}_q^{n+1} + \boldsymbol{\varsigma}_q^n\right) \\
&= \frac{\varphi_q}{\Delta t} a_{-1}\left(\boldsymbol{\chi}^m, \boldsymbol{\varsigma}_q^m\right) - \frac{1}{2\Delta t} a_{-1}\left(\boldsymbol{\varsigma}_q^m, \boldsymbol{\varsigma}_q^m\right) - \frac{\tau_q}{\Delta t^2} \sum_{n=0}^{m-1} a_{-1}\left(\boldsymbol{\varsigma}_q^{n+1} - \boldsymbol{\varsigma}_q^n, \boldsymbol{\varsigma}_q^{n+1} - \boldsymbol{\varsigma}_q^n\right) \\
&\quad + \frac{\tau_q}{\Delta t} a_{-1}\left(\boldsymbol{E}_q^{m-1}, \boldsymbol{\varsigma}_q^m\right) - \frac{\tau_q}{\Delta t} \sum_{n=0}^{m-1} a_{-1}\left(\boldsymbol{E}_q^{n+1} - \boldsymbol{E}_q^n, \boldsymbol{\varsigma}_q^{n+1}\right) \qquad (4.2.30)
\end{aligned}$$

for any q by the fact $\boldsymbol{\varsigma}_q^0 = \mathbf{0}$ and summation by parts.

By substitution of (4.2.30) into (4.2.29), multiplying Δt implies

$$\begin{aligned}
&\frac{\rho}{2} \|\boldsymbol{\varpi}^m\|_{L_2(\Omega)}^2 + \frac{1}{2} \|\boldsymbol{\chi}^m\|_{\mathcal{V}}^2 + \frac{1}{2} \sum_{q=1}^{N_\varphi} \frac{1}{\varphi_q} a_{-1}\left(\boldsymbol{\varsigma}_q^m, \boldsymbol{\varsigma}_q^m\right) \\
&\quad + \Delta t \sum_{n=0}^{m-1} \sum_{q=1}^{N_\varphi} \frac{\tau_q}{\varphi_q} a_{-1}\left(\frac{\boldsymbol{\varsigma}_q^{n+1} - \boldsymbol{\varsigma}_q^n}{\Delta t}, \frac{\boldsymbol{\varsigma}_q^{n+1} - \boldsymbol{\varsigma}_q^n}{\Delta t}\right) \\
&\quad + \frac{\Delta t}{4} \sum_{n=0}^{m-1} J_0^{\alpha_0, \beta_0}\left(\boldsymbol{\varpi}^{n+1} + \boldsymbol{\varpi}^n, \boldsymbol{\varpi}^{n+1} + \boldsymbol{\varpi}^n\right) \\
&= \frac{\rho}{2} \|\boldsymbol{\varpi}^0\|_{L_2(\Omega)}^2 + \frac{1}{2} \|\boldsymbol{\chi}^0\|_{\mathcal{V}}^2 + \frac{\rho}{2} \sum_{n=0}^{m-1} \left(\dot{\boldsymbol{\theta}}^{n+1} - \dot{\boldsymbol{\theta}}^n, \boldsymbol{\varpi}^{n+1} + \boldsymbol{\varpi}^n\right)_{L_2(\Omega)} \\
&\quad - \rho \sum_{n=0}^{m-1} \left(\dot{\boldsymbol{\theta}}^{n+1} - \dot{\boldsymbol{\theta}}^n, \boldsymbol{\mathcal{E}}_2^n\right)_{L_2(\Omega)} - \rho \sum_{n=0}^{m-1} \left(\dot{\boldsymbol{\theta}}^{n+1} - \dot{\boldsymbol{\theta}}^n, \boldsymbol{\mathcal{E}}_3^n\right)_{L_2(\Omega)} \\
&\quad + \frac{\rho}{2} \Delta t \sum_{n=0}^{m-1} \left(\boldsymbol{\mathcal{E}}_1^n, \boldsymbol{\varpi}^{n+1} + \boldsymbol{\varpi}^n\right)_{L_2(\Omega)} - \rho \Delta t \sum_{n=0}^{m-1} \left(\boldsymbol{\mathcal{E}}_1^n, \boldsymbol{\mathcal{E}}_2^n\right)_{L_2(\Omega)} - \rho \Delta t \sum_{n=0}^{m-1} \left(\boldsymbol{\mathcal{E}}_1^n, \boldsymbol{\mathcal{E}}_3^n\right)_{L_2(\Omega)} \\
&\quad + \rho \sum_{n=0}^{m-1} \left(\boldsymbol{\varpi}^{n+1} - \boldsymbol{\varpi}^n, \boldsymbol{\mathcal{E}}_2^n\right)_{L_2(\Omega)} + \rho \sum_{n=0}^{m-1} \left(\boldsymbol{\varpi}^{n+1} - \boldsymbol{\varpi}^n, \boldsymbol{\mathcal{E}}_3^n\right)_{L_2(\Omega)} + \sum_{q=1}^{N_\varphi} a_{-1}\left(\boldsymbol{\chi}^m, \boldsymbol{\varsigma}_q^m\right) \\
&\quad + \sum_{q=1}^{N_\varphi} \frac{\tau_q}{\varphi_q} a_{-1}\left(\boldsymbol{E}_q^{m-1}, \boldsymbol{\varsigma}_q^m\right) - \sum_{n=0}^{m-1} \sum_{q=1}^{N_\varphi} \frac{\tau_q}{\varphi_q} a_{-1}\left(\boldsymbol{E}_q^{n+1} - \boldsymbol{E}_q^n, \boldsymbol{\varsigma}_q^{n+1}\right) + \frac{1}{2} \sum_{n=0}^{m-1} B(\boldsymbol{\chi}^{n+1}, \boldsymbol{\chi}^n).
\end{aligned}$$

Hence the definition of SIPG and coercivity allow us to have

$$\frac{\rho}{2} \|\boldsymbol{\varpi}^m\|_{L_2(\Omega)}^2 + \frac{1}{2} \|\boldsymbol{\chi}^m\|_{\mathcal{V}}^2 + \frac{1}{2} \sum_{q=1}^{N_\varphi} \frac{1}{\varphi_q} \|\boldsymbol{\varsigma}_q^m\|_{\mathcal{V}}^2 + \Delta t \sum_{n=0}^{m-1} \sum_{q=1}^{N_\varphi} \frac{\kappa \tau_q}{\varphi_q} \left\|\frac{\boldsymbol{\varsigma}_q^{n+1} - \boldsymbol{\varsigma}_q^n}{\Delta t}\right\|_{\mathcal{V}}^2$$

$$
\begin{aligned}
&+\frac{\Delta t}{4}\sum_{n=0}^{m-1} J_0^{\alpha_0,\beta_0}\left(\boldsymbol{\varpi}^{n+1}+\boldsymbol{\varpi}^{n},\boldsymbol{\varpi}^{n+1}+\boldsymbol{\varpi}^{n}\right)\\
=&\Bigg|\frac{\rho}{2}\left\|\boldsymbol{\varpi}^0\right\|^2_{L_2(\Omega)}+\frac{1}{2}\left\|\boldsymbol{\chi}^0\right\|^2_{\mathcal{V}}+\frac{\rho}{2}\sum_{n=0}^{m-1}\left(\dot{\boldsymbol{\theta}}^{n+1}-\dot{\boldsymbol{\theta}}^{n},\boldsymbol{\varpi}^{n+1}+\boldsymbol{\varpi}^{n}\right)_{L_2(\Omega)}\\
&-\rho\sum_{n=0}^{m-1}\left(\dot{\boldsymbol{\theta}}^{n+1}-\dot{\boldsymbol{\theta}}^{n},\boldsymbol{\mathcal{E}}_2^n\right)_{L_2(\Omega)}-\rho\sum_{n=0}^{m-1}\left(\dot{\boldsymbol{\theta}}^{n+1}-\dot{\boldsymbol{\theta}}^{n},\boldsymbol{\mathcal{E}}_3^n\right)_{L_2(\Omega)}\\
&+\frac{\rho}{2}\Delta t\sum_{n=0}^{m-1}\left(\boldsymbol{\mathcal{E}}_1^n,\boldsymbol{\varpi}^{n+1}+\boldsymbol{\varpi}^{n}\right)_{L_2(\Omega)}-\rho\Delta t\sum_{n=0}^{m-1}\left(\boldsymbol{\mathcal{E}}_1^n,\boldsymbol{\mathcal{E}}_2^n\right)_{L_2(\Omega)}-\rho\Delta t\sum_{n=0}^{m-1}\left(\boldsymbol{\mathcal{E}}_1^n,\boldsymbol{\mathcal{E}}_3^n\right)_{L_2(\Omega)}\\
&+\rho\sum_{n=0}^{m-1}\left(\boldsymbol{\varpi}^{n+1}-\boldsymbol{\varpi}^{n},\boldsymbol{\mathcal{E}}_2^n\right)_{L_2(\Omega)}+\rho\sum_{n=0}^{m-1}\left(\boldsymbol{\varpi}^{n+1}-\boldsymbol{\varpi}^{n},\boldsymbol{\mathcal{E}}_3^n\right)_{L_2(\Omega)}+\sum_{q=1}^{N_\varphi}a_{-1}\left(\boldsymbol{\chi}^m,\boldsymbol{\varsigma}_q^m\right)\\
&+\sum_{q=1}^{N_\varphi}\frac{\tau_q}{\varphi_q}a_{-1}\left(\boldsymbol{E}_q^{m-1},\boldsymbol{\varsigma}_q^m\right)-\sum_{n=0}^{m-2}\sum_{q=1}^{N_\varphi}\frac{\tau_q}{\varphi_q}a_{-1}\left(\boldsymbol{E}_q^{n+1}-\boldsymbol{E}_q^{n},\boldsymbol{\varsigma}_q^{n+1}\right)+\frac{1}{2}\sum_{n=0}^{m-1}B(\boldsymbol{\chi}^{n+1},\boldsymbol{\chi}^n)\\
&+\sum_{q=1}^{N_\varphi}\frac{1}{\varphi_q}\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e\{\underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{\varsigma}_q^m)\cdot\boldsymbol{n}_e\}\cdot[\boldsymbol{\varsigma}_q^m]de\Bigg|. \qquad (4.2.31)
\end{aligned}
$$

Now, one can show the bounds for (4.2.31) as following the similar arguments in the bounds for (3.3.20) in vector-valued. But we should also use (4.2.7), (4.2.6) and elliptic approximation properties (4.2.22)-(4.2.24) for vector-valued cases. Consequently, taking into account maximum, we can obtain for large α_0

$$\max_{0\le n\le N}\|\boldsymbol{\varpi}^n\|_{L_2(\Omega)}+\max_{0\le n\le N}\|\boldsymbol{\chi}^n\|_{\mathcal{V}}\le C(h^{\min(k+1,s)-1}+\Delta t^2),$$

additionally, if Ω is convex or elliptic regularity is satisfied

$$\max_{0\le n\le N}\|\boldsymbol{\varpi}^n\|_{L_2(\Omega)}+\max_{0\le n\le N}\|\boldsymbol{\chi}^n\|_{\mathcal{V}}\le C(h^{\min(k+1,s)}+\Delta t^2),$$

where C is a positive constant independent of $h,\ \Delta t$, not increasing exponentially with respect to T. □

In case of the velocity form, we have similar results as following.

Lemma 4.5. *Suppose $\boldsymbol{u}\in H^4(0,T;[C^2(\Omega)]^d)\cap W^1_\infty(0,T;[H^s(\mathcal{E}_h)]^d)$ and $\beta_0(d-1)\ge 1$ for $s>3/2$. Let us consider the fully discrete solution of* **(S2)***. For large enough α_0, there exists a positive constant C such that*

$$\max_{0\le n\le N}\|\boldsymbol{\varpi}^n\|_{L_2(\Omega)}+\max_{0\le n\le N}\|\boldsymbol{\chi}^n\|_{\mathcal{V}}\le C(h^{\min(k+1,s)-1}+\Delta t^2).$$

Furthermore, if Ω is convex or elliptic regularity is given with the super-penalisation, we have

$$\max_{0\le n\le N}\|\boldsymbol{\varpi}^n\|_{L_2(\Omega)}+\max_{0\le n\le N}\|\boldsymbol{\chi}^n\|_{\mathcal{V}}\le C(h^{\min(k+1,s)}+\Delta t^2).$$

Proof. A proof is shown by the extension of Lemma 3.8. For average between $t = t_{n+1}$ and $t = t_n$, subtracting (4.2.3) from (4.2.12) gives

$$\begin{aligned}
&\frac{\rho}{\Delta t}\left(\boldsymbol{\varpi}^{n+1}-\boldsymbol{\varpi}^{n},\boldsymbol{v}\right)_{L_2(\Omega)}+\frac{\varphi_0}{2}a_1\left(\boldsymbol{\chi}^{n+1}+\boldsymbol{\chi}^{n},\boldsymbol{v}\right)+\frac{1}{2}\sum_{q=1}^{N_\varphi}a_{-1}\left(\boldsymbol{\Upsilon}_q^{n+1}+\boldsymbol{\Upsilon}_q^{n},\boldsymbol{v}\right)\\
&\quad+\frac{1}{2}J_0^{\alpha_0,\beta_0}(\boldsymbol{\varpi}^{n+1}+\boldsymbol{\varpi}^{n},\boldsymbol{v})\\
&=\frac{\rho}{\Delta t}\left(\dot{\boldsymbol{\theta}}^{n+1}-\dot{\boldsymbol{\theta}}^{n},\boldsymbol{v}\right)_{L_2(\Omega)}+\rho\left(\boldsymbol{\mathcal{E}}_1^n,\boldsymbol{v}\right)_{L_2(\Omega)}
\end{aligned}$$

for $n \in \{0,\dots,N-1\}$, $\forall \boldsymbol{v} \in [\mathcal{D}_k(\mathcal{E}_h)]^d$, where $\boldsymbol{\mathcal{E}}_1(t) := \frac{\ddot{\boldsymbol{u}}(t+\Delta t)+\ddot{\boldsymbol{u}}(t)}{2} - \frac{\dot{\boldsymbol{u}}(t+\Delta t)-\dot{\boldsymbol{u}}(t)}{\Delta t}$ by Galerkin orthogonality. By choosing $\boldsymbol{v} = \frac{\boldsymbol{\chi}^{n+1}-\boldsymbol{\chi}^n}{\Delta t}$, it is written by

$$\begin{aligned}
&\frac{\rho}{2\Delta t}\left(\|\boldsymbol{\varpi}^{n+1}\|_{L_2(\Omega)}^2-\|\boldsymbol{\varpi}^{n}\|_{L_2(\Omega)}^2\right)+\frac{\varphi_0}{2\Delta t}\left(\|\boldsymbol{\chi}^{n+1}\|_{\mathcal{V}}^2-\|\boldsymbol{\chi}^{n}\|_{\mathcal{V}}^2\right)\\
&\quad+\frac{1}{2\Delta t}\sum_{q=1}^{N_\varphi}a_{-1}\left(\boldsymbol{\Upsilon}_q^{n+1}+\boldsymbol{\Upsilon}_q^{n},\boldsymbol{\chi}^{n+1}-\boldsymbol{\chi}^{n}\right)+\frac{1}{4}J_0^{\alpha_0,\beta_0}(\boldsymbol{\varpi}^{n+1}+\boldsymbol{\varpi}^{n},\boldsymbol{\varpi}^{n+1}+\boldsymbol{\varpi}^{n})\\
&=\frac{\rho}{2\Delta t}\left(\dot{\boldsymbol{\theta}}^{n+1}-\dot{\boldsymbol{\theta}}^{n},\boldsymbol{\varpi}^{n+1}+\boldsymbol{\varpi}^{n}\right)_{L_2(\Omega)}+\frac{\rho}{2}\left(\boldsymbol{\mathcal{E}}_1^n,\boldsymbol{\varpi}^{n+1}+\boldsymbol{\varpi}^{n}\right)_{L_2(\Omega)}\\
&\quad-\frac{\rho}{\Delta t}\left(\dot{\boldsymbol{\theta}}^{n+1}-\dot{\boldsymbol{\theta}}^{n},\boldsymbol{\mathcal{E}}_2^n\right)_{L_2(\Omega)}-\frac{\rho}{\Delta t}\left(\dot{\boldsymbol{\theta}}^{n+1}-\dot{\boldsymbol{\theta}}^{n},\boldsymbol{\mathcal{E}}_3^n\right)_{L_2(\Omega)}\\
&\quad-\rho\left(\boldsymbol{\mathcal{E}}_1^n,\boldsymbol{\mathcal{E}}_2^n\right)_{L_2(\Omega)}-\rho\left(\boldsymbol{\mathcal{E}}_1^n,\boldsymbol{\mathcal{E}}_3^n\right)_{L_2(\Omega)}+\frac{\varphi_0}{2\Delta t}B(\boldsymbol{\chi}^{n+1},\boldsymbol{\chi}^{n}),
\end{aligned}\tag{4.2.32}$$

by (4.2.25).

On the other hand, a difference of (4.2.4) and (4.2.13) yields for any $q \in \{1,\dots,N_\varphi\}$, $\forall \boldsymbol{v} \in [\mathcal{D}_k(\mathcal{E}_h)]^d$

$$\begin{aligned}
&\frac{\tau_q}{\Delta t}a_{-1}\left(\boldsymbol{\Upsilon}_q^{n+1}-\boldsymbol{\Upsilon}_q^{n},\boldsymbol{v}\right)+\frac{1}{2}a_{-1}\left(\boldsymbol{\Upsilon}_q^{n+1}+\boldsymbol{\Upsilon}_q^{n},\boldsymbol{v}\right)-\frac{\tau_q\varphi_q}{\Delta t}a_{-1}\left(\boldsymbol{\chi}^{n+1}-\boldsymbol{\chi}^{n},\boldsymbol{v}\right)\\
&=\tau_q a_{-1}\left(\boldsymbol{E}_q^n,\boldsymbol{v}\right)-\tau_q\varphi_q a_{-1}\left(\boldsymbol{\mathcal{E}}_3^n,\boldsymbol{v}\right)
\end{aligned}$$

by Garlerkin orthogonality where for each q

$$\boldsymbol{E}_q(t) := \frac{\dot{\boldsymbol{\zeta}}_q(t+\Delta t)+\dot{\boldsymbol{\zeta}}_q(t)}{2}-\frac{\boldsymbol{\zeta}_q(t+\Delta t)-\boldsymbol{\zeta}_q(t)}{\Delta t}.$$

When we put $\boldsymbol{v} = \frac{\boldsymbol{\Upsilon}_q^{n+1}+\boldsymbol{\Upsilon}_q^n}{2}$ and divide it by $2\tau_q\varphi_q$, symmetry of SIPG implies

$$\begin{aligned}
&\frac{1}{2\Delta t}a_{-1}\left(\boldsymbol{\Upsilon}_q^{n+1}+\boldsymbol{\Upsilon}_q^{n},\boldsymbol{\chi}^{n+1}-\boldsymbol{\chi}^{n}\right)\\
&=\frac{1}{2\Delta t\varphi_q}\left(a_{-1}\left(\boldsymbol{\Upsilon}_q^{n+1},\boldsymbol{\Upsilon}_q^{n+1}\right)-a_{-1}\left(\boldsymbol{\Upsilon}_q^{n},\boldsymbol{\Upsilon}_q^{n}\right)\right)+\frac{1}{2\tau_q\varphi_q}a_{-1}\left(\boldsymbol{\Upsilon}_q^{n+1}+\boldsymbol{\Upsilon}_q^{n},\boldsymbol{\Upsilon}_q^{n+1}+\boldsymbol{\Upsilon}_q^{n}\right)\\
&\quad-\frac{1}{2\varphi_q}a_{-1}\left(\boldsymbol{E}_q^n,\boldsymbol{\Upsilon}_q^{n+1}+\boldsymbol{\Upsilon}_q^{n}\right)+\frac{1}{2}a_{-1}\left(\boldsymbol{\mathcal{E}}_3^n,\boldsymbol{\Upsilon}_q^{n+1}+\boldsymbol{\Upsilon}_q^{n}\right).
\end{aligned}\tag{4.2.33}$$

Hence taking into account substitution of (4.2.33) into (4.2.32), summation for $n = 0, \ldots, m-1$, where $m \in \{1, \ldots, N\}$ and multiplication by Δt gives us to obtain

$$
\begin{aligned}
&\frac{\rho}{2}\left\|\boldsymbol{\varpi}^m\right\|_{L_2(\Omega)}^2+\frac{\varphi_0}{2}\left\|\boldsymbol{\chi}^m\right\|_{\mathcal{V}}^2+\frac{1}{2}\sum_{q=1}^{N_\varphi}\frac{1}{\varphi_q}a_{-1}\left(\boldsymbol{\Upsilon}_q^m,\boldsymbol{\Upsilon}_q^m\right)\\
&\quad+\Delta t\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi}\frac{1}{2\tau_q\varphi_q}a_{-1}\left(\boldsymbol{\Upsilon}_q^{n+1}+\boldsymbol{\Upsilon}_q^n,\boldsymbol{\Upsilon}_q^{n+1}+\boldsymbol{\Upsilon}_q^n\right)\\
&\quad+\frac{\Delta t}{4}\sum_{n=0}^{m-1}J_0^{\alpha_0,\beta_0}(\boldsymbol{\varpi}^{n+1}+\boldsymbol{\varpi}^n,\boldsymbol{\varpi}^{n+1}+\boldsymbol{\varpi}^n)\\
=&\frac{\rho}{2}\left\|\boldsymbol{\varpi}^0\right\|_{L_2(\Omega)}^2+\frac{\varphi_0}{2}\left\|\boldsymbol{\chi}^0\right\|_{\mathcal{V}}^2+\frac{\rho}{2}\sum_{n=0}^{m-1}\left(\dot{\boldsymbol{\theta}}^{n+1}-\dot{\boldsymbol{\theta}}^n,\boldsymbol{\varpi}^{n+1}+\boldsymbol{\varpi}^n\right)_{L_2(\Omega)}\\
&\quad+\frac{\rho}{2}\Delta t\sum_{n=0}^{m-1}\left(\boldsymbol{\mathcal{E}}_1^n,\boldsymbol{\varpi}^{n+1}+\boldsymbol{\varpi}^n\right)_{L_2(\Omega)}-\rho\sum_{n=0}^{m-1}\left(\dot{\boldsymbol{\theta}}^{n+1}-\dot{\boldsymbol{\theta}}^n,\boldsymbol{\mathcal{E}}_2^n\right)_{L_2(\Omega)}\\
&\quad-\rho\sum_{n=0}^{m-1}\left(\dot{\boldsymbol{\theta}}^{n+1}-\dot{\boldsymbol{\theta}}^n,\boldsymbol{\mathcal{E}}_3^n\right)_{L_2(\Omega)}-\rho\Delta t\sum_{n=0}^{m-1}\left(\boldsymbol{\mathcal{E}}_1^n,\boldsymbol{\mathcal{E}}_2^n\right)_{L_2(\Omega)}-\rho\Delta t\sum_{n=0}^{m-1}\left(\boldsymbol{\mathcal{E}}_1^n,\boldsymbol{\mathcal{E}}_3^n\right)_{L_2(\Omega)}\\
&\quad+\frac{\Delta t}{2}\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi}\frac{1}{\varphi_q}a_{-1}\left(\boldsymbol{E}_q^n,\boldsymbol{\Upsilon}_q^{n+1}+\boldsymbol{\Upsilon}_q^n\right)-\frac{\Delta t}{2}\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi}a_{-1}\left(\boldsymbol{\mathcal{E}}_3^n,\boldsymbol{\Upsilon}_q^{n+1}+\boldsymbol{\Upsilon}_q^n\right)\\
&\quad+\frac{\varphi_0}{2}\sum_{n=0}^{m-1}B(\boldsymbol{\chi}^{n+1},\boldsymbol{\chi}^n).
\end{aligned}
$$

Then use of coercivity of SIPG leads us to have

$$
\begin{aligned}
&\frac{\rho}{2}\left\|\boldsymbol{\varpi}^m\right\|_{L_2(\Omega)}^2+\frac{\varphi_0}{2}\left\|\boldsymbol{\chi}^m\right\|_{\mathcal{V}}^2+\frac{1}{2}\sum_{q=1}^{N_\varphi}\frac{\kappa}{\varphi_q}\left\|\boldsymbol{\Upsilon}_q^m\right\|_{\mathcal{V}}^2+\Delta t\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi}\frac{\kappa}{2\tau_q\varphi_q}\left\|\boldsymbol{\Upsilon}_q^{n+1}+\boldsymbol{\Upsilon}_q^n\right\|_{\mathcal{V}}^2\\
&\quad+\frac{\Delta t}{4}\sum_{n=0}^{m-1}J_0^{\alpha_0,\beta_0}(\boldsymbol{\varpi}^{n+1}+\boldsymbol{\varpi}^n,\boldsymbol{\varpi}^{n+1}+\boldsymbol{\varpi}^n)\\
\leq&\Bigg|\frac{\rho}{2}\left\|\boldsymbol{\varpi}^0\right\|_{L_2(\Omega)}^2+\frac{\varphi_0}{2}\left\|\boldsymbol{\chi}^0\right\|_{\mathcal{V}}^2+\frac{\rho}{2}\sum_{n=0}^{m-1}\left(\dot{\boldsymbol{\theta}}^{n+1}-\dot{\boldsymbol{\theta}}^n,\boldsymbol{\varpi}^{n+1}+\boldsymbol{\varpi}^n\right)_{L_2(\Omega)}\\
&\quad+\frac{\rho}{2}\Delta t\sum_{n=0}^{m-1}\left(\boldsymbol{\mathcal{E}}_1^n,\boldsymbol{\varpi}^{n+1}+\boldsymbol{\varpi}^n\right)_{L_2(\Omega)}-\rho\sum_{n=0}^{m-1}\left(\dot{\boldsymbol{\theta}}^{n+1}-\dot{\boldsymbol{\theta}}^n,\boldsymbol{\mathcal{E}}_2^n\right)_{L_2(\Omega)}\\
&\quad-\rho\sum_{n=0}^{m-1}\left(\dot{\boldsymbol{\theta}}^{n+1}-\dot{\boldsymbol{\theta}}^n,\boldsymbol{\mathcal{E}}_3^n\right)_{L_2(\Omega)}-\rho\Delta t\sum_{n=0}^{m-1}\left(\boldsymbol{\mathcal{E}}_1^n,\boldsymbol{\mathcal{E}}_2^n\right)_{L_2(\Omega)}-\rho\Delta t\sum_{n=0}^{m-1}\left(\boldsymbol{\mathcal{E}}_1^n,\boldsymbol{\mathcal{E}}_3^n\right)_{L_2(\Omega)}\\
&\quad+\frac{\Delta t}{2}\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi}\frac{1}{\varphi_q}a_{-1}\left(\boldsymbol{E}_q^n,\boldsymbol{\Upsilon}_q^{n+1}+\boldsymbol{\Upsilon}_q^n\right)-\frac{\Delta t}{2}\sum_{n=0}^{m-1}\sum_{q=1}^{N_\varphi}a_{-1}\left(\boldsymbol{\mathcal{E}}_3^n,\boldsymbol{\Upsilon}_q^{n+1}+\boldsymbol{\Upsilon}_q^n\right)
\end{aligned}
$$

$$+\frac{\varphi_0}{2}\sum_{n=0}^{m-1} B(\boldsymbol{\chi}^{n+1},\boldsymbol{\chi}^n)\Bigg|. \tag{4.2.34}$$

As seen in the bound for (3.3.33), we can observe the right hand side of (4.2.34) is bounded. More precisely, use of Cauchy-Schwarz inequalities, Young's inequalities, integration by parts (also summation by parts), continuity of SIPG, (4.2.22)-(4.2.24), (4.2.7) and Crank-Nicolson approximations allows us to obtain the following result. If α_0 is sufficiently large, we have

$$\max_{0\le n\le N}\|\boldsymbol{\varpi}^n\|_{L_2(\Omega)}+\max_{0\le n\le N}\|\boldsymbol{\chi}^n\|_{\mathcal{V}}\le C(h^{\min(k+1,s)-1}+\Delta t^2),$$

where C is a positive constant independent of h and Δt. Once elliptic regularity estimates hold, we have

$$\max_{0\le n\le N}\|\boldsymbol{\varpi}^n\|_{L_2(\Omega)}+\max_{0\le n\le N}\|\boldsymbol{\chi}^n\|_{\mathcal{V}}\le C(h^{\min(k+1,s)}+\Delta t^2).$$

□

By Lemmas 4.4 and 4.5, we can derive a numerical error estimates theorem.

Theorem 4.9. *Suppose* $\boldsymbol{u}\in H^4(0,T;[C^2(\Omega)]^d)\cap W^1_\infty(0,T;[H^s(\mathcal{E}_h)]^d)$ *and the discrete solutions in* $[\mathcal{D}_k(\mathcal{E}_h)]^d$ *satisfy either* **(S1)** *or* **(S2)** *for* $s>3/2, s\in\mathbb{N}$. *If we assume the conditions of* Lemmas 4.4 *and* 4.5 *are satisfied, then we have*

$$\max_{0\le n\le N}\|\boldsymbol{u}(t_n)-\boldsymbol{U}_h^n\|_{\mathcal{V}}\le C(h^{\min(k+1,s)-1}+\Delta t^2)$$

and

$$\max_{0\le n\le N}\|\dot{\boldsymbol{u}}(t_n)-\boldsymbol{W}_h^n\|_{L_2(\Omega)}\le C(h^{\min(k+1,s)-1}+\Delta t^2)$$

for some positive C. *With elliptic regularity, it is also observed that*

$$\max_{0\le n\le N}\|\dot{\boldsymbol{u}}(t_n)-\boldsymbol{W}_h^n\|_{L_2(\Omega)}\le C(h^{\min(k+1,s)}+\Delta t^2)$$

for some positive C. *In addition, we can also see* L_2 *error estimates of a displacement vector*

$$\max_{0\le n\le N}\|\boldsymbol{u}(t_n)-\boldsymbol{U}_h^n\|_{L_2(\Omega)}\le C(h^{\min(k+1,s)-1}+\Delta t^2).$$

If elliptic regularity is given, it shows

$$\max_{0\le n\le N}\|\boldsymbol{u}(t_n)-\boldsymbol{U}_h^n\|_{L_2(\Omega)}\le C(h^{\min(k+1,s)}+\Delta t^2).$$

Proof. Our claim have resulted in Lemmas 4.4 and 4.5 with triangular inequalities as shown in other error estimates theorems. L_2 norm error estimation of a displacement vector, however, requires Poincaré's inequality for piecewise H^1 vector field. For example, (1.4.10) can be extended by

$$\forall \boldsymbol{v} \in [H^1(\mathcal{E}_h)]^d, \ \|\boldsymbol{v}\|_{L_2(\Omega)} \leq C \left(|||\nabla \boldsymbol{v}|||^2_{H^0(\mathcal{E}_h)} + \sum_{e \subset \Gamma_h \cup \Gamma_D} \frac{1}{|e|^{\beta_0}} \|[\boldsymbol{v}]\|^2_{L_2(e)} \right)^{1/2}$$

and therefore

$$\forall \boldsymbol{v} \in [H^1(\mathcal{E}_h)]^d, \ \|\boldsymbol{v}\|_{L_2(\Omega)} \leq C \|\boldsymbol{v}\|_{\mathcal{V}}$$

by (4.2.5) with the definition of the DG energy norm. Hence we have

$$\|\boldsymbol{u}(t_n) - \boldsymbol{U}_h^n\|_{L_2(\Omega)} \leq \|\boldsymbol{\theta}(t_n)\|_{L_2(\Omega)} + \|\boldsymbol{\chi}^n\|_{L_2(\Omega)} \leq \|\boldsymbol{\theta}(t_n)\|_{L_2(\Omega)} + C \|\boldsymbol{\chi}^n\|_{\mathcal{V}}.$$

Consequently, Lemmas 4.4 and 4.5 lead us to obtain optimal and suboptimal L_2 error estimates. □

For the existence and uniqueness of fully discrete solutions regardless of internal variables, sufficiently large penalty parameters are required. Also it is essential for proper convergence orders. In particular, coercivity, continuity, and the bound of the skew symmetric part have resulted in a large α_0 and $\beta_0(d-1) \geq 1$. Furthermore, optimal L_2 error estimates need the super-penalisation $\beta_0(d-1) \geq 3$ due to NIPG.

4.2.2 Numerical Experiments

As in CGFEM, we recall the sufficiently smooth strong solution $\boldsymbol{u}$ the on unit square. In addition, we set all coefficients as in Section 4.1.2. Note that our spatial domain guarantees elliptic regularity estimates and hence it is able to observe optimal L_2 error estimates if super-penalised.

First of all, we would like to solve elastic problem with DG. We want to check the exactness and elliptic error estimates. We want to solve a simple elastic problem as follow:

$$-\nabla \cdot \underline{\boldsymbol{\varepsilon}}(\boldsymbol{u}) = \boldsymbol{f}.$$

Example 4.1.

Let $\boldsymbol{u} = (x, x)$ on the unit square. Using both SIPG and NIPG, we can approximate the discrete solution $\boldsymbol{U}_h$. Then the numerical error $\boldsymbol{e}_h = \boldsymbol{u} - \boldsymbol{U}_h$ can be computed with in L_2 norm. In Table 4.5, it is described that our numerical scheme shows the exactness but it requires sufficiently large penalty parameter α_0. For the both methods, $\|\boldsymbol{e}_h\|_{L_2(\Omega)}$ is small enough to see the exactness when $\alpha_0 = 10$.

α_0	β_0	SIPG	NIPG
0.001	1	1.848×10^{-11}	4.004×10^{-11}
10	1	1.779×10^{-15}	3.0919×10^{-15}

Table 4.5: Numerical error of elastic problems: $\boldsymbol{u} = (x, x)$, $h = 1$

Example 4.2.

Consider a quadratic polynomial as our strong solution. Let $\boldsymbol{u} = (xy, 0)$. We solve the elastic problem by SIPG and NIPG with either linear or quadratic polynomial basis. Obviously, a numerical solution would be exact to the strong solution when $k \geq 2$. As shown in Table 4.6, whence we take $k = 2$, the numerical error in L_2 norm becomes quite small even if there are only two triangles on the mesh.

		$k=1$		$k=2$	
α_0	β_0	SIPG	NIPG	SIPG	NIPG
10	1	6.525×10^{-2}	7.023×10^{-2}	4.538×10^{-15}	3.586×10^{-15}
100	1	9.232×10^{-2}	9.277×10^{-2}	1.064×10^{-15}	4.231×10^{-15}

Table 4.6: Numerical error of elastic problems: $\boldsymbol{u} = (xy, 0)$, $h = 1$

Remark In [67, 16], an adaptive DGFEM for linear elasticity problem is presented. In particular, as a matter of choice for the penalty parameter, α_0 can be selected by $\alpha_0 = O(10)$ [16]. More details in terms of how large α_0 must be, are shown in [67].

Example 4.3.

Now, we consider a hyperbolic problem without internal variables, i.e. solve

$$\ddot{\boldsymbol{u}} - \nabla \cdot \underline{\varepsilon}(\boldsymbol{u}) = \boldsymbol{f}.$$

We set an exact solution by $\boldsymbol{u} = (t^2 xy, 0)$. According to our error estimates theorems our numerical solution has first order accuracy with linear polynomial basis and the exactness with higher degree of polynomials with respect to spatial domain meshes, since we use the second order scheme in time. For $k = 1$, Table 4.7 indicates that the approximate solution converges with almost second order in L_2 norm, respectively for spatial meshes and it has the exactness in time. On the other hands, with quadratic polynomial basis, Table 4.8 shows the numerical solution has exactness in time and space.

SIPG

h \ Δt	1	1/2	1/4	1/8
1	7.6669e-02	7.2265e-02	7.0970e-02	7.0639e-02
1/2	2.6789e-02	2.4357e-02	2.4052e-02	2.4073e-02
1/4	7.7162e-03	6.7678e-03	6.6023e-03	6.5882e-03
1/8	2.1078e-03	1.8321e-03	1.7802e-03	1.7720e-03

NIPG

h \ Δt	1	1/2	1/4	1/8
1	7.6070e-02	7.2027e-02	7.0825e-02	7.0518e-02
1/2	2.6252e-02	2.4116e-02	2.3895e-02	2.3937e-02
1/4	7.4285e-03	6.6318e-03	6.5088e-03	6.5053e-03
1/8	2.0055e-03	1.7812e-03	1.7439e-03	1.7394e-03

Table 4.7: Numerical error of dynamic elastic problems: $\alpha_0 = 10$, $\beta_0 = 1$, $k = 1$

SIPG

h \ Δt	1	1/2	1/4	1/8
1	1.9955e-15	1.5804e-15	1.7381e-15	7.1551e-15
1/2	3.2284e-15	1.0502e-15	4.5765e-15	9.8627e-15
1/4	5.6848e-15	3.4841e-15	4.3158e-15	4.3146e-15
1/8	1.7948e-14	1.6016e-14	1.6074e-14	1.4737e-14

NIPG

h \ Δt	1	1/2	1/4	1/8
1	1.8634e-15	1.4061e-15	1.8040e-15	7.9326e-15
1/2	1.2710e-15	2.2216e-15	1.3763e-15	1.3226e-14
1/4	6.5714e-15	5.0503e-15	7.8005e-15	9.3680e-15
1/8	3.4030e-14	2.5324e-14	1.9076e-14	1.8736e-14

Table 4.8: Numerical error of dynamic elastic problems: $\alpha_0 = 10$, $\beta_0 = 1$, $k = 2$

Remark Recall the matter of condition number in the previous scalar DG problem. [64] presents the performance of various DG methods including standard/super-penalised NIPG. The condition number of the stiffness matrix follows $O(h^{-(\beta_0+1)})$. However, it is necessary for optimal L_2 error estimates to introduce the super-penalisation. While we want to get L_2 optimality, we have severe difficulty in solving the linear system by iterative methods. More precisely, our linear solvers in FEniCS (*biconjugate gradient method* and *GMRES*) encounter critical issues for fine meshes, despite theoretical stability bounds. In practice, increasing condition numbers by smaller h force the performance of iterative methods to be deteriorated. Therefore, it is essential to improve linear solvers. For instance, we can develop and use some Krylov methods e.g. multigrid algorithms [65, 68] and preconditioners such as Schwarz algorithms [69] in FEniCS.

Turning back to viscoelastic problems, we set the model problem as in CGFEM. Let us define

$$\boldsymbol{u}(x,y,t) = (xye^{1-t}, \cos(t)\sin(xy))$$

on the unit square with two internal variables where

$$\varphi_0 = 0.5,\ \varphi_1 = 0.1,\ \varphi_2 = 0.4,\ \tau_1 = 0.5,\ \tau_2 = 1.5.$$

Moreover, we assume an identity fourth order tensor as our $\underline{\boldsymbol{D}}$. As seen in Example 4.3, the penalty parameter may follow $\alpha_0 = O(10)$ and so we will choose $\alpha_0 = 50$ and β_0 with varying penalisation (standard one $\beta_0 = 1$ and super one $\beta_0 = 3$).

We solve it in two ways; the displacement form **(S1)** and velocity form **(S2)**, respectively. The resulting linear system is dealt with by *biconjugate gradient method* as a linear solver and *incomplete LU* as a preconditioner provided in FEniCS.

Due to the sufficiently smooth exact solution, Theorem 4.9 gives

$$\|\boldsymbol{e}_h^N\|_{\mathcal{V}} = O(h^k + \Delta t^2),\ \|\tilde{\boldsymbol{e}}_h^N\|_{L_2(\Omega)} = O(h^k + \Delta t^2),\ \|\boldsymbol{e}_h^N\|_{L_2(\Omega)} = O(h^k + \Delta t^2)$$

when $\beta_0 = 1$. If it is super-penalised, we can achieve optimal L_2 error estimates, which gives higher order $k+1$. Here is the list of numerical results:

- Standard penalisation ($\beta_0 = 1$): Tables 4.9, 4.10, 4.13, 4.14
- Super-penalisation ($\beta_0 = 3$): Tables 4.11, 4.12, 4.15, 4.16
- Linear polynomial basis ($k = 1$): Tables 4.9, 4.10, 4.11, 4.12
- Quadratic polynomial basis ($k = 2$): Tables 4.13, 4.14, 4.15, 4.16
- Displacement form **(S1)**: Tables 4.9, 4.11, 4.13, 4.15
- Velocity form **(S2)**: Tables 4.10, 4.12, 4.14, 4.16

Remark Although DG energy error estimates have been shown, the DG energy norm is defined but depends on penalty parameters. Large penalty parameters force to obtain bad numerical errors. However, if we consider the broken Sobolev norm $|||\cdot|||_{H^1(\mathcal{E}_h)}$, the broken Sobolev norm of error is independent of α_0 and β_0. Note that we have known $|||\boldsymbol{v}|||_{H^1(\mathcal{E}_h)} \leq C \|\boldsymbol{v}\|_{\mathcal{V}}$ for any $\boldsymbol{v} \in [H^1(\mathcal{E}_h)]^d$. Thus, we may want to use the broken H^1 norm for the sake of energy error estimates, instead of DG energy norm.

In Tables 4.9 and 4.10, numerical errors are seen with respect to broken H^1 norm and L_2 norm. In spite of the standard penalisation, L_2 optimality is observed for odd k (here $k = 1$) as in the scalar analogue. The numerical convergence orders are given by $\left|\left|\left|\boldsymbol{e}_h^N\right|\right|\right|_{H^1(\mathcal{E}_h)} = O(h + \Delta t^2)$ and $\left\|\tilde{\boldsymbol{e}}_h^N\right\|_{L_2(\Omega)} + \left\|\boldsymbol{e}_h^N\right\|_{L_2(\Omega)} = O(h^2 + \Delta t^2)$ for both forms of internal variables, respectively. In a similar way, Tables 4.13, 4.14, 4.15 and 4.16 exhibit higher order of accuracy with respect to spatial meshes.

$\lVert\!\lvert \boldsymbol{e}_h^N \rvert\!\rVert_{H^1(\mathcal{E}_h)}$

h \ Δt	1/2	1/4	1/8	1/16	1/32	1/64
1/2	1.0868e-01	9.1713e-02	8.0739e-02	7.7117e-02	7.5810e-02	7.5383e-02
1/4	6.6597e-02	5.4713e-02	5.1119e-02	4.9823e-02	4.9430e-02	4.9260e-02
1/8	4.3360e-02	2.3067e-02	1.9834e-02	1.9466e-02	1.9070e-02	1.8900e-02
1/16	3.4807e-02	1.1931e-02	7.6583e-03	6.9356e-03	6.7179e-03	6.5676e-03
1/32	3.2742e-02	9.0545e-03	3.5144e-03	2.5530e-03	2.4159e-03	2.3699e-03
1/64	3.2331e-02	8.3849e-03	2.3490e-03	1.0793e-03	8.7818e-04	8.4373e-04

$\lVert \tilde{\boldsymbol{e}}_h^N \rVert_{L_2(\Omega)}$

h \ Δt	1/2	1/4	1/8	1/16	1/32	1/64
1/2	1.0868e-01	9.1713e-02	8.0739e-02	7.7117e-02	7.5810e-02	7.5383e-02
1/4	6.6597e-02	5.4713e-02	5.1119e-02	4.9823e-02	4.9430e-02	4.9260e-02
1/8	4.3360e-02	2.3067e-02	1.9834e-02	1.9466e-02	1.9070e-02	1.8900e-02
1/16	3.4807e-02	1.1931e-02	7.6583e-03	6.9356e-03	6.7179e-03	6.5676e-03
1/32	3.2742e-02	9.0545e-03	3.5144e-03	2.5530e-03	2.4159e-03	2.3699e-03
1/64	3.2331e-02	8.3849e-03	2.3490e-03	1.0793e-03	8.7818e-04	8.4373e-04

$\lVert \boldsymbol{e}_h^N \rVert_{L_2(\Omega)}$

h \ Δt	1/2	1/4	1/8	1/16	1/32	1/64
1/2	6.5933e-02	6.3264e-02	6.2418e-02	6.2184e-02	6.2123e-02	6.2107e-02
1/4	3.1698e-02	2.6774e-02	2.5048e-02	2.4598e-02	2.4484e-02	2.4456e-02
1/8	1.4192e-02	9.3287e-03	7.9063e-03	7.5300e-03	7.4401e-03	7.4180e-03
1/16	9.3382e-03	3.9591e-03	2.4566e-03	2.1083e-03	2.0255e-03	2.0062e-03
1/32	8.3825e-03	2.7346e-03	1.0234e-03	6.2661e-04	5.4132e-04	5.2186e-04
1/64	8.1828e-03	2.4836e-03	7.0961e-04	2.5686e-04	1.5626e-04	1.3601e-04

Table 4.9: Numerical errors of **(S1)**: $k = 1,\ \alpha_0 = 50,\ \beta_0 = 1$

$$\| e_h^N \|_{H^1(\mathcal{E}_h)}$$

h \ Δt	1/2	1/4	1/8	1/16	1/32	1/64
1/2	2.5888e-01	2.6378e-01	2.6514e-01	2.6528e-01	2.6529e-01	2.6529e-01
1/4	1.1966e-01	1.1654e-01	1.1596e-01	1.1605e-01	1.1608e-01	1.1609e-01
1/8	5.7056e-02	5.3453e-02	5.3247e-02	5.3104e-02	5.3002e-02	5.2962e-02
1/16	3.0760e-02	2.5697e-02	2.5168e-02	2.5107e-02	2.5121e-02	2.5121e-02
1/32	2.1402e-02	1.3106e-02	1.2186e-02	1.2134e-02	1.2120e-02	1.2117e-02
1/64	1.8624e-02	7.6873e-03	6.0814e-03	5.9586e-03	5.9482e-03	5.9451e-03

$$\| \tilde{e}_h^N \|_{L_2(\Omega)}$$

h \ Δt	1/2	1/4	1/8	1/16	1/32	1/64
1/2	8.6421e-02	6.7983e-02	6.1000e-02	5.8659e-02	5.8115e-02	5.8028e-02
1/4	4.9095e-02	3.5258e-02	3.1542e-02	3.0497e-02	3.0133e-02	3.0006e-02
1/8	3.2486e-02	1.4991e-02	1.1592e-02	1.0991e-02	1.0722e-02	1.0604e-02
1/16	2.7437e-02	8.6092e-03	4.5458e-03	3.7999e-03	3.6148e-03	3.5266e-03
1/32	2.6216e-02	7.0585e-03	2.3508e-03	1.4147e-03	1.2696e-03	1.2320e-03
1/64	2.5945e-02	6.7006e-03	1.7977e-03	6.6628e-04	4.6503e-04	4.3286e-04

$$\| e_h^N \|_{L_2(\Omega)}$$

h \ Δt	1/2	1/4	1/8	1/16	1/32	1/64
1/2	4.1788e-02	4.2084e-02	4.1996e-02	4.1922e-02	4.1898e-02	4.1892e-02
1/4	1.8372e-02	1.6265e-02	1.5572e-02	1.5404e-02	1.5361e-02	1.5351e-02
1/8	8.3800e-03	5.4831e-03	4.6907e-03	4.4926e-03	4.4443e-03	4.4321e-03
1/16	5.8529e-03	2.3815e-03	1.4468e-03	1.2419e-03	1.1955e-03	1.1847e-03
1/32	5.3824e-03	1.7074e-03	6.1798e-04	3.6936e-04	3.1831e-04	3.0705e-04
1/64	5.2859e-03	1.5730e-03	4.4313e-04	1.5611e-04	9.3088e-05	8.0447e-05

Table 4.10: Numerical errors of **(S2)**: $k = 1,\ \alpha_0 = 50,\ \beta_0 = 1$

$$\left|\left|\left|\boldsymbol{e}_h^N\right|\right|\right|_{H^1(\mathcal{E}_h)}$$

h \ Δt	1/2	1/4	1/8	1/16	1/32	1/64
1/2	3.1149e-01	3.0486e-01	3.0433e-01	3.0412e-01	3.0404e-01	3.0402e-01
1/4	1.5639e-01	1.3894e-01	1.3420e-01	1.3360e-01	1.3350e-01	1.3348e-01
1/8	7.3551e-02	6.3391e-02	6.2308e-02	6.1776e-02	6.1388e-02	6.1259e-02
1/16	4.0271e-02	3.0343e-02	2.8698e-02	2.8458e-02	2.8439e-02	2.8457e-02
1/32	3.0349e-02	1.5681e-02	1.3621e-02	1.3481e-02	1.3425e-02	1.3406e-02
1/64	2.7796e-02	9.9357e-03	6.8325e-03	6.5400e-03	6.5091e-03	6.4974e-03

$$\left\|\tilde{\boldsymbol{e}}_h^N\right\|_{L_2(\Omega)}$$

h \ Δt	1/2	1/4	1/8	1/16	1/32	1/64
1/2	1.0868e-01	9.1713e-02	8.0739e-02	7.7117e-02	7.5810e-02	7.5383e-02
1/4	6.6597e-02	5.4713e-02	5.1119e-02	4.9823e-02	4.9430e-02	4.9260e-02
1/8	4.3360e-02	2.3067e-02	1.9834e-02	1.9466e-02	1.9070e-02	1.8900e-02
1/16	3.4807e-02	1.1931e-02	7.6583e-03	6.9356e-03	6.7179e-03	6.5676e-03
1/32	3.2742e-02	9.0545e-03	3.5144e-03	2.5530e-03	2.4159e-03	2.3699e-03
1/64	3.2331e-02	8.3849e-03	2.3490e-03	1.0793e-03	8.7818e-04	8.4373e-04

$$\left\|\boldsymbol{e}_h^N\right\|_{L_2(\Omega)}$$

h \ Δt	1/2	1/4	1/8	1/16	1/32	1/64
1/2	6.5933e-02	6.3264e-02	6.2418e-02	6.2184e-02	6.2123e-02	6.2107e-02
1/4	3.1698e-02	2.6774e-02	2.5048e-02	2.4598e-02	2.4484e-02	2.4456e-02
1/8	1.4192e-02	9.3287e-03	7.9063e-03	7.5300e-03	7.4401e-03	7.4180e-03
1/16	9.3382e-03	3.9591e-03	2.4566e-03	2.1083e-03	2.0255e-03	2.0062e-03
1/32	8.3825e-03	2.7346e-03	1.0234e-03	6.2661e-04	5.4132e-04	5.2186e-04
1/64	8.1828e-03	2.4836e-03	7.0961e-04	2.5686e-04	1.5626e-04	1.3601e-04

Table 4.11: Numerical errors of **(S1)**: $k = 1,\ \alpha_0 = 50,\ \beta_0 = 3$

$$\left\|\left\|\boldsymbol{e}_h^N\right\|\right\|_{H^1(\mathcal{E}_h)}$$

h \ Δt	1/2	1/4	1/8	1/16	1/32	1/64
1/2	2.5887e-01	2.6376e-01	2.6516e-01	2.6530e-01	2.6531e-01	2.6532e-01
1/4	1.1968e-01	1.1654e-01	1.1594e-01	1.1603e-01	1.1606e-01	1.1607e-01
1/8	5.7106e-02	5.3510e-02	5.3303e-02	5.3162e-02	5.3058e-02	5.3017e-02
1/16	3.0794e-02	2.5732e-02	2.5204e-02	2.5145e-02	2.5159e-02	2.5159e-02
1/32	2.1416e-02	1.3125e-02	1.2207e-02	1.2155e-02	1.2141e-02	1.2138e-02
1/64	1.8629e-02	7.6962e-03	6.0923e-03	5.9696e-03	5.9594e-03	5.9562e-03

$$\left\|\tilde{\boldsymbol{e}}_h^N\right\|_{L_2(\Omega)}$$

h \ Δt	1/2	1/4	1/8	1/16	1/32	1/64
1/2	8.6214e-02	6.8003e-02	6.1261e-02	5.8967e-02	5.8148e-02	5.8026e-02
1/4	4.9060e-02	3.5369e-02	3.1727e-02	3.0748e-02	3.0462e-02	3.0300e-02
1/8	3.2479e-02	1.5001e-02	1.1633e-02	1.1050e-02	1.0792e-02	1.0692e-02
1/16	2.7436e-02	8.6099e-03	4.5516e-03	3.8116e-03	3.6295e-03	3.5439e-03
1/32	2.6215e-02	7.0585e-03	2.3513e-03	1.4168e-03	1.2729e-03	1.2362e-03
1/64	2.5944e-02	6.7007e-03	1.7979e-03	6.6606e-04	4.6582e-04	4.3346e-04

$$\left\|\boldsymbol{e}_h^N\right\|_{L_2(\Omega)}$$

h \ Δt	1/2	1/4	1/8	1/16	1/32	1/64
1/2	4.1681e-02	4.1968e-02	4.1884e-02	4.1812e-02	4.1788e-02	4.1782e-02
1/4	1.8325e-02	1.6204e-02	1.5509e-02	1.5340e-02	1.5297e-02	1.5287e-02
1/8	8.3699e-03	5.4700e-03	4.6758e-03	4.4773e-03	4.4289e-03	4.4166e-03
1/16	5.8501e-03	2.3780e-03	1.4433e-03	1.2384e-03	1.1921e-03	1.1813e-03
1/32	5.3816e-03	1.7065e-03	6.1703e-04	3.6844e-04	3.1739e-04	3.0616e-04
1/64	5.2857e-03	1.5727e-03	4.4309e-04	1.5519e-04	9.3352e-05	7.8896e-05

Table 4.12: Numerical errors of **(S2)**: $k = 1,\ \alpha_0 = 50,\ \beta_0 = 3$

$$\|\|\boldsymbol{e}_h^N\|\|_{H^1(\mathcal{E}_h)}$$

h \ Δt	1/2	1/4	1/8	1/16	1/32	1/64
1/2	3.1943e-02	1.7812e-02	1.7455e-02	1.7428e-02	1.7164e-02	1.6942e-02
1/4	2.5935e-02	7.6937e-03	4.2905e-03	4.2251e-03	4.3743e-03	4.5193e-03
1/8	2.6585e-02	7.1461e-03	1.8255e-03	1.0447e-03	1.0211e-03	1.0254e-03
1/16	2.6859e-02	7.3545e-03	1.8200e-03	4.5171e-04	2.4497e-04	2.5126e-04
1/32	2.6934e-02	7.4221e-03	1.8716e-03	4.5524e-04	1.1366e-04	5.9667e-05
1/64	2.6953e-02	7.4400e-03	1.8882e-03	4.6886e-04	1.1378e-04	2.7975e-05

$$\|\tilde{\boldsymbol{e}}_h^N\|_{L_2(\Omega)}$$

h \ Δt	1/2	1/4	1/8	1/16	1/32	1/64
1/2	3.3635e-02	1.2567e-02	1.0570e-02	9.5225e-03	9.4485e-03	9.7771e-03
1/4	3.2163e-02	8.4701e-03	3.1325e-03	2.3636e-03	2.3057e-03	2.1951e-03
1/8	3.2181e-02	8.2049e-03	2.1029e-03	7.3171e-04	5.6389e-04	5.4555e-04
1/16	3.2206e-02	8.2117e-03	2.0608e-03	5.2569e-04	1.7826e-04	1.3336e-04
1/32	3.2214e-02	8.2184e-03	2.0646e-03	5.1604e-04	1.3064e-04	4.3885e-05
1/64	3.2216e-02	8.2203e-03	2.0664e-03	5.1716e-04	1.2908e-04	3.2662e-05

$$\|\boldsymbol{e}_h^N\|_{L_2(\Omega)}$$

h \ Δt	1/2	1/4	1/8	1/16	1/32	1/64
1/2	6.6017e-03	2.5598e-03	2.8489e-03	3.0997e-03	3.1661e-03	3.1822e-03
1/4	7.4754e-03	1.8607e-03	5.7956e-04	7.4937e-04	8.3085e-04	8.5300e-04
1/8	7.9479e-03	2.2464e-03	4.8079e-04	1.4440e-04	1.9171e-04	2.1239e-04
1/16	8.0766e-03	2.3674e-03	5.8420e-04	1.2114e-04	3.6291e-05	4.8422e-05
1/32	8.1096e-03	2.3989e-03	6.1470e-04	1.4739e-04	3.0337e-05	9.1024e-06
1/64	8.1179e-03	2.4069e-03	6.2257e-04	1.5504e-04	3.6929e-05	7.5861e-06

Table 4.13: Numerical errors of **(S1)**: $k = 2,\ \alpha_0 = 50,\ \beta_0 = 1$

$$|||\boldsymbol{e}_h^N|||_{H^1(\mathcal{E}_h)}$$

h \ Δt	1/2	1/4	1/8	1/16	1/32	1/64
1/2	2.2940e-02	1.3229e-02	1.3050e-02	1.2830e-02	1.2719e-02	1.2609e-02
1/4	1.7586e-02	5.4845e-03	3.2666e-03	3.2082e-03	3.1778e-03	3.2170e-03
1/8	1.7543e-02	4.8059e-03	1.3438e-03	8.0480e-04	7.7457e-04	7.7799e-04
1/16	1.7621e-02	4.8465e-03	1.2223e-03	3.3462e-04	1.9624e-04	1.9163e-04
1/32	1.7647e-02	4.8700e-03	1.2344e-03	3.0604e-04	8.3816e-05	4.8398e-05
1/64	1.7653e-02	4.8768e-03	1.2404e-03	3.0938e-04	7.6517e-05	2.0840e-05

$$\|\tilde{\boldsymbol{e}}_h^N\|_{L_2(\Omega)}$$

h \ Δt	1/2	1/4	1/8	1/16	1/32	1/64
1/2	2.6318e-02	8.2155e-03	5.6917e-03	5.0936e-03	5.0372e-03	5.1985e-03
1/4	2.5832e-02	6.6781e-03	2.0591e-03	1.2787e-03	1.2123e-03	1.1599e-03
1/8	2.5845e-02	6.5935e-03	1.6702e-03	4.9586e-04	3.0660e-04	2.8850e-04
1/16	2.5857e-02	6.5976e-03	1.6568e-03	4.1799e-04	1.2233e-04	7.3264e-05
1/32	2.5860e-02	6.6003e-03	1.6583e-03	4.1484e-04	1.0429e-04	3.0356e-05
1/64	2.5861e-02	6.6010e-03	1.6589e-03	4.1525e-04	1.0376e-04	2.6078e-05

$$\|\boldsymbol{e}_h^N\|_{L_2(\Omega)}$$

h \ Δt	1/2	1/4	1/8	1/16	1/32	1/64
1/2	4.2771e-03	1.4817e-03	1.6788e-03	1.8449e-03	1.8913e-03	1.9030e-03
1/4	4.8533e-03	1.1642e-03	3.0221e-04	4.3095e-04	4.8799e-04	5.0324e-04
1/8	5.1486e-03	1.4278e-03	2.9889e-04	7.3457e-05	1.1006e-04	1.2476e-04
1/16	5.2287e-03	1.5064e-03	3.7055e-04	7.5180e-05	1.8396e-05	2.7821e-05
1/32	5.2492e-03	1.5267e-03	3.9054e-04	9.3449e-05	1.8814e-05	4.6119e-06
1/64	5.2543e-03	1.5318e-03	3.9565e-04	9.8477e-05	2.3410e-05	4.7038e-06

Table 4.14: Numerical errors of **(S2)**: $k = 2,\ \alpha_0 = 50,\ \beta_0 = 1$

$\lVert\!\lvert \boldsymbol{e}_h^N \rVert\!\rvert_{H^1(\mathcal{E}_h)}$

h \ Δt	1/2	1/4	1/8	1/16	1/32	1/64
1/2	2.5887e-01	2.6376e-01	2.6516e-01	2.6530e-01	2.6531e-01	2.6532e-01
1/4	1.1968e-01	1.1654e-01	1.1594e-01	1.1603e-01	1.1606e-01	1.1607e-01
1/8	5.7106e-02	5.3510e-02	5.3303e-02	5.3162e-02	5.3058e-02	5.3017e-02
1/16	3.0794e-02	2.5732e-02	2.5204e-02	2.5145e-02	2.5159e-02	2.5159e-02
1/32	2.1416e-02	1.3125e-02	1.2207e-02	1.2155e-02	1.2141e-02	1.2138e-02
1/64	1.8629e-02	7.6962e-03	6.0923e-03	5.9696e-03	5.9594e-03	5.9562e-03

$\lVert \tilde{\boldsymbol{e}}_h^N \rVert_{L_2(\Omega)}$

h \ Δt	1/2	1/4	1/8	1/16	1/32	1/64
1/2	8.6214e-02	6.8003e-02	6.1261e-02	5.8967e-02	5.8148e-02	5.8026e-02
1/4	4.9060e-02	3.5369e-02	3.1727e-02	3.0748e-02	3.0462e-02	3.0300e-02
1/8	3.2479e-02	1.5001e-02	1.1633e-02	1.1050e-02	1.0792e-02	1.0692e-02
1/16	2.7436e-02	8.6099e-03	4.5516e-03	3.8116e-03	3.6295e-03	3.5439e-03
1/32	2.6215e-02	7.0585e-03	2.3513e-03	1.4168e-03	1.2729e-03	1.2362e-03
1/64	2.5944e-02	6.7007e-03	1.7979e-03	6.6606e-04	4.6582e-04	4.3346e-04

$\lVert \boldsymbol{e}_h^N \rVert_{L_2(\Omega)}$

h \ Δt	1/2	1/4	1/8	1/16	1/32	1/64
1/2	4.1681e-02	4.1968e-02	4.1884e-02	4.1812e-02	4.1788e-02	4.1782e-02
1/4	1.8325e-02	1.6204e-02	1.5509e-02	1.5340e-02	1.5297e-02	1.5287e-02
1/8	8.3699e-03	5.4700e-03	4.6758e-03	4.4773e-03	4.4289e-03	4.4166e-03
1/16	5.8501e-03	2.3780e-03	1.4433e-03	1.2384e-03	1.1921e-03	1.1813e-03
1/32	5.3816e-03	1.7065e-03	6.1703e-04	3.6844e-04	3.1739e-04	3.0616e-04
1/64	5.2857e-03	1.5727e-03	4.4309e-04	1.5519e-04	9.3352e-05	7.8896e-05

Table 4.15: Numerical errors of **(S1)**: $k = 2,\ \alpha_0 = 50,\ \beta_0 = 3$

$$|||\boldsymbol{e}_h^N|||_{H^1(\mathcal{E}_h)}$$

h \ Δt	1/2	1/4	1/8	1/16	1/32	1/64
1/2	2.3011e-02	1.3311e-02	1.3120e-02	1.2902e-02	1.2787e-02	1.2675e-02
1/4	1.7601e-02	5.5138e-03	3.3086e-03	3.2496e-03	3.2198e-03	3.2593e-03
1/8	1.7544e-02	4.8096e-03	1.3525e-03	8.1838e-04	7.8770e-04	7.9125e-04
1/16	1.7622e-02	4.8470e-03	1.2233e-03	3.3720e-04	2.0004e-04	1.9540e-04
1/32	1.7647e-02	4.8701e-03	1.2346e-03	3.0632e-04	8.4497e-05	4.9422e-05
1/64	1.7653e-02	4.8769e-03	1.2400e-03	3.0934e-04	7.5534e-05	2.0786e-05

$$\|\tilde{\boldsymbol{e}}_h^N\|_{L_2(\Omega)}$$

h \ Δt	1/2	1/4	1/8	1/16	1/32	1/64
1/2	2.6321e-02	8.2226e-03	5.7066e-03	5.1113e-03	5.0550e-03	5.2138e-03
1/4	2.5832e-02	6.6793e-03	2.0602e-03	1.2794e-03	1.2157e-03	1.1620e-03
1/8	2.5845e-02	6.5938e-03	1.6704e-03	4.9546e-04	3.0576e-04	2.8778e-04
1/16	2.5857e-02	6.5976e-03	1.6569e-03	4.1802e-04	1.2215e-04	7.2862e-05
1/32	2.5860e-02	6.6003e-03	1.6583e-03	4.1486e-04	1.0432e-04	3.0344e-05
1/64	2.5861e-02	6.6010e-03	1.6586e-03	4.1536e-04	1.0269e-04	2.6212e-05

$$\|\boldsymbol{e}_h^N\|_{L_2(\Omega)}$$

h \ Δt	1/2	1/4	1/8	1/16	1/32	1/64
1/2	4.2814e-03	1.4907e-03	1.6835e-03	1.8483e-03	1.8942e-03	1.9058e-03
1/4	4.8557e-03	1.1667e-03	2.9989e-04	4.2688e-04	4.8390e-04	4.9915e-04
1/8	5.1496e-03	1.4290e-03	2.9986e-04	7.2021e-05	1.0818e-04	1.2289e-04
1/16	5.2290e-03	1.5067e-03	3.7091e-04	7.5461e-05	1.7972e-05	2.7277e-05
1/32	5.2493e-03	1.5268e-03	3.9063e-04	9.3546e-05	1.8890e-05	4.5296e-06
1/64	5.2544e-03	1.5319e-03	3.9544e-04	9.8483e-05	2.2740e-05	4.6244e-06

Table 4.16: Numerical errors of **(S2)**: $k = 2,\ \alpha_0 = 50,\ \beta_0 = 3$

As we mentioned before in Chapter 3.4, on account of super-penalisation to get optimal L_2 errors, there may exist a difficulty in solving large linear systems. Poor condition numbers degrade the performance of DGFEM for fine spatial meshes. In our computational works, as h decreasing, iterative solvers have serious difficulty in getting appropriate solutions. Especially, super-penalisation yields much worse condition numbers of its global matrix. Table 4.17 and Figure 4.1 illustrate the comparison of condition numbers between the standard and super-penalisation. As a result, ill-conditioned matrices by super-penalised degenerate numerical convergence for fine spatial meshes. More precisely, we can experimentally observe that the condition number of the global matrix is of order $O(h^{-2} + \Delta t)$ and of order $O(h^{-4} + \Delta t)$ for standard and super-penalisation, respectively. Even though $h = 1/32$, the condition number of super-penalisation is quite big about 10^8 so that the iterative methods may not work properly for small h. For instance, if $h = 1/512$, the condition number becomes $O(10^{12})$. Accordingly, in order to resolve this issue, we need to enhance linear solvers in FEniCS.

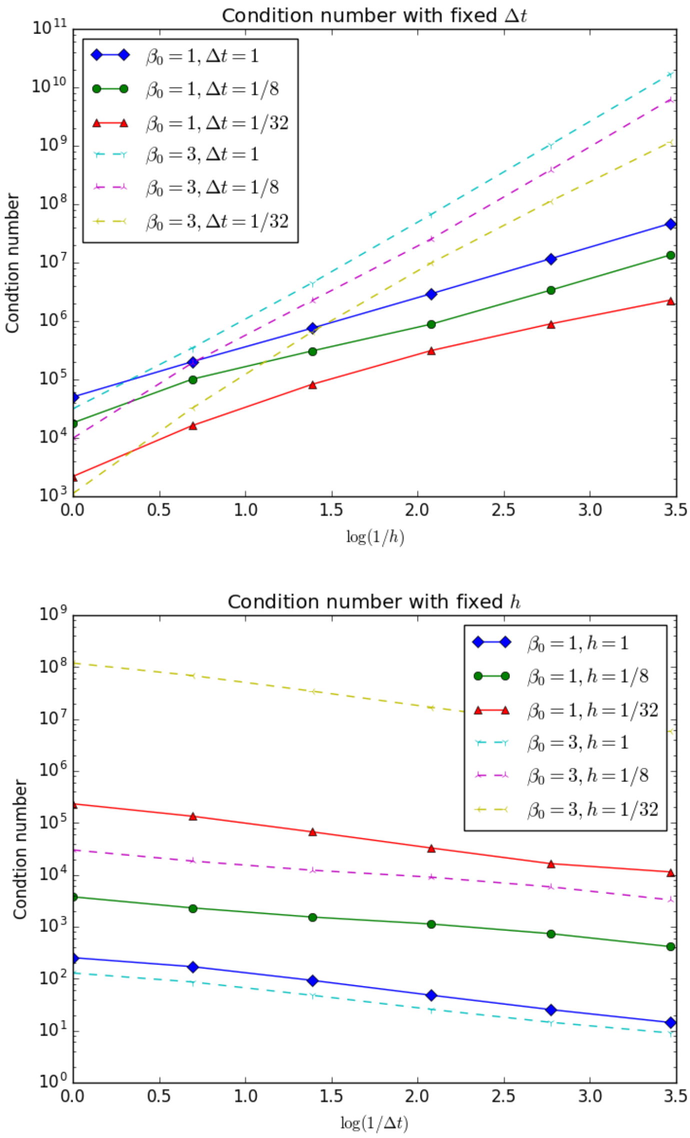

Figure 4.1: Graph of condition numbers with respect to space/time meshes; Standard(solid line) and Super-penalised(dash line)

Standard penalisation

h \ Δt	1	1/2	1/4	1/8	1/16	1/32
1	2.54e+02	1.71e+02	9.32e+01	4.81e+01	2.54e+01	1.44e+01
1/2	1.01e+03	7.47e+02	5.10e+02	3.00e+02	1.65e+02	8.58e+01
1/4	3.77e+03	2.31e+03	1.54e+03	1.13e+03	7.37e+02	4.16e+02
1/8	1.47e+04	8.54e+03	4.43e+03	2.97e+03	2.26e+03	1.57e+03
1/16	5.85e+04	3.37e+04	1.69e+04	8.45e+03	5.79e+03	4.48e+03
1/32	2.34e+05	1.35e+05	6.74e+04	3.28e+04	1.64e+04	1.14e+04

Super-penalisation

h \ Δt	1	1/2	1/4	1/8	1/16	1/32
1	1.29e+02	8.66e+01	4.80e+01	2.57e+01	1.45e+01	9.03e+00
1/2	2.02e+03	1.49e+03	1.02e+03	5.97e+02	3.25e+02	1.67e+02
1/4	3.01e+04	1.84e+04	1.23e+04	9.02e+03	5.87e+03	3.29e+03
1/8	4.71e+05	2.73e+05	1.42e+05	9.50e+04	7.24e+04	5.00e+04
1/16	7.48e+06	4.32e+06	2.17e+06	1.08e+06	7.40e+05	5.73e+05
1/32	1.20e+08	6.89e+07	3.45e+07	1.68e+07	8.41e+06	5.83e+06

Table 4.17: Condition numbers of a global matrix with $\alpha_0 = 50$

Summary

We have studied linear viscoelastic problems with CGFEM and DGFEM. We have formulated variational problems with respect to two types of internal variables for each finite element method. In the meantime, using similar arguments for proofs in Chapter 2 and 3, stability analysis as well as error analysis have been presented. Regardless of finite element methods and forms of internal variables, well-posedness and optimal error estimates have been shown without Grönwall constants as seen in scalar analogue. In terms of numerical simulations, optimal convergence orders are observed. However, as we concerned before, ill-conditioned linear system arises in super-penalised DG for spatially fine meshes. The number of degrees of freedom for vector-valued problem is dimension d times more than that of scalar cases. Hence the size of linear system is bigger than scalar case, so improvement of linear solvers is necessary for DG approximations.

Chapter 5

Fractional Order Viscoelastic Wave Propagations

Many phenomena in reality are modelled as integro-differential equations. Viscoelastic materials are also able to be described by the integro-diffential equations with fractional order [1, 2, 4, 70]. We consider the viscoelastic models with fractional order based on numerical approaches such as FEM and FDMs (e.g. see [71, 33, 34, 72, 73, 74, 10]).

In reality, many experimental results have shown that viscoelastic materials exhibit approximately linear response of relaxation over large time range on log-log scale [4]. It was a reasonable choice to propose the power law form by Nutting [75]. In [41], the authors presented an elastomer 3M-467 obeys the power law form and Koller described applications of fractional calculus to viscoelastic phenomena [70]. As a result, it is possible to generate a rich variety of relaxation functions of much more complexity than the power law form which provided the original stimulus for the approach [4].

Mittag-Leffler type kernels were employed in a natural manner to formulate the fractional order viscoelastic model, on account of analytic solutions of fractional order differential equations, e.g. see [71, 33] and more references therein. A variety of numerical approaches based on spatial Galerkin methods [71, 33, 34], have also been presented. Numerical simulation of the quasi-static and damped responses of a viscoelastic ballast material was investigated in [76].

In contrast, power law type kernels can be used for the sake of conciseness. McLean and Thomée studied a parabolic type equation with a positive type memory term [8]. Recently, improved works of fractional order viscoelasticity were regarded in [77, 9]. More general cases, also known as a time fractional Oldroyd-B fluid problem, were studied, see e.g. [78] and references therein.

In this chapter, we formulate numerical schemes to solve the fractional order viscoelasticity problem of the power law type kernel in the same manner as generalised Maxwell solid. Moreover, we introduce some numerical technique for the integral form of constitutive relation. *A priori* error estimates are also presented. Finally, we carry out numerical experiments of fractional order viscoelasticity based on CGFEM and DGFEM, respectively.

5.1 Preliminary

According to [28, 29, 79], we can define the following definitions to give a framework of fractional calculus.

Definition Gamma function
Gamma function is defined by

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt, \qquad \mathbf{Re}\ z > 0,\ z \in \mathbb{C}.$$

Definition Beta function
Beta function $\mathcal{B}(z_1, z_2)$ is defined by

$$\mathcal{B}(z_1, z_2) = \int_0^1 t^{z_1-1}(1-t)^{z_2-1}dt, \qquad \text{for } \mathbf{Re}\ z_1 > 0 \text{ and } \mathbf{Re}\ z_2 > 0.$$

$\mathcal{B}(z_1, z_2)$ is symmetric and it has the following property [80, pp. 253-294]

$$\mathcal{B}(z_1, z_2) = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)}.$$

Definition Riemann-Liouville fractional derivative and integral
Let f be a function defined on $[a, b]$ and $\alpha \in \mathbb{R}^+$. α can be written uniquely by $\alpha = n_\alpha + q_\alpha$ for $n_\alpha \in \mathbb{N} \cup \{0\}$, $q_\alpha \in [0, 1)$. A left Riemann-Liouville derivative of order α is defined by

$${}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t f(t')(t-t')^{n-\alpha-1}dt',\ t > a,$$

where $n = n_\alpha + 1$. In this manner, the right Riemann-Liouville derivative of order α is defined by

$${}_tD_b^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dt}\right)^n \int_t^b f(t')(t'-t)^{n-\alpha-1}dt',\ t < b.$$

If $f \in L_1[a, b]$, we can define

$${}_aI_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t f(t')(t-t')^{\alpha-1}dt',\ t > a,$$

and

$${}_tI_b^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b f(t')(t'-t)^{\alpha-1}dt',\ t < b.$$

Remark Let $\alpha \in (0, 1)$. For a function f on $[a, b]$, it holds

$${}_aD_t^\alpha f(t) = \frac{d}{dt}{}_aI_t^{1-\alpha} f(t), \qquad {}_tD_b^\alpha f(t) = -\frac{d}{dt}{}_tI_b^{1-\alpha} f(t).$$

Also, ${}_aD_t^\alpha f(t)$ can be expressed by

$$\begin{aligned}
{}_aD_t^\alpha f(t) =& \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t f(t')(t-t')^{-\alpha} dt' \\
=& \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^{t-a} f(t-u) u^{-\alpha} du \\
& (\because \text{ let } u = t - t') \\
=& \frac{1}{\Gamma(1-\alpha)} \left(f(a)(t-a)^{-\alpha} + \int_0^{t-a} \frac{d}{dt} f(t-u) u^{-\alpha} du \right) \\
& (\because \text{ by Leibniz integral rule}) \\
=& \frac{1}{\Gamma(1-\alpha)} \left(f(a)(t-a)^{-\alpha} + \int_0^{t-a} \dot{f}(t-u) u^{-\alpha} du \right) \\
& (\because \text{ where } \dot{f}(t) = \frac{d}{dt} f(t)) \\
=& \frac{1}{\Gamma(1-\alpha)} \left(f(a)(t-a)^{-\alpha} + \int_a^t \dot{f}(t')(t-t')^{-\alpha} dt' \right) \\
=& \frac{f(a)(t-a)^{-\alpha}}{\Gamma(1-\alpha)} + {}_aI_t^{1-\alpha} \dot{f}(t).
\end{aligned}$$

Definition Caputo fractional derivatives
A left Caputo fractional derivative of order $\alpha \in \mathbb{R}^+ \backslash \mathbb{N}$ on $[a, b]$ is denoted by ${}_a^CD_t^\alpha f(t)$ which is defined by

$${}_a^CD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t f^{(n)}(t')(t-t')^{n-\alpha-1} dt' = {}_aI_t^{n-\alpha} f^{(n)}(t), \qquad t > a,$$

where $f^{(n)}$ is n-th derivative of f and $n = n_\alpha + 1$. In a similar way, a right Caputo fractional derivative is given by

$${}_t^CD_b^\alpha f(t) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_t^b f^{(n)}(t')(t-t')^{n-\alpha-1} dt' = (-1)^n {}_tI_b^{n-\alpha} f^{(n)}(t), \quad t < b.$$

If $\alpha \in \mathbb{N} \cup \{0\}$, Caputo derivatives are defined by

$${}_a^CD_t^\alpha f(t) = f^{(\alpha)}(t), \text{ and } {}_t^CD_b^\alpha f(t) = (-1)^\alpha f^{(\alpha)}(t).$$

Remark Let $\alpha \in (0, 1)$. We can observe the relation between Riemann-Liouville differential operator and Caputo differential operator as following. By Leibniz integral rule, we have

$${}_aD_t^\alpha (f(t) - f(a)) = {}_a^CD_t^\alpha f(t) \qquad \text{and} \qquad {}_tD_b^\alpha (f(t) - f(b)) = {}_t^CD_b^\alpha f(t).$$

Remark Let us consider fractional order derivatives for $\alpha \in \mathbb{R}^+ \backslash \mathbb{N}$. We have the following properties.

- If $f \in L_\infty[a,b]$, then

$$ {}_a^C D_t^\alpha {}_a I_t^\alpha f(t) = f(t) \text{ and } {}_t^C D_b^\alpha {}_t I_b^\alpha f(t) = f(t). \tag{5.1.1} $$

- If f is absolutely continuous on $[a,b]$ and its m-th derivatives are also absolutely continuous on $[a,b]$ for $m = 1, \ldots, n-1$, then

$$ {}_a I_t^\alpha {}_a^C D_t^\alpha f(t) = f(t) - \sum_{m=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k, \tag{5.1.2} $$

and

$$ {}_t I_b^\alpha {}_t^C D_b^\alpha f(t) = f(t) - \sum_{m=0}^{n-1} \frac{(-1)^k f^{(k)}(b)}{k!}(b-x)^k. \tag{5.1.3} $$

- If $\alpha \in (0,1)$, integration by parts in fractional order gives

$$ \int_a^b g(t)\, {}_a^C D_t^\alpha f(t)\, dt = \int_a^b f(t)\, {}_a^C D_t^\alpha g(t)\, dt + f(t) {}_t I_b^{1-\alpha} g(t)\big|_{t=a}^{t=b}, \tag{5.1.4} $$

and

$$ \int_a^b g(t)\, {}_t^C D_b^\alpha f(t)\, dt = \int_a^b f(t)\, {}_t^C D_b^\alpha g(t)\, dt - f(t) {}_a I_t^{1-\alpha} g(t)\big|_{t=a}^{t=b}. \tag{5.1.5} $$

Remark Fractional integration of polynomials
Let $\alpha > 0$ and $k > -1$. Then we have

$$ \begin{aligned} {}_a I_t^\alpha (t-a)^k =& \frac{1}{\Gamma(\alpha)} \int_a^t (t-t')^{\alpha-1}(t'-a)^k dt' \\ =& \frac{(t-a)^{j+\alpha}}{\Gamma(\alpha)} \int_0^1 (1-u)^{\alpha-1} u^k du \\ & (\because \text{let } u = \frac{t'-a}{t-a} \text{ then } dt' = (t-a)du) \\ =& \frac{(t-a)^{j+\alpha}}{\Gamma(\alpha)} \mathcal{B}(\alpha, k+1). \end{aligned} $$

By a property of Beta function, we have

$$ {}_a I_t^\alpha (t-a)^k = \frac{\Gamma(k+1)}{\Gamma(\alpha+k+1)}(t-a)^{j+\alpha}. \tag{5.1.6} $$

Definition Mittag-Leffler function
We can define a Mittag-Leffler function by for $z \in \mathbb{C}$

$$ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}, \qquad \alpha, \beta > 0. $$

However, we consider only $\beta = 1$ hence we have

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + 1)}, \qquad \alpha > 0.$$

Mittag-Leffler function is so important to solve fractional differential equations. It could be the exact solution to a fractional differential equation[30, 31]. For instance, the solution of the fractional differential equation ${}_0D_t^\alpha y = ay$ is $y = AE_\alpha(at^\alpha)$ for given constant a and $0 < \alpha < 1$ where A is an arbitrary constant. Moreover, fractional order differential equations can be represented by convolutions involved by Mittag-Leffler type kernel [81, 82, 10].

We now introduce numerical approaches for the fractional integration and derivative. Consider a time discretisation for $[0, T]$ such that

$$0 = t_0 < t_1 < \cdots < t_N = T, \;\; \Delta t = T/N, \;\; t_i = i\Delta t \text{ for } i = 0, 1, \ldots, N.$$

Theorem 5.1. Linear Interpolation to a Fractional Integral [74]
Let $y \in C^2[0, t_n]$ and $\alpha > 0$ for $N \geq n \in \mathbb{N}$. Then the fractional integral of order α for y can be written by

$$ {}_0I_{t_n}^\alpha y(t) = \frac{\Delta t^\alpha}{\Gamma(\alpha + 2)} \sum_{i=0}^{n} B_{n,i} y(t_i) + O(\Delta t^2), \tag{5.1.7}$$

where

$$B_{n,i} = \begin{cases} n^\alpha(\alpha + 1 - n) + (n-1)^{\alpha+1}, & i = 0, \\ (n-i-1)^{\alpha+1} + (n-i+1)^{\alpha+1} - 2(n-i)^{\alpha+1}, & i = 1, \ldots, n-1, \\ 1, & i = n. \end{cases}$$

By Theorem 5.1, we can approximate the fractional integration by

$${}_0I_t^\alpha y(t) \approx \frac{\Delta t^\alpha}{\Gamma(\alpha + 2)} \sum_{i=0}^{n} B_{n,i} y(t_i) \qquad \text{for } t \in [t_{n-1}, t_n].$$

Lemma 5.1. *Let $(B_{n,i})_{i=0}^n$ be given in* Theorem 5.1. *Then $B_{n,i}$ is positive and bounded for any i such that $0 < B_{n,i} < 2$.*

Proof. For $i = n$, our claim is clearly true. Let us consider $i = 0$. $B_{n,0}$ can be rewritten as $B_{n,0} = n^\alpha(\alpha + 1) - n^{\alpha+1} + (n-1)^{\alpha+1}$. Define a function f by $f(x) = x^{\alpha+1}$. Then we can write

$$B_{n,0} = f'(n) - f(n) + f(n-1).$$

Note that for $x > 0$

$$f(x), f'(x), f''(x) > 0.$$

In addition, mean value theorem leads us to have

$$B_{n,0} = f'(n) - f(n) + f(n-1)$$

$$\begin{aligned}
&= f'(n) - \frac{f(n) - f(n-1)}{1} \\
&= f'(n) - f'(x_n)
\end{aligned}$$

for some $x_n \in (n-1, n)$. Since f' is increasing when $x > 0$ and $n > x_n$,

$$0 < f'(n) - f'(x_n) = B_{n,0}.$$

Similarly, for $i = 1, \ldots, n-1$ $B_{n,i}$ can be rewritten as

$$B_{n,i} = f(m+1) - 2f(m) + f(m-1),$$

where $m = n - i \geq 1$. Thus the mean value theorem implies

$$\begin{aligned}
B_{n,i} &= f(m+1) - 2f(m) + f(m-1) \\
&= \frac{f(m+1) - f(m)}{1} - \frac{f(m) - f(m-1)}{1} \\
&= f'(x_{m+1}) - f'(x_m) > 0,
\end{aligned}$$

since $f''(x) > 0$ for $x > 0$ and $x_{m+1} > x_m$ where $x_m \in (m-1, m)$ and $x_{m+1} \in (m, m+1)$.

Furthermore, let us consider their upper bounds. When $i = n$, $B_{n,n} = 1$ clearly. If $n = 1$, $B_{1,0} = \alpha < 1$. Suppose $2 \leq n$. Use of Taylor theorem implies that

$$B_{n,0} = f'(n) - f(n) + f(n-1) = \frac{1}{2} f''(x_n),$$

where $x_n \in (n-1, n)$. Since $1 \leq n-1 < x_n$ and $f''(x) = \alpha(\alpha+1)x^{\alpha-1}$,

$$B_{n,0} = \frac{1}{2}\alpha(\alpha+1)x_n^{\alpha-1} \leq \frac{1}{2}\alpha(\alpha+1) < 1$$

for $0 < \alpha < 1$. When we take into account $i = 1, \ldots, n-1$, Taylor theorem leads us to have

$$B_{n,i} = f'(m) + \frac{1}{2} f''(x_{m+1}) - f'(m) + \frac{1}{2} f''(x_m) = \frac{1}{2}(f''(x_{m+1}) + f''(x_m)),$$

where $m = n - i \geq 1$, for some $x_m \in (m-1, m)$ and $x_{m+1} \in (m, m+1)$. If $m = 1$, that is $i = n - 1$,

$$B_{n,i} = 2^{\alpha+1} - 2 < 2.$$

Otherwise, since $1 < x_m < x_{m+1}$,

$$B_{n,i} = \frac{1}{2}\alpha(\alpha+1)(x_{m+1}^{\alpha-1} + x_m^{\alpha-1}) < \alpha(\alpha+1) < 2.$$

Therefore, we can conclude that for any $0 \leq i \leq n \in \mathbb{N}$ and $0 < \alpha < 1$

$$0 < B_{n,i} < 2.$$

□

On the other hand, we can also use quadrature rules to approximate the fractional calculus. In [83], we can observe that Crank-Nicolson method is applied to a fractional derivative. Hence the numerical scheme is given as follows.

Theorem 5.2. Crank-Nicolson Method for a Fractional Derivative [83] *Suppose $y \in C^2(0,T)$ and $\alpha \in (0,1)$. Then we can derive for $n = 0, \ldots, N-1$*

$$
{}_0^C D_t^\alpha \bar{y}^n = \frac{1}{\Delta t^\alpha \Gamma(2-\alpha)} \sum_{i=0}^{n+1} w_i y(t_i) + O(\Delta t^{2-\alpha}), \tag{5.1.8}
$$

where

$$
w_i = \begin{cases} (n-1/2)^{1-\alpha} - (n+1/2)^{1-\alpha}, & i = 0, \\ (n-i-1/2)^{1-\alpha} - 2(n-i+1/2)^{1-\alpha} + (n-i+3/2)^{1-\alpha}, & i = 1, \ldots, n-1, \\ (3/2)^{1-\alpha} - (1/2)^{1-\alpha} - (1/2)^{1-\alpha}, & i = n, \\ (1/2)^{1-\alpha}, & i = n+1. \end{cases}
$$

Thus, we can have the following approximation

$$
{}_0^C D_t^\alpha \bar{y}^n \approx \frac{{}_0^C D_t^\alpha y(t_{n+1}) + {}_0^C D_t^\alpha y(t_n)}{2} \approx \frac{\Delta t^\alpha}{\Gamma(\alpha+2)} \sum_{i=0}^{n+1} w_i y(t_i).
$$

5.2 Fractional Order Viscoelastic Models by Power Law

Recall the equation of viscoelastic model problem. We have

$$
\rho \ddot{\boldsymbol{u}} - \nabla \cdot \underline{\boldsymbol{\sigma}} = \boldsymbol{f},
$$

where ρ is a density of mass, $\boldsymbol{u}$ is a displacement vector, $\underline{\boldsymbol{\sigma}}$ is stress and $\boldsymbol{f}$ is a volume load. A constitutive equation between stress and strain is defined with respect to a given model. In particular, the model by power law contains a fractional order derivative [4, 1, 10]. In a intermediate sense between elasticity and viscosity, for example stress is proportional to strain in elastic solid or to rate of strain in viscous liquid, we can formulate the constitutive law in viscoelastic materials by

$$
\underline{\boldsymbol{\sigma}}(t) = \underline{\hat{\boldsymbol{D}}} \underline{\boldsymbol{\varepsilon}}(t) + {}_0D_t^\alpha (\underline{\tilde{\boldsymbol{D}}} \underline{\boldsymbol{\varepsilon}}(t)), \tag{5.2.1}
$$

where $\underline{\hat{\boldsymbol{D}}}$ and $\underline{\tilde{\boldsymbol{D}}}$ are fourth order tensors, $\underline{\boldsymbol{\varepsilon}}$ is strain and $\alpha \in (0,1)$, since the stress is proportional to the strain in solid and the stress is proportional to the rate of the strain in fluid (see e.g. [3, 1, 2]). To simplify , we assume $\underline{\hat{\boldsymbol{D}}}$ and $\underline{\tilde{\boldsymbol{D}}}$ are piecewise constants, which means the fourth order tensors are independent of a spatial variable. For example, in a classical elastic models, these fourth order tensors denotes Hooke's tensors. In this manner, $\underline{\hat{\boldsymbol{D}}}$ and $\underline{\tilde{\boldsymbol{D}}}$ are defined by

$$
\hat{D}_{ijkl} = 2\hat{\mu}\delta_{ik}\delta_{jl} + \hat{\lambda}\delta_{ij}\delta_{kl} \text{ and } \tilde{D}_{ijkl} = 2\tilde{\mu}\delta_{ik}\delta_{jl} + \tilde{\lambda}\delta_{ij}\delta_{kl} \text{ for } i,j,k,l = 1, \ldots, d,
$$

where $\hat{\mu}, \hat{\lambda}, \tilde{\mu}, \tilde{\lambda}$ are Lamé parameters [5], hence we have

$$(\underline{\hat{\boldsymbol{D}}}\underline{\boldsymbol{\varepsilon}}(t))_{ij} = 2\hat{\mu}\varepsilon_{ij} + \hat{\lambda}\mathrm{tr}(\underline{\boldsymbol{\varepsilon}}(t))\delta_{ij}, \qquad (\underline{\tilde{\boldsymbol{D}}}\underline{\boldsymbol{\varepsilon}}(t))_{ij} = 2\tilde{\mu}\varepsilon_{ij} + \tilde{\lambda}\mathrm{tr}(\underline{\boldsymbol{\varepsilon}}(t))\delta_{ij}, \ \forall i, j = 1, \ldots, d.$$

Whereas (5.2.1) consists of a fractional derivative in Riemann-Liouville derivative, Mittag-Leffler type kernels can be introduced as in [10, 71, 33, 34]. The constitutive equation is given by

$$\underline{\boldsymbol{\sigma}}(t) = \underline{\hat{\boldsymbol{D}}}\underline{\boldsymbol{\varepsilon}}(t) - \int_0^t \beta(t-t')\underline{\tilde{\boldsymbol{D}}}\underline{\boldsymbol{\varepsilon}}(t')dt', \tag{5.2.2}$$

where

$$\beta(t) = -\frac{d}{dt}E_\alpha\left(-\left(t/\tau\right)^\alpha\right) = \frac{\alpha}{\tau}\left(\frac{t}{\tau}\right)^{\alpha-1}\dot{E}_\alpha\left(-\left(t/\tau\right)^\alpha\right) \text{ for some positive } \tau.$$

The hyperbolic equations with the constitutive law with respect to the Mittag-Leffler type kernels have been dealt by finite element methods in [10, 71, 33, 34]. In a similar way, we will take into account (5.2.1) by introducing finite element methods. One can give a weak form then a stability analysis and an error analysis would be shown. However, (5.2.1) can be also written with fractional integration by

$$\underline{\boldsymbol{\sigma}}(t) = \underline{\hat{\boldsymbol{D}}}\underline{\boldsymbol{\varepsilon}}(t) + \frac{\underline{\tilde{\boldsymbol{D}}}\underline{\boldsymbol{\varepsilon}}(0)}{\Gamma(1-\alpha)}t^{-\alpha} + {}_0I_t^{1-\alpha}(\underline{\tilde{\boldsymbol{D}}}\underline{\dot{\boldsymbol{\varepsilon}}}(t)). \tag{5.2.3}$$

Comparing (5.2.1) with (5.2.3), we could observe that (5.2.1) consists of the strain of the displacement vector in the integral form. (5.2.3), however, contains the strain of the velocity vector involved in fractional integration. Note that when the stress consists of the strain of displacement vector in memory term, we call it the displacement form. Otherwise if the strain-rate tensor is in memory terms, then we call it the velocity form. Thus, in a general sense of the constitutive relations (1.3.10) and (1.3.11), we can call the constitutive equations, (5.2.1) and (5.2.3), a displacement form and a velocity form in a fractional order, respectively.

Interestingly, if we suppose $\underline{\hat{\boldsymbol{D}}}$ is a zero tensor, we can reduce the order of differentiation. For example, let us denote $\boldsymbol{w} = \dot{\boldsymbol{u}}$ then the model problem is rewritten as

$$\rho\dot{\boldsymbol{w}}(t) - \nabla\cdot\underline{\boldsymbol{\sigma}}(t) = \boldsymbol{f}(t), \tag{5.2.4}$$

$$\underline{\boldsymbol{\sigma}}(t) = \frac{\underline{\tilde{\boldsymbol{D}}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{u}_0)}{\Gamma(1-\alpha)}t^{-\alpha} + {}_0I_t^{1-\alpha}(\underline{\tilde{\boldsymbol{D}}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{w}(t))), \tag{5.2.5}$$

with sufficiently smooth $\boldsymbol{u}_0$. For simplicity, we assume that $\boldsymbol{u}_0 = \boldsymbol{0}$. Thus, we consider

$$\rho\dot{\boldsymbol{w}}(t) - \nabla\cdot {}_0I_t^{1-\alpha}(\underline{\tilde{\boldsymbol{D}}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{w}(t))) = \boldsymbol{f}(t), \tag{5.2.6}$$

and taking into account fractional differentiation and (5.1.1), using identity property, yields

$$\rho\, {}_0^CD_t^{1-\alpha}\dot{\boldsymbol{w}}(t) - \nabla\cdot(\underline{\tilde{\boldsymbol{D}}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{w}(t))) = {}_0^CD_t^{1-\alpha}\boldsymbol{f}(t) := \tilde{\boldsymbol{f}}(t). \tag{5.2.7}$$

It is necessary to assume sufficiently smooth and bounded $\boldsymbol{w}$ and $\boldsymbol{f}$ for the existence of (5.2.7).

We now formulate a weak form of (5.2.6) in two ways using CGFEM and DGFEM. Hence we recall the CG/DG bilinear form and the test spaces with regular subdivisions with respect to vector-valued function spaces as we concerned in Chapter 4.

5.3 Model Problem with Fractional Integral

Let us assume a spatial domain and a time domain as before. For convenience of notation, let $\underline{\boldsymbol{D}} \leftarrow \tilde{\underline{\boldsymbol{D}}}$. Hence we have the following model problem such that

$$\rho\dot{\boldsymbol{w}}(t) - \nabla \cdot {}_0I_t^{1-\alpha}(\underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{w}(t))) = \boldsymbol{f}(t), \qquad \text{on } (0,T] \times \Omega, \tag{5.3.1}$$

$${}_0I_t^{1-\alpha}(\underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{w}(t))) \cdot \boldsymbol{n} = \boldsymbol{g}_N(t), \qquad \text{on } [0,T] \times \Gamma_N, \tag{5.3.2}$$

$$\boldsymbol{w}(t) = \boldsymbol{0}, \qquad \text{on } [0,T] \times \Gamma_D, \tag{5.3.3}$$

$$\boldsymbol{w}(0) = \boldsymbol{w}_0, \qquad \text{on } \Omega, \tag{5.3.4}$$

where $\alpha \in (0,1)$, $\underline{\boldsymbol{D}}$ is a symmetric positive definite piecewise constant fourth order tensor and data terms, $\boldsymbol{f}$, $\boldsymbol{g}_N$ and $\boldsymbol{w}_0$, are sufficiently smooth as in Chapter 4.

5.3.1 CGFEM for Fractional Order Viscoelastic Problem

In a typical way, we can derive a variational form by multiplying by H^1 functions and using integration by parts. First of all, let us recall $\boldsymbol{V} = \{\boldsymbol{v} \in [H^1(\Omega)]^d \mid \boldsymbol{v}(\boldsymbol{x}) = 0 \text{ on } \Gamma_D\}$. When we suppose $\boldsymbol{w}(t), \boldsymbol{v} \in \boldsymbol{V}$, we have

$$\int_\Omega -\nabla \cdot {}_0I_t^{1-\alpha}(\underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{w}(t))) \cdot \boldsymbol{v}\, d\Omega = \int_\Omega {}_0I_t^{1-\alpha}(\underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{w}(t))) : \underline{\boldsymbol{\varepsilon}}(\boldsymbol{v})\, d\Omega - (\boldsymbol{g}_N(t), \boldsymbol{v})_{L_2(\Gamma_N)}$$

by integration by parts.

Remark By Leibniz integral rule, we can observe that

$${}_0I_t^{1-\alpha}(\underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{w}(t))) = \underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}({}_0I_t^{1-\alpha}\boldsymbol{w}(t)).$$

Hence we can obtain the following weak problem:
(T) Find $\boldsymbol{w}(t) \in \boldsymbol{V}$ for all $t \in [0,T]$ such that satisfies

$$(\rho\dot{\boldsymbol{w}}(t), \boldsymbol{v})_{L_2(\Omega)} + a\left({}_0I_t^{1-\alpha}\boldsymbol{w}(t), \boldsymbol{v}\right) = F(t;\boldsymbol{v}),\ \forall t \in (0,T], \tag{5.3.5}$$

$$a\left(\boldsymbol{w}(0), \boldsymbol{v}\right) = a\left(\boldsymbol{w}_0, \boldsymbol{v}\right), \tag{5.3.6}$$

for any $\boldsymbol{v} \in \boldsymbol{V}$ where $a(\cdot,\cdot)$ and F are defined by

$$a\left(\boldsymbol{v}, \boldsymbol{w}\right) = \int_\Omega \underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{v}) : \underline{\boldsymbol{\varepsilon}}(\boldsymbol{w}) d\Omega$$

and

$$F(t;\boldsymbol{v}) = (\boldsymbol{f}(t), \boldsymbol{v})_{L_2(\Omega)} + (\boldsymbol{g}_N(t), \boldsymbol{v})_{L_2(\Gamma_N)}.$$

Remark Recall the facts regarding the bilinear form and the linear form in the previous Chapter. Since (4.1.12) and (4.1.13) hold, the bilinear form is coercive and continuous. Also, the energy norm induced by the bilinear form is equivalent to H^1 norm. Furthermore, the linear form is continuous.

Note that the fractional integral is defined as Volterra integral equation with a weakly singular kernel. Hence we should deal with it very carefully in stability and error analysis. More precisely, we have to use the following remark.

Definition Positive definite kernel
Let $\beta(t)$ be a real valued kernel in $L_1(0,T)$ for $T > 0$. Then the kernel is positive definite if

$$\int_0^T \phi(t) \int_0^t \beta(t-t')\phi(t')dt'dt \geq 0, \qquad \forall \phi \in C[0,T].$$

Remark According to [84], we have a positive definite kernel $t^{-\alpha}$ for $0 < \alpha < 1$. Consequently, we can derive for $T > 0$

$$\int_0^T \phi(t) \int_0^t (t-t')^{-\alpha}\phi(t')dt'dt = \int_0^T \int_0^t (t-t')^{-\alpha}\phi(t')\phi(t)dt'dt \geq 0, \; \forall \phi \in C[0,T], \tag{5.3.7}$$

and hence

$$\int_0^T {}_0I_t^{1-\alpha}\phi(t)\phi(t)dt = \frac{1}{\Gamma(1-\alpha)} \int_0^T \int_0^t (t-t')^{-\alpha}\phi(t')\phi(t)dt'dt \geq 0. \tag{5.3.8}$$

Theorem 5.3. *Suppose that $\boldsymbol{f}$ and $\boldsymbol{w}_0$ are smooth enough. In addition, to simplify, we assume either $\boldsymbol{g}_N = \boldsymbol{0}$ or zero measure of Γ_N. Then there exists a positive constant C such that*

$$\rho \|\boldsymbol{w}\|^2_{L_\infty(0,T;L_2(\Omega))} \leq C \left(\rho \|\boldsymbol{w}_0\|^2_V + \|\boldsymbol{f}\|^2_{L_2(0,T;L_2(\Omega))} \right).$$

Proof. Let $\boldsymbol{v} = \boldsymbol{w}(t)$ in (5.3.5) then it gives

$$\frac{\rho}{2}\frac{d}{dt} \|\boldsymbol{w}(t)\|^2_{L_2(\Omega)} + a\left({}_0I_t^{1-\alpha}\boldsymbol{w}(t), \boldsymbol{w}(t)\right) = F(\boldsymbol{w}(t)). \tag{5.3.9}$$

Taking into account the second term of the left hand side of (5.3.9), the definition of the fractional integral gives

$$a\left({}_0I_t^{1-\alpha}\boldsymbol{w}(t), \boldsymbol{w}(t)\right) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-t')^{-\alpha} a\left(\boldsymbol{w}(t'), \boldsymbol{w}(t)\right) dt', \tag{5.3.10}$$

by Leibniz integral rule. By substitution of (5.3.10) into (5.3.9), integrating over time yields

$$\frac{\rho}{2}(\|\boldsymbol{w}(\tau)\|^2_{L_2(\Omega)} - \|\boldsymbol{w}(0)\|^2_{L_2(\Omega)}) + \frac{1}{\Gamma(1-\alpha)} \int_0^\tau \int_0^t (t-t')^{-\alpha} a\left(\boldsymbol{w}(t'), \boldsymbol{w}(t)\right) dt'dt$$

$$= \int_0^{\tau} F(\boldsymbol{w}(t))dt, \tag{5.3.11}$$

for $0 < \tau \le T$. In the double integral, we can expand the bilinear form and take spatial integration outside so that we have

$$\begin{aligned}
&\int_0^{\tau}\int_0^{t} (t-t')^{-\alpha} a\left(\boldsymbol{w}(t'), \boldsymbol{w}(t)\right) dt'dt \\
&= \int_0^{\tau}\int_0^{t} (t-t')^{-\alpha} \int_{\Omega} \underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{w}(t')) : \underline{\boldsymbol{\varepsilon}}(\boldsymbol{w}(t)) d\Omega dt'dt \\
&= \int_{\Omega}\int_0^{\tau}\int_0^{t} (t-t')^{-\alpha} \underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{w}(t')) : \underline{\boldsymbol{\varepsilon}}(\boldsymbol{w}(t)) dt'dtd\Omega \\
&= \int_{\Omega}\int_0^{\tau}\int_0^{t} (t-t')^{-\alpha} \underline{\boldsymbol{D}}^{1/2}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{w}(t')) : \underline{\boldsymbol{D}}^{1/2}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{w}(t)) dt'dtd\Omega \\
&\ge 0
\end{aligned}$$

where $\underline{\boldsymbol{D}}^{1/2}\underline{\boldsymbol{D}}^{1/2} = \underline{\boldsymbol{D}}$ for $i, j, k, l, = 1, \ldots, d$, by (5.3.7). As a consequence (5.3.11) yields

$$\frac{\rho}{2} \|\boldsymbol{w}(\tau)\|^2_{L_2(\Omega)} \le \frac{\rho}{2} \|\boldsymbol{w}(0)\|^2_{L_2(\Omega)} + \int_0^{\tau} F(\boldsymbol{w}(t))dt. \tag{5.3.12}$$

As we concerned before, we can observe a bound of the last term in (5.3.12) such that

$$\begin{aligned}
\int_0^{\tau} F(\boldsymbol{w}(t))dt &\le \int_0^{\tau} \|\boldsymbol{f}(t)\|_{L_2(\Omega)} \|\boldsymbol{w}(t)\|_{L_2(\Omega)} dt \\
&\le \|\boldsymbol{w}\|_{L_\infty(0,T;L_2(\Omega))} \int_0^{T} \|\boldsymbol{f}(t)\|_{L_2(\Omega)} dt \\
&\le \|\boldsymbol{w}\|_{L_\infty(0,T;L_2(\Omega))} \sqrt{T} \|\boldsymbol{f}\|_{L_2(0,T;L_2(\Omega))} \\
&\le \frac{\epsilon_a}{2} \|\boldsymbol{w}\|^2_{L_\infty(0,T;L_2(\Omega))} + \frac{T}{2\epsilon_a} \|\boldsymbol{f}\|^2_{L_2(0,T;L_2(\Omega))}
\end{aligned}$$

by Cauchy-Schwarz and Young's inequality for any positive ϵ_a. Since τ is arbitrary, we can complete the proof by choice of $\epsilon_a = \rho/2$ and therefore we have

$$\frac{\rho}{4} \|\boldsymbol{w}\|^2_{L_\infty(0,T;L_2(\Omega))} \le C\left(\rho \|\boldsymbol{w}(0)\|^2_{L_2(\Omega)} + \|\boldsymbol{f}\|^2_{L_2(0,T;L_2(\Omega))}\right), \tag{5.3.13}$$

where C is a positive constant depending on the final time T but not exponentially increasing. Moreover, coercivity and (5.3.6) imply that $\|\boldsymbol{w}(0)\|^2_{L_2(\Omega)} \le C \|\boldsymbol{w}_0\|^2_V$ hence the theorem is proved. □

Remark Here, we need to assume zero traction $\boldsymbol{g}_N$ or pure Dirichlet boundary problem. When we, at a first glance, consider the bound for its trace, it is essential to use trace inequality, especially (4.1.23). More precisely, we can obtain

$$\int_0^{\tau} (\boldsymbol{g}_N(t), \boldsymbol{w}(t))_{L_2(\Gamma_N)} dt \le \int_0^{\tau} \|\boldsymbol{g}_N(t)\|_{L_2(\Gamma_N)} \|\boldsymbol{w}(t)\|_{L_2(\Gamma_N)} dt$$

$$\leq \int_0^\tau \|\boldsymbol{g}_N(t)\|_{L_2(\Gamma_N)} C \|\boldsymbol{w}(t)\|_V \, dt$$

by Cauchy-Schwarz inequality and (4.1.23). However, as seen in (5.3.12), we only have L_2 norm of $\boldsymbol{w}$ so that we cannot reduce the energy norm part. Nevertheless, in case of a discrete problem, we can deal with the energy norm of $\boldsymbol{w}$ by inverse polynomial trace theorem although we use Continuous Galerkin method.

Once we recall the finite dimensional test space $\boldsymbol{V}^h$ which is a set of a continuous piecewise polynomial of degree k from Chapter 4, Theorem 5.3 shows the well-posedness of a semidiscrete formulation as well. The key of the proof is using the positive definiteness of kernel (5.3.7). However, in a fully discrete problem, it is necessary to use numerical integration for the fractional integral and hence the weak singularity of the kernel may matter.

In comparison with the linear viscoelastic models with internal variables in Chapter 4, the power-type Volterra integral should be dealt by quadrature rules or some other numerical integrations rather than use of auxiliary equations governed by internal variables, for example (4.2.2), (4.2.4), etc. However, in terms of fractional integral ${}_0I_t^{1-\alpha}$, our kernel is weakly singular at $t = 0$. Therefore, we should be cautious when using numerical integrations with the singular kernel. As in Theorem 5.1, we can choose the following numerical approach by linear interpolation such that

$${}_0I_{t_n}^{1-\alpha}\boldsymbol{w}(t) = \frac{\Delta t^{1-\alpha}}{\Gamma(3-\alpha)} \sum_{i=0}^{n} B_{n,i}\boldsymbol{w}(t_i) + \boldsymbol{O}(\Delta t^2) = \boldsymbol{q}_n(\boldsymbol{w}) + \boldsymbol{O}(\Delta t^2),$$

where

$$B_{n,i} = \begin{cases} n^{1-\alpha}(2-\alpha-n) + (n-1)^{2-\alpha}, & i = 0, \\ (n-i-1)^{2-\alpha} + (n-i+1)^{2-\alpha} - 2(n-i)^{2-\alpha}, & i = 1, \dots, n-1, \\ 1, & i = n. \end{cases}$$

Consequently, we can formulate the fully discrete formulation with the above numerical integration for fractional order integral. Let us denote our fully discrete solution by $\boldsymbol{W}_h^n$ for $n = 0, \dots, N$ when $\Delta t = T/N > 0$. Finally, Crank-Nicolson method yields a fully discrete form of **(T)** as follows.

(T) Find $\boldsymbol{W}_h^n \in \boldsymbol{V}^h$ for $n = 0, \dots, N$ such that satisfying for any $\boldsymbol{v} \in \boldsymbol{V}^h$,

$$\left(\rho \frac{\boldsymbol{W}_h^{n+1} - \boldsymbol{W}_h^n}{\Delta t}, \boldsymbol{v}\right)_{L_2(\Omega)} + a\left(\frac{\boldsymbol{q}_{n+1}(\boldsymbol{W}_h) + \boldsymbol{q}_n(\boldsymbol{W}_h)}{2}, \boldsymbol{v}\right) = \bar{F}^n(\boldsymbol{v}), \tag{5.3.14}$$

$\forall n = 0, \dots, N-1$, and

$$a\left(\boldsymbol{W}_h^0, \boldsymbol{v}\right) = a\left(\boldsymbol{w}_0, \boldsymbol{v}\right). \tag{5.3.15}$$

Next, we want to carry out its stability analysis. One can show the discrete stable bound then we also obtain the existence and uniqueness of the solution. We use the same arguments as before to show bounds for linear form but we have to deal with more the numerical integration part.

Theorem 5.4. *Let $\boldsymbol{W}_h^n$ be a fully discrete solution to* **(T)** *for $n = 0, \ldots, N$. Suppose data terms, ($\boldsymbol{g}_N, \boldsymbol{w}_0,$ and $\boldsymbol{f}$), are sufficiently smooth then there exists a positive constant C such that*

$$\begin{aligned}
&\max_{0\leq n\leq N} \|\boldsymbol{W}_h^n\|^2_{L_2(\Omega)} + \frac{\Delta t^{2-\alpha}}{\Gamma(3-\alpha)} \sum_{n=0}^{N-1} \left\|\boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n\right\|_V^2 \\
\leq & C\left(\|\boldsymbol{w}_0\|_V^2 + \Delta t \sum_{n=0}^{N} \|\boldsymbol{f}(t_n)\|^2_{L_2(\Omega)} + \Delta t h^{-1} \sum_{n=0}^{N} \|\boldsymbol{g}_N(t_n)\|^2_{L_2(\Gamma_N)} \right) \\
\leq & C\left(\|\boldsymbol{w}_0\|_V^2 + \|\boldsymbol{f}\|^2_{L_\infty(0,T;L_2(\Omega))} + h^{-1} \|\boldsymbol{g}_N\|^2_{L_\infty(0,T;L_2(\Gamma_N))} \right).
\end{aligned}$$

Proof. Let $m \in \{1, \ldots, N\}$. A choice of $\boldsymbol{v} = 2\Delta t(\boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n)$ in (5.3.14) with summing from $n = 0$ to $n = m-1$ yields

$$\begin{aligned}
&2\rho\left(\|\boldsymbol{W}_h^m\|^2_{L_2(\Omega)} - \left\|\boldsymbol{W}_h^0\right\|^2_{L_2(\Omega)}\right) + \Delta t \sum_{n=0}^{m-1} a\left(\boldsymbol{q}_{n+1}(\boldsymbol{W}_h) + \boldsymbol{q}_n(\boldsymbol{W}_h), \boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n\right) \\
=&\Delta t \sum_{n=0}^{m-1} \left(\boldsymbol{f}(t_{n+1}) + \boldsymbol{f}(t_n), \boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n\right)_{L_2(\Omega)} \\
&+ \Delta t \sum_{n=0}^{m-1} \left(\boldsymbol{g}_N(t_{n+1}) + \boldsymbol{g}_N(t_n), \boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n\right)_{L_2(\Gamma_N)}
\end{aligned} \tag{5.3.16}$$

Expanding $\boldsymbol{q}_n$ allows us to rewrite (5.3.16) as

$$\begin{aligned}
&2\rho \|\boldsymbol{W}_h^m\|^2_{L_2(\Omega)} + \frac{\Delta t^{2-\alpha}}{\Gamma(3-\alpha)} \sum_{n=0}^{m-1} \left\|\boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n\right\|_V^2 \\
=&2\rho \left\|\boldsymbol{W}_h^0\right\|^2_{L_2(\Omega)} + \Delta t \sum_{n=0}^{m-1} \left(\boldsymbol{f}(t_{n+1}) + \boldsymbol{f}(t_n), \boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n\right)_{L_2(\Omega)} \\
&+ \Delta t \sum_{n=0}^{m-1} \left(\boldsymbol{g}_N(t_{n+1}) + \boldsymbol{g}_N(t_n), \boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n\right)_{L_2(\Gamma_N)} \\
&- \frac{\Delta t^{2-\alpha}}{\Gamma(3-\alpha)} \sum_{n=0}^{m-1} a\left(\sum_{i=0}^{n} B_{n+1,i} \boldsymbol{W}_h^i + \sum_{i=0}^{n-1} B_{n,i} \boldsymbol{W}_h^i, \boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n \right).
\end{aligned} \tag{5.3.17}$$

From now on, in a similar technique, we shall find the bounds of the right hand side of (5.3.17).

- $\left\|\boldsymbol{W}_h^0\right\|^2_{L_2(\Omega)}$
  Since (5.3.15) holds, we have

$$\left\|\boldsymbol{W}_h^0\right\|_V^2 \leq \left\|\boldsymbol{W}_h^0\right\|_V \|\boldsymbol{w}_0\|_V$$

by Cauchy-Schwarz inequality and so

$$\left\|\boldsymbol{W}_h^0\right\|_{L_2(\Omega)} \le \left\|\boldsymbol{W}_h^0\right\|_{H^1(\Omega)} \le C\left\|\boldsymbol{W}_h^0\right\|_V \le C\left\|\boldsymbol{w}_0\right\|_V$$

for some positive C by (4.1.12).

- $\Delta t \sum_{n=0}^{m-1}\left(\boldsymbol{f}(t_{n+1})+\boldsymbol{f}(t_n), \boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n\right)_{L_2(\Omega)}$
  Use of Cauchy-Schwarz and Young's inequalities gives

$$\begin{aligned}
&\Delta t \sum_{n=0}^{m-1}\left(\boldsymbol{f}(t_{n+1})+\boldsymbol{f}(t_n), \boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n\right)_{L_2(\Omega)} \\
\le&\Delta t \sum_{n=0}^{m}\left(2\epsilon_a\left\|\boldsymbol{f}(t_n)\right\|_{L_2(\Omega)}^2+\frac{2}{\epsilon_a}\left\|\boldsymbol{W}_h^n\right\|_{L_2(\Omega)}^2\right) \\
\le&\Delta t \sum_{n=0}^{N} 2\epsilon_a\left\|\boldsymbol{f}(t_n)\right\|_{L_2(\Omega)}^2+\frac{2(T+\Delta t)}{\epsilon_a} \max_{0\le n\le N}\left\|\boldsymbol{W}_h^n\right\|_{L_2(\Omega)}^2
\end{aligned}$$

  for any positive ϵ_a.

- $\Delta t \sum_{n=0}^{m-1}\left(\boldsymbol{g}_N(t_{n+1})+\boldsymbol{g}_N(t_n), \boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n\right)_{L_2(\Gamma_N)}$
  While using the same approach as the above, we can also derive

$$\begin{aligned}
&\Delta t \sum_{n=0}^{m-1}\left(\boldsymbol{g}_N(t_{n+1})+\boldsymbol{g}_N(t_n), \boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n\right)_{L_2(\Gamma_N)} \\
\le&\Delta t \sum_{n=0}^{m} 2\epsilon_b\left\|\boldsymbol{g}_N(t_n)\right\|_{L_2(\Gamma_N)}^2+\Delta t \sum_{n=0}^{m} \frac{2}{\epsilon_b}\left\|\boldsymbol{W}_h^n\right\|_{L_2(\Gamma_N)}^2 \\
\le&\Delta t \sum_{n=0}^{m} 2\epsilon_b\left\|\boldsymbol{g}_N(t_n)\right\|_{L_2(\Gamma_N)}^2+\Delta t \sum_{n=0}^{m} \frac{2}{\epsilon_b} C h^{-1}\left\|\boldsymbol{W}_h^n\right\|_{L_2(\Omega)}^2 \\
\le&\Delta t \sum_{n=0}^{N} 2\epsilon_b\left\|\boldsymbol{g}_N(t_n)\right\|_{L_2(\Gamma_N)}^2+C h^{-1} \frac{2(T+\Delta t)}{\epsilon_b} \max_{0\le n\le N}\left\|\boldsymbol{W}_h^n\right\|_{L_2(\Omega)}^2
\end{aligned}$$

  by inverse polynomial trace theorem, for any positive ϵ_b. Note that if we use a general trace inequality such as (4.1.23), it is necessary to deal with the energy norm estimates but we cannot handle the energy norm by L_2 norm. Hence we have to introduce inverse polynomial trace theorem rather than (4.1.23).

From the above bounds, (5.3.17) can be written by

$$2\rho\left\|\boldsymbol{W}_h^m\right\|_{L_2(\Omega)}^2+\frac{\Delta t^{2-\alpha}}{\Gamma(3-\alpha)} \sum_{n=0}^{m-1}\left\|\boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n\right\|_V^2$$

$$
\begin{aligned}
\leq& \rho C \|\boldsymbol{w}_0\|_V^2 + \Delta t \sum_{n=0}^{N} 2\epsilon_a \|\boldsymbol{f}(t_n)\|_{L_2(\Omega)}^2 + \frac{2(T+\Delta t)}{\epsilon_a} \max_{0\leq n\leq N} \|\boldsymbol{W}_h^n\|_{L_2(\Omega)}^2 \\
&+ \Delta t \sum_{n=0}^{N} 2\epsilon_b \|\boldsymbol{g}_N(t_n)\|_{L_2(\Gamma_N)}^2 + \frac{Ch^{-1}2(T+\Delta t)}{\epsilon_b} \max_{0\leq n\leq N} \|\boldsymbol{W}_h^n\|_{L_2(\Omega)}^2 \\
&- \frac{\Delta t^{2-\alpha}}{\Gamma(3-\alpha)} \sum_{n=0}^{m-1} a\left(\sum_{i=0}^{n} B_{n+1,i}\boldsymbol{W}_h^i + \sum_{i=0}^{n-1} B_{n,i}\boldsymbol{W}_h^i, \boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n\right) \\
:=& \mathcal{R} - \frac{\Delta t^{2-\alpha}}{\Gamma(3-\alpha)} \sum_{n=0}^{m-1} a\left(\sum_{i=0}^{n} B_{n+1,i}\boldsymbol{W}_h^i + \sum_{i=0}^{n-1} B_{n,i}\boldsymbol{W}_h^i, \boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n\right). \qquad (5.3.18)
\end{aligned}
$$

Now, the last term of (5.3.18) remains to show its boundedness. Note that $\mathcal{R}$ in (5.3.18) is independent of m. Hereafter, we would like to use mathematical induction to derive the upper bound of the last term. Our claim to be shown by induction is

$$
2\rho \|\boldsymbol{W}_h^m\|_{L_2(\Omega)}^2 + \frac{\Delta t^{2-\alpha}}{2\Gamma(3-\alpha)} \sum_{n=0}^{m-1} \left\|\boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n\right\|_V^2 \leq C\big(\mathcal{R} + \Delta t^{2-\alpha} \left\|\boldsymbol{W}_h^0\right\|_V^2\big), \quad (5.3.19)
$$

for some positive C, $\forall m$. For $m = 1$ in the last term of (5.3.18), we have

$$
-a\left(B_{1,0}\boldsymbol{W}_h^0, \boldsymbol{W}_h^1 + \boldsymbol{W}_h^0\right) \leq \frac{B_{1,0}^2\epsilon}{2} \left\|\boldsymbol{W}_h^0\right\|_V^2 + \frac{1}{2\epsilon} \left\|\boldsymbol{W}_h^1 + \boldsymbol{W}_h^0\right\|_V^2
$$

by Cauchy-Schwarz and Young's inequality with any positive ϵ. Hence, proper ϵ, for example $\epsilon = 1$, allows us to have

$$
2\rho \left\|\boldsymbol{W}_h^1\right\|_{L_2(\Omega)}^2 + \frac{\Delta t^{2-\alpha}}{2\Gamma(3-\alpha)} \left\|\boldsymbol{W}_h^1 + \boldsymbol{W}_h^0\right\|_V^2 \leq \mathcal{R} + \frac{B_{1,0}^2}{2} \frac{\Delta t^{2-\alpha}}{\Gamma(3-\alpha)} \left\|\boldsymbol{W}_h^0\right\|_V^2. \quad (5.3.20)
$$

In case of $m = 2$, (5.3.18) gives

$$
\begin{aligned}
&2\rho \left\|\boldsymbol{W}_h^2\right\|_{L_2(\Omega)}^2 + \frac{\Delta t^{2-\alpha}}{\Gamma(3-\alpha)}(\left\|\boldsymbol{W}_h^2 + \boldsymbol{W}_h^1\right\|_V^2 + \left\|\boldsymbol{W}_h^1 + \boldsymbol{W}_h^0\right\|_V^2) \\
\leq& \mathcal{R} - \frac{\Delta t^{2-\alpha}}{\Gamma(3-\alpha)} a\left(B_{2,1}\boldsymbol{W}_h^1 + B_{2,0}\boldsymbol{W}_h^0 + B_{1,0}\boldsymbol{W}_h^0, \boldsymbol{W}_h^2 + \boldsymbol{W}_h^1\right) \\
&- \frac{\Delta t^{2-\alpha}}{\Gamma(3-\alpha)} a\left(B_{1,0}\boldsymbol{W}_h^0, \boldsymbol{W}_h^1 + \boldsymbol{W}_h^0\right).
\end{aligned}
$$

Using Cauchy-Schwarz and Young's inequalities, we can have

$$
\begin{aligned}
&-a\left(B_{2,1}\boldsymbol{W}_h^1 + B_{2,0}\boldsymbol{W}_h^0 + B_{1,0}\boldsymbol{W}_h^0, \boldsymbol{W}_h^2 + \boldsymbol{W}_h^1\right) \\
\leq& \frac{(\max(B_{2,1}, B_{2,0} + B_{1,0}))^2}{2\epsilon} \left\|\boldsymbol{W}_h^1 + \boldsymbol{W}_h^0\right\|_V^2 + \frac{\epsilon}{2} \left\|\boldsymbol{W}_h^2 + \boldsymbol{W}_h^1\right\|_V^2,
\end{aligned}
$$

and

$$-a\left(B_{1,0}\boldsymbol{W}_h^0, \boldsymbol{W}_h^1 + \boldsymbol{W}_h^0\right) \leq \frac{B_{1,0}^2}{2\epsilon}\left\|\boldsymbol{W}_h^0\right\|_V^2 + \frac{\epsilon}{2}\left\|\boldsymbol{W}_h^1 + \boldsymbol{W}_h^0\right\|_V^2$$

for any positive ϵ. Hence coupling with (5.3.20) which provides the bound for $\left\|\boldsymbol{W}_h^1 + \boldsymbol{W}_h^0\right\|_V^2$, and choosing $\epsilon = 1$, (5.3.18) for $m = 2$ becomes

$$2\rho\left\|\boldsymbol{W}_h^2\right\|_{L_2(\Omega)}^2 + \frac{\Delta t^{2-\alpha}}{2\Gamma(3-\alpha)}\sum_{n=0}^{1}\left\|\boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n\right\|_V^2 \leq C\big(\mathcal{R} + \Delta t^{2-\alpha}\left\|\boldsymbol{W}_h^0\right\|_V^2\big),$$

for some positive C. As following induction method, let us assume that (5.3.19) holds for $m = j < N$ so that

$$2\rho\left\|\boldsymbol{W}_h^j\right\|_{L_2(\Omega)}^2 + \frac{\Delta t^{2-\alpha}}{2\Gamma(3-\alpha)}\sum_{n=0}^{j-1}\left\|\boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n\right\|_V^2 \leq C\big(\mathcal{R} + \Delta t^{2-\alpha}\left\|\boldsymbol{W}_h^0\right\|_V^2\big).$$

Taking into account the case of $m = j + 1$, we have

$$\begin{aligned}
&\sum_{n=0}^{j} a\left(\sum_{i=0}^{n} B_{n+1,i}\boldsymbol{W}_h^i + \sum_{i=0}^{n-1} B_{n,i}\boldsymbol{W}_h^i, \boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n\right)\\
&= \sum_{n=0}^{j} a\left(\sum_{i=1}^{n-1} B_{n,i}(\boldsymbol{W}_h^{i+1} + \boldsymbol{W}_h^i), \boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n\right)\\
&\quad + \sum_{n=0}^{j} a\left(B_{n,0}\boldsymbol{W}_h^0 + B_{n+1,0}\boldsymbol{W}_h^0 + B_{n+1,1}\boldsymbol{W}_h^1, \boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n\right)\\
&= \sum_{n=0}^{j}\sum_{i=1}^{n-1} B_{n,i}\, a\left(\boldsymbol{W}_h^{i+1} + \boldsymbol{W}_h^i, \boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n\right)\\
&\quad + \sum_{n=0}^{j} a\left((B_{n,0} + B_{n+1,0} - B_{n+1,1})\boldsymbol{W}_h^0, \boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n\right)\\
&\quad + \sum_{n=0}^{j} a\left(B_{n+1,1}(\boldsymbol{W}_h^1 + \boldsymbol{W}_h^0), \boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n\right)\\
&\leq \sum_{n=0}^{j}\sum_{i=1}^{n-1}\left(\frac{G^2\epsilon}{2}\left\|\boldsymbol{W}_h^{i+1} + \boldsymbol{W}_h^i\right\|_V^2 + \frac{1}{2\epsilon}\left\|\boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n\right\|_V^2\right)\\
&\quad + \sum_{n=0}^{j}\left(\frac{(3G)^2\tilde{\epsilon}}{2}\left\|\boldsymbol{W}_h^0\right\|_V^2 + \frac{1}{2\tilde{\epsilon}}\left\|\boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n\right\|_V^2\right)\\
&\quad + \sum_{n=0}^{j}\left(\frac{G^2\check{\epsilon}}{2}\left\|\boldsymbol{W}_h^1 + \boldsymbol{W}_h^0\right\|_V^2 + \frac{1}{2\check{\epsilon}}\left\|\boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n\right\|_V^2\right)
\end{aligned}$$

where $0 < G = \max_{0\le i\le n\le N} B_{n,i} < 2$ by Lemma 5.1, for any positive $\epsilon, \tilde{\epsilon}, \check{\epsilon}$. Thus, we can obtain the boundedness of $\sum_{n=0}^{j}\sum_{i=1}^{n-1} \left\|\boldsymbol{W}_h^{i+1} + \boldsymbol{W}_h^i\right\|_V^2$ since $\sum_{i=1}^{n-1} \left\|\boldsymbol{W}_h^{i+1} + \boldsymbol{W}_h^i\right\|_{\mathcal{V}}^2$ is bounded for $0 \le i \le j$ by the induction assumption. Consequently, appropriate choice of $(\epsilon, \tilde{\epsilon}, \check{\epsilon})$ yields

$$2\rho \left\|\boldsymbol{W}_h^{j+1}\right\|_{L_2(\Omega)}^2 + \frac{\Delta t^{2-\alpha}}{2\Gamma(3-\alpha)} \sum_{n=0}^{j} \left\|\boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n\right\|_V^2 \le C\Big(\mathcal{R} + \Delta t^{2-\alpha} \left\|\boldsymbol{W}_h^0\right\|_V^2\Big).$$

Thus we can complete the induction and hence (5.3.19) holds. Turning to main goal, when we consider maximum in (5.3.19), since $\left\|\boldsymbol{W}_h^0\right\|_V \le \|\boldsymbol{w}_0\|_V$ as well as m is arbitrary, (5.3.19) can be written as

$$\begin{aligned}
&2\rho \max_{0\le n\le N} \|\boldsymbol{W}_h^n\|_{L_2(\Omega)}^2 + \frac{\kappa\Delta t^{2-\alpha}}{2\Gamma(3-\alpha)} \sum_{n=0}^{N-1} \left\|\boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n\right\|_V^2 \\
\le& 2C\Bigg(\|\boldsymbol{w}_0\|_V^2 + \Delta t \sum_{n=0}^{N} 2\epsilon_a \|\boldsymbol{f}(t_n)\|_{L_2(\Omega)}^2 + \frac{2(T+\Delta t)}{\epsilon_a} \max_{0\le n\le N} \|\boldsymbol{W}_h^n\|_{L_2(\Omega)}^2 \\
&+ \Delta t \sum_{n=0}^{N} 2\epsilon_b \|\boldsymbol{g}_N(t_n)\|_{L_2(\Gamma_N)}^2 + \frac{h^{-1}2(T+\Delta t)}{\epsilon_b} \max_{0\le n\le N} \|\boldsymbol{W}_h^n\|_{L_2(\Omega)}^2 \Bigg).
\end{aligned}$$

Therefore, choosing $\epsilon_a = 8C(T+\Delta t)/\rho$ and $\epsilon_b = 8Ch^{-1}(T+\Delta t)/\rho$ gives

$$\begin{aligned}
&\rho \max_{0\le n\le N} \|\boldsymbol{W}_h^n\|_{L_2(\Omega)}^2 + \frac{\Delta t^{2-\alpha}}{2\Gamma(3-\alpha)} \sum_{n=0}^{N-1} \left\|\boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n\right\|_V^2 \\
\le& C\Bigg(\|\boldsymbol{w}_0\|_V^2 + \Delta t \sum_{n=0}^{N} \|\boldsymbol{f}(t_n)\|_{L_2(\Omega)}^2 + \Delta t \sum_{n=0}^{N} h^{-1} \|\boldsymbol{g}_N(t_n)\|_{L_2(\Gamma_N)}^2 \Bigg) \\
\le& C\Bigg(\|\boldsymbol{w}_0\|_V^2 + \|\boldsymbol{f}\|_{L_\infty(0,T;L_2(\Omega))}^2 + h^{-1} \|\boldsymbol{g}_N\|_{L_\infty(0,T;L_2(\Gamma_N))}^2 \Bigg)
\end{aligned}$$

for some positive C. □

In spite of using CGFEM, the stability bound in Theorem 5.4 contains h^{-1} term. Unless using inverse polynomial trace theorem, we cannot analyse trace of the discrete solution in $L_2(\Omega)$ norm. However, as we concerned before, h^{-1} term has no effect on the well-posedness. Therefore, Theorem 5.4 implies the existence and uniqueness of the discrete solution.

For a fully discrete problem, we use the Crank-Nicolson finite difference method in time discretisation. However, in order to attain optimal second order accuracy, it is essential to assume sufficient smoothness of an exact solution, for instance

$$\left|\frac{u(t+\Delta t) - u(t)}{\Delta t} - \frac{\dot{u}(t+\Delta t) + \dot{u}(t)}{2}\right| = O(\Delta t^2),$$

when $u \in C^3$. Hence the optimal convergence order in time requires H^3 smoothness in time. More precisely, the bound of $|u^{(3)}(t)|$ is necessary. However, our primal model problem (5.3.1) contains weak singularity on the fractional order integration so that it is necessary to check regularity of solutions before giving a supposition on the smoothness.

Remark (Regularity of solutions)

Let us recall the primal equation (5.3.1). We can rewrite

$$\begin{aligned}\rho\dot{\boldsymbol{w}}(t) =& \nabla \cdot {}_0I_t^{1-\alpha}(\underline{\boldsymbol{D}}\boldsymbol{\varepsilon}(\boldsymbol{w}(t))) + \boldsymbol{f}(t) \\ =& \beta * \mathcal{D}\boldsymbol{w}(t) + \boldsymbol{f}(t)\end{aligned}$$

where $\beta(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}$ is a weakly singular kernel, $\mathcal{D} = \nabla \cdot \underline{\boldsymbol{D}}\boldsymbol{\varepsilon}$ is a linear differential operator on the spatial domain and $*$ denotes Laplace convolution such that

$$f_1 * f_2 = \int_0^t f_1(t-t')f_2(t')dt'.$$

By Young's inequality for the convolution, we can observe that

$$\|\rho\dot{\boldsymbol{w}}\|_{L_2(0,T)} \leq \|\beta\|_{L_1(0,T)}\|\mathcal{D}\boldsymbol{w}\|_{L_2(0,T)} + \|\boldsymbol{f}\|_{L_2(0,T)}.$$

Since β is L_1 integrable, if $\boldsymbol{w}$ and $\boldsymbol{f}$ are L_2 integrable in time, so is $\dot{\boldsymbol{w}}$. Differentiating (5.3.1) in time gives

$$\rho\ddot{\boldsymbol{w}}(t) = \beta(t)\mathcal{D}\boldsymbol{w}(0) + \beta * \mathcal{D}\dot{\boldsymbol{w}}(t) + \dot{\boldsymbol{f}}(t).$$

$\ddot{\boldsymbol{w}}$ is L_1 integrable with sufficiently smooth $\boldsymbol{f}$ but it is also L_2 integrable only if $\mathcal{D}\boldsymbol{w}(0) = \dot{\boldsymbol{w}}(0) = \boldsymbol{0}$. In a similar way, we can consider a third time derivative of $\boldsymbol{w}$. Then we have

$$\rho\boldsymbol{w}^{(3)}(t) = \dot{\beta}(t)\mathcal{D}\boldsymbol{w}(0) + \beta(t)\mathcal{D}\dot{\boldsymbol{w}}(0) + \beta * \mathcal{D}\ddot{\boldsymbol{w}}(t) + \ddot{\boldsymbol{f}}(t).$$

Note that $\dot{\beta}(t)$ is non-integrable in L_1 and L_2 so is the third derivative if $\mathcal{D}\boldsymbol{w}(0) \neq \boldsymbol{0}$. That is, we cannot make sure boundedness of the third derivative. As concerned, due to the weakly singular kernel, it is unable to take full advantage of the second order schemes in terms of time discretisation. However, in [8], the spatial L_2 norm of the third time derivative is bounded by the initial condition and $\boldsymbol{f}$. In our sense, we have

$$\left|\frac{\boldsymbol{w}(t+\Delta t) - \boldsymbol{w}(t)}{\Delta t} - \frac{\dot{\boldsymbol{w}}(t+\Delta t) + \dot{\boldsymbol{w}}(t)}{2}\right| = \boldsymbol{O}(\Delta t^{2-\alpha}), \tag{5.3.21}$$

for any $t \in [0, T-\Delta t]$ where $\boldsymbol{f} \in W_1^3(0,T;L_2(\Omega)) \cap H^2(0,T;L_2(\Omega))$. Furthermore, the optimal convergent order, which is of Δt^2, is only given when we can ignore the singularity on $t=0$. For example, if $\boldsymbol{w}(0) = \dot{\boldsymbol{w}}(0) = \boldsymbol{0}$,

$$\left|\frac{\boldsymbol{w}(t+\Delta t) - \boldsymbol{w}(t)}{\Delta t} - \frac{\dot{\boldsymbol{w}}(t+\Delta t) + \dot{\boldsymbol{w}}(t)}{2}\right| = \boldsymbol{O}(\Delta t^2). \tag{5.3.22}$$

To sum up, as we suppose a sufficiently smooth $\boldsymbol{f}$, the solution satisfies (5.3.21) with our initial and boundary conditions. However, the certain assumption such as zero initial data or $\boldsymbol{w} \in \ker(\mathcal{D})$ leads the solution to fulfil (5.3.22) where $\ker(\mathcal{D})$ is a kernel set of the differential operator $\mathcal{D}$.

Next, we state and prove *a priori* error estimates by recalling elliptic approximations (4.1.31) and (4.1.32). Hence recall the elliptic projection operator $\boldsymbol{R}$ in (4.1.30) and define

$$\boldsymbol{\theta}(t) := \boldsymbol{w}(t) - \boldsymbol{R}\boldsymbol{w}(t)\ t \in [0, T], \qquad \boldsymbol{\chi}^n := \boldsymbol{W}_h^n - \boldsymbol{R}\boldsymbol{w}(t_n) \text{ for } n = 0, \dots, N.$$

Besides, we assume the elliptic regularity for optimal L_2 error estimates.

Lemma 5.2. *Suppose*

$$\boldsymbol{w} \in C^2(0, T; [H^s(\Omega)]^d) \cap W^1_\infty(0, T; \boldsymbol{V})$$

and $(\boldsymbol{W}_h^n)_{n=0}^N$ *satisfies the fully discrete formula* **(T)**. *Then we have*

$$\max_{0 \le n \le N} \|\boldsymbol{\chi}^n\|_{L_2(\Omega)} + \left(\Delta t^{2-\alpha} \sum_{n=0}^{N-1} \left\|\boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^n\right\|_V^2\right)^{1/2} = O(h^r + \Delta t^{2-\alpha}),$$

where $r = \min(s, k+1)$*. Moreover, if* $\boldsymbol{w}(0) = \dot{\boldsymbol{w}}(0) = \boldsymbol{0}$ *or* $\boldsymbol{w} \in \ker(\mathcal{D})$*, then we have*

$$\max_{0 \le n \le N} \|\boldsymbol{\chi}^n\|_{L_2(\Omega)} + \left(\Delta t^{2-\alpha} \sum_{n=0}^{N-1} \left\|\boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^n\right\|_V^2\right)^{1/2} = O(h^r + \Delta t^2).$$

Proof. For $m \in \{1, \dots, N\}$, subtracting (5.3.5) for average between $t = t_{n+1}$ and $t = t_n$ from (5.3.14) where $0 \le n \le m-1$ gives

$$\begin{aligned}
&\rho \left(\frac{\boldsymbol{W}_h^{n+1} - \boldsymbol{W}_h^n}{\Delta t} - \frac{\dot{\boldsymbol{w}}^{n+1} + \dot{\boldsymbol{w}}^n}{2}, \boldsymbol{v} \right)_{L_2(\Omega)} \\
&\quad + a \left(\frac{\boldsymbol{q}_{n+1}(\boldsymbol{W}_h) + \boldsymbol{q}_n(\boldsymbol{W}_h)}{2} - \frac{{}_0I_{t_{n+1}}^{1-\alpha}\boldsymbol{w} + {}_0I_{t_n}^{1-\alpha}\boldsymbol{w}}{2}, \boldsymbol{v} \right) = 0
\end{aligned}$$

for any $\boldsymbol{v} \in \boldsymbol{V}^h$. By definitions of $\boldsymbol{\theta}$ and $\boldsymbol{\chi}$, as adding numerical integration of the fractional integral of $\boldsymbol{w}$, we can rewrite it by

$$\begin{aligned}
&\frac{\rho}{\Delta t} \left(\boldsymbol{\chi}^{n+1} - \boldsymbol{\chi}^n, \boldsymbol{v}\right)_{L_2(\Omega)} + \frac{1}{2} a \left(\boldsymbol{q}_{n+1}(\boldsymbol{\chi}) + \boldsymbol{q}_n(\boldsymbol{\chi}), \boldsymbol{v}\right) \\
&= \frac{\rho}{\Delta t} \left(\boldsymbol{\theta}^{n+1} - \boldsymbol{\theta}^n, \boldsymbol{v}\right)_{L_2(\Omega)} + \frac{1}{2} a \left(\boldsymbol{q}_{n+1}(\boldsymbol{\theta}) + \boldsymbol{q}_n(\boldsymbol{\theta}), \boldsymbol{v}\right) + \frac{1}{2} a \left(\boldsymbol{e}^{n+1} + \boldsymbol{e}^n, \boldsymbol{v}\right) \\
&\quad + \rho \left(\boldsymbol{\mathcal{E}}^n, \boldsymbol{v}\right)_{L_2(\Omega)},
\end{aligned} \tag{5.3.23}$$

where $\boldsymbol{e}^n := \boldsymbol{q}_n(\boldsymbol{w}) - {}_0I_{t_n}^{1-\alpha}\boldsymbol{w}$ and $\boldsymbol{\mathcal{E}}(t) := \frac{\dot{\boldsymbol{w}}(t+\Delta t) + \dot{\boldsymbol{w}}(t)}{2} - \frac{\boldsymbol{w}(t+\Delta t) - \boldsymbol{w}(t)}{\Delta t}$ for $t \in [0, T - \Delta t]$. Galerkin orthogonality reduces (5.3.23) to

$$\frac{\rho}{\Delta t} \left(\boldsymbol{\chi}^{n+1} - \boldsymbol{\chi}^n, \boldsymbol{v}\right)_{L_2(\Omega)} + \frac{1}{2} a \left(\boldsymbol{q}_{n+1}(\boldsymbol{\chi}) + \boldsymbol{q}_n(\boldsymbol{\chi}), \boldsymbol{v}\right)$$

$$=\frac{\rho}{\Delta t}\left(\boldsymbol{\theta}^{n+1}-\boldsymbol{\theta}^{n},\boldsymbol{v}\right)_{L_2(\Omega)}+\frac{1}{2}a\left(\boldsymbol{e}^{n+1}+\boldsymbol{e}^{n},\boldsymbol{v}\right)+\rho\left(\boldsymbol{\mathcal{E}}^{n},\boldsymbol{v}\right)_{L_2(\Omega)}, \tag{5.3.24}$$

Once we put $\boldsymbol{v}=2\Delta t(\boldsymbol{\chi}^{n+1}+\boldsymbol{\chi}^{n})$ in (5.3.24), summing from $n=0$ to $n=m-1$ implies

$$\begin{aligned}
&2\rho\left\|\boldsymbol{\chi}^{m}\right\|_{L_2(\Omega)}^{2}+\frac{\Delta t^{2-\alpha}}{\Gamma(3-\alpha)}\sum_{n=0}^{m-1}\left\|\boldsymbol{\chi}^{n+1}+\boldsymbol{\chi}^{n}\right\|_{V}^{2}\\
=&2\rho\left\|\boldsymbol{\chi}^{0}\right\|_{L_2(\Omega)}^{2}+2\rho\sum_{n=0}^{m-1}\left(\boldsymbol{\theta}^{n+1}-\boldsymbol{\theta}^{n},\boldsymbol{\chi}^{n+1}+\boldsymbol{\chi}^{n}\right)_{L_2(\Omega)}\\
&+\Delta t\sum_{n=0}^{m-1}a\left(\boldsymbol{e}^{n+1}+\boldsymbol{e}^{n},\boldsymbol{\chi}^{n+1}+\boldsymbol{\chi}^{n}\right)+2\rho\Delta t\sum_{n=0}^{m-1}\left(\boldsymbol{\mathcal{E}}^{n},\boldsymbol{\chi}^{n+1}+\boldsymbol{\chi}^{n}\right)_{L_2(\Omega)}\\
&-\frac{\Delta t^{2-\alpha}}{\Gamma(3-\alpha)}\sum_{n=0}^{m-1}a\left(\sum_{i=0}^{n}B_{n+1,i}\boldsymbol{\chi}^{i}+\sum_{i=0}^{n-1}B_{n,i}\boldsymbol{\chi}^{i},\boldsymbol{\chi}^{n+1}+\boldsymbol{\chi}^{n}\right).
\end{aligned} \tag{5.3.25}$$

For the sake of estimations, we shall show the bounds of (5.3.25) as following.

- $\left\|\boldsymbol{\chi}^{0}\right\|_{L_2(\Omega)}^{2}$
  (5.3.14) and Galerkin orthogonality lead us to have

$$\begin{aligned}
a\left(\boldsymbol{\chi}^{0},\boldsymbol{v}\right)=&a\left(\boldsymbol{W}_h^0-\boldsymbol{R}\boldsymbol{w}_0,\boldsymbol{v}\right)\\
=&a\left(\boldsymbol{W}_h^0-\boldsymbol{w}_0+\boldsymbol{w}_0-\boldsymbol{R}\boldsymbol{w}_0,\boldsymbol{v}\right)\\
=&a\left(\boldsymbol{W}_h^0-\boldsymbol{w}_0,\boldsymbol{v}\right)+a\left(\boldsymbol{w}_0-\boldsymbol{R}\boldsymbol{w}_0,\boldsymbol{v}\right)\\
=&0
\end{aligned}$$

  for any $\boldsymbol{v}\in\boldsymbol{V}^h$. It implies that

$$\left\|\boldsymbol{\chi}^{0}\right\|_{L_2(\Omega)}^{2}\leq\left\|\boldsymbol{\chi}^{0}\right\|_{H^1(\Omega)}^{2}\leq C\left\|\boldsymbol{\chi}^{0}\right\|_{V}^{2}=0,$$

  by the coercivity (4.1.12).

- $\sum_{n=0}^{m-1}\left(\boldsymbol{\theta}^{n+1}-\boldsymbol{\theta}^{n},\boldsymbol{\chi}^{n+1}+\boldsymbol{\chi}^{n}\right)_{L_2(\Omega)}$
  Since $\boldsymbol{w}$ belongs to H^1 in time, we can write

$$\begin{aligned}
&\sum_{n=0}^{m-1}\left(\boldsymbol{\theta}^{n+1}-\boldsymbol{\theta}^{n},\boldsymbol{\chi}^{n+1}+\boldsymbol{\chi}^{n}\right)_{L_2(\Omega)}\\
&=\sum_{n=0}^{m-1}\int_{t_n}^{t_{n+1}}\left(\dot{\boldsymbol{\theta}}(t'),\boldsymbol{\chi}^{n+1}+\boldsymbol{\chi}^{n}\right)_{L_2(\Omega)}dt'\\
&\leq\sum_{n=0}^{m-1}\int_{t_n}^{t_{n+1}}\left\|\dot{\boldsymbol{\theta}}(t')\right\|_{L_2(\Omega)}\left\|\boldsymbol{\chi}^{n+1}+\boldsymbol{\chi}^{n}\right\|_{L_2(\Omega)}dt'
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{\epsilon_a}{2}\int_0^{t_m}\left\|\dot{\boldsymbol{\theta}}(t')\right\|^2_{L_2(\Omega)}dt' + \frac{\Delta t}{2\epsilon_a}\sum_{n=0}^{m-1}\left\|\boldsymbol{\chi}^{n+1}+\boldsymbol{\chi}^n\right\|^2_{L_2(\Omega)}\\
&\le \frac{\epsilon_a}{2}\left\|\dot{\boldsymbol{\theta}}\right\|^2_{L_2(0,T;L_2(\Omega))} + \frac{\Delta t}{2\epsilon_a}4N\max_{0\le n\le N}\left\|\boldsymbol{\chi}^n\right\|^2_{L_2(\Omega)}\\
&= \frac{\epsilon_a}{2}\left\|\dot{\boldsymbol{\theta}}\right\|^2_{L_2(0,T;L_2(\Omega))} + \frac{2T}{\epsilon_a}\max_{0\le n\le N}\left\|\boldsymbol{\chi}^n\right\|^2_{L_2(\Omega)}\\
&\le \frac{\epsilon_a}{2}O(h^{2r}) + \frac{2T}{\epsilon_a}\max_{0\le n\le N}\left\|\boldsymbol{\chi}^n\right\|^2_{L_2(\Omega)}
\end{aligned}
$$

by Cauchy-Schwarz inequalities, Young's inequality and (4.1.32) for any positive ϵ_a where $r = \min(k+1, s)$.

- $\Delta t \sum_{n=0}^{m-1} a\left(\boldsymbol{e}^{n+1}+\boldsymbol{e}^n, \boldsymbol{\chi}^{n+1}+\boldsymbol{\chi}^n\right)$
  We follows the simple fact:

$$
a\left(\boldsymbol{e}^{n+1}+\boldsymbol{e}^n, \boldsymbol{v}\right) = \left(-\nabla\cdot\underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{e}^{n+1}+\boldsymbol{e}^n), \boldsymbol{v}\right)_{L_2(\Omega)}
$$

by integration by parts. Hence using Cauchy-Schwarz inequalities and Young's inequality, we can obtain

$$
\begin{aligned}
&\Delta t\sum_{n=0}^{m-1}\left(-\nabla\cdot\underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{e}^{n+1}+\boldsymbol{e}^n), \boldsymbol{\chi}^{n+1}+\boldsymbol{\chi}^n\right)_{L_2(\Omega)}\\
&\le \Delta t\sum_{n=0}^{m-1}\left\|\nabla\cdot\underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{e}^{n+1}+\boldsymbol{e}^n)\right\|_{L_2(\Omega)}\left\|\boldsymbol{\chi}^{n+1}+\boldsymbol{\chi}^n\right\|_{L_2(\Omega)}\\
&\le \Delta t\sum_{n=0}^{m-1}\frac{\epsilon_b}{2}\left\|\nabla\cdot\underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{e}^{n+1}+\boldsymbol{e}^n)\right\|^2_{L_2(\Omega)} + \Delta t\sum_{n=0}^{m-1}\frac{1}{2\epsilon_b}\left\|\boldsymbol{\chi}^{n+1}+\boldsymbol{\chi}^n\right\|^2_{L_2(\Omega)}\\
&\le \Delta t\sum_{n=0}^{N-1}\frac{\epsilon_b}{2}\left\|\nabla\cdot\underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{e}^{n+1}+\boldsymbol{e}^n)\right\|^2_{L_2(\Omega)} + \frac{2T}{\epsilon_b}\max_{0\le n\le N}\left\|\boldsymbol{\chi}^n\right\|^2_{L_2(\Omega)}
\end{aligned}
$$

for any positive ϵ_b, since $m\Delta t \le N\Delta t = T$. Recall the linear interpolation error then we have $\boldsymbol{e}^n = \boldsymbol{O}(\Delta t^2)$. Note that each component of $\boldsymbol{O}(\Delta t^2)$ is an order of Δt^2 and our domain is bounded and so L_2 norm of $\boldsymbol{O}(\Delta t^2)$ with respect to the spatial domain is bounded by $C\Delta t^2$ for some positive C. Therefore, we have

$$
\begin{aligned}
&\Delta t\sum_{n=0}^{m-1}\left(-\nabla\cdot\underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{e}^{n+1}+\boldsymbol{e}^n), \boldsymbol{\chi}^{n+1}+\boldsymbol{\chi}^n\right)_{L_2(\Omega)}\\
&\le \Delta t\sum_{n=0}^{N-1}\frac{\epsilon_b}{2}O(\Delta t^4) + \frac{2T}{\epsilon_b}\max_{0\le n\le N}\left\|\boldsymbol{\chi}^n\right\|^2_{L_2(\Omega)}\\
&= \epsilon_b T O(\Delta t^4) + \frac{2T}{\epsilon_b}\max_{0\le n\le N}\left\|\boldsymbol{\chi}^n\right\|^2_{L_2(\Omega)}
\end{aligned}
$$

- $\Delta t \sum_{n=0}^{m-1} \left(\boldsymbol{\mathcal{E}}^n, \boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^n\right)_{L_2(\Omega)}$
  Note that (5.3.21) implies $\|\boldsymbol{\mathcal{E}}^n\|_{L_2(\Omega)} = O(\Delta t^{2-\alpha})$ for any $n = 0, \ldots, N-1$. In this manner, Cauchy-Schwarz inequalities and Young's inequality yield

$$
\begin{aligned}
&\Delta t \sum_{n=0}^{m-1} \left(\boldsymbol{\mathcal{E}}^n, \boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^n\right)_{L_2(\Omega)} \\
\leq &\Delta t \sum_{n=0}^{m-1} \|\boldsymbol{\mathcal{E}}^n\|_{L_2(\Omega)} \left\|\boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^n\right\|_{L_2(\Omega)} \\
\leq &\Delta t \sum_{n=0}^{m-1} \frac{\epsilon_c}{2} \|\boldsymbol{\mathcal{E}}^n\|_{L_2(\Omega)}^2 + \Delta t \sum_{n=0}^{m-1} \frac{1}{2\epsilon_c} \left\|\boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^n\right\|_{L_2(\Omega)}^2 \\
\leq &\Delta t \sum_{n=0}^{N-1} \frac{\epsilon_c}{2} \|\boldsymbol{\mathcal{E}}^n\|_{L_2(\Omega)}^2 + \Delta t \sum_{n=0}^{N-1} \frac{2}{\epsilon_c} \max_{0\leq n\leq N} \|\boldsymbol{\chi}^n\|_{L_2(\Omega)}^2 \\
\leq &C\frac{T\epsilon_c}{2} O(\Delta t^{4-2\alpha}) + \frac{2T}{\epsilon_c} \max_{0\leq n\leq N} \|\boldsymbol{\chi}^n\|_{L_2(\Omega)}^2
\end{aligned}
$$

  for any positive ϵ_c.

Combining the above results then (5.3.25) has a bound as

$$
\begin{aligned}
&2\rho \|\boldsymbol{\chi}^m\|_{L_2(\Omega)}^2 + \frac{\Delta t^{2-\alpha}}{\Gamma(3-\alpha)} \sum_{n=0}^{m-1} \left\|\boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^n\right\|_V^2 \\
\leq &\rho\epsilon_a O(h^{2r}) + \epsilon_b O(\Delta t^4) + \epsilon_c O(\Delta t^{4-2\alpha}) + \frac{4\rho T}{\epsilon_a} \max_{0\leq n\leq N} \|\boldsymbol{\chi}^n\|_{L_2(\Omega)}^2 \\
&+ \frac{2T}{\epsilon_b} \max_{0\leq n\leq N} \|\boldsymbol{\chi}^n\|_{L_2(\Omega)}^2 + \frac{4\rho T}{\epsilon_c} \max_{0\leq n\leq N} \|\boldsymbol{\chi}^n\|_{L_2(\Omega)}^2 \\
&- \frac{\Delta t^{2-\alpha}}{\Gamma(3-\alpha)} \sum_{n=0}^{m-1} a\left(\sum_{i=0}^{n} B_{n+1,i}\boldsymbol{\chi}^i + \sum_{i=0}^{n-1} B_{n,i}\boldsymbol{\chi}^i, \boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^n\right).
\end{aligned}
\tag{5.3.26}
$$

As seen in the proof of Theorem 5.4, using mathematical induction shows the bound of the last term of (5.3.26). As proved before, coupled with $\left\|\boldsymbol{\chi}^0\right\|_V = 0$, we can obtain

$$
\begin{aligned}
&2\rho \|\boldsymbol{\chi}^m\|_{L_2(\Omega)}^2 + \frac{\Delta t^{2-\alpha}}{2\Gamma(3-\alpha)} \sum_{n=0}^{m-1} \left\|\boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^n\right\|_V^2 \\
\leq &C\left(\rho\epsilon_a O(h^{2r}) + \epsilon_b O(\Delta t^4) + \epsilon_c O(\Delta t^{4-2\alpha}) + \left(\frac{4\rho T}{\epsilon_a} + \frac{2T}{\epsilon_b} + \frac{4\rho T}{\epsilon_c}\right) \max_{0\leq n\leq N} \|\boldsymbol{\chi}^n\|_{L_2(\Omega)}^2\right)
\end{aligned}
\tag{5.3.27}
$$

for some positive C. Whence we consider maximum on (5.3.27), we have

$$
2\rho \max_{0\leq n\leq N} \|\boldsymbol{\chi}^n\|_{L_2(\Omega)}^2 + \frac{\Delta t^{2-\alpha}}{2\Gamma(3-\alpha)} \sum_{n=0}^{N-1} \left\|\boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^n\right\|_V^2
$$

$$\leq 2C\Bigg(\rho\epsilon_a O(h^{2r}) + \epsilon_b O(\Delta t^4) + \epsilon_c O(\Delta t^{4-2\alpha}) + \left(\frac{4\rho T}{\epsilon_a} + \frac{2T}{\epsilon_b} + \frac{4\rho T}{\epsilon_c}\right) \max_{0\leq n\leq N} \|\boldsymbol{\chi}^n\|^2_{L_2(\Omega)}\Bigg),$$

therefore choosing $\epsilon_a, \epsilon_c = 32CT$ and $\epsilon_b = 8CT/\rho$ implies

$$\begin{aligned}&\rho \max_{0\leq n\leq N} \|\boldsymbol{\chi}^n\|^2_{L_2(\Omega)} + \frac{\Delta t^{2-\alpha}}{2\Gamma(3-\alpha)} \sum_{n=0}^{N-1} \left\|\boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^n\right\|^2_V \\ \leq& O(h^{2r}) + O(\Delta t^4) + O(\Delta t^{4-2\alpha}).\end{aligned}$$

As a consequence, we can conclude that

$$\max_{0\leq n\leq N} \|\boldsymbol{\chi}^n\|_{L_2(\Omega)} + \left(\Delta t^{2-\alpha} \sum_{n=0}^{N-1} \left\|\boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^n\right\|^2_V\right)^{1/2} = O(h^r + \Delta t^{2-\alpha}).$$

Besides, with higher regularity of the solution in time such that (5.3.22) is fulfilled, then we could take second order accuracy in time. To be specific, when we suppose $\boldsymbol{w}(0) = \dot{\boldsymbol{w}}(0) = \mathbf{0}$ or $\boldsymbol{w} \in \ker(\mathcal{D})$, (5.3.22) implies $\|\boldsymbol{\mathcal{E}}^n\|_{L_2(\Omega)} = O(\Delta t^2)$. Therefore, $O(\Delta t^{2-\alpha})$ can be replaced by $O(\Delta t^2)$ so that we have

$$\max_{0\leq n\leq N} \|\boldsymbol{\chi}^n\|_{L_2(\Omega)} + \left(\Delta t^{2-\alpha} \sum_{n=0}^{N-1} \left\|\boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^n\right\|^2_V\right)^{1/2} = O(h^r + \Delta t^2).$$

□

Next, we state that our discrete solution has optimal energy error as well as L_2 error with respect to the space but its numerical error regarding time is suboptimal. We prove it by Lemma 5.2, and we can also observe second order error in time with further suppositions.

Theorem 5.5. *Assume that*

$$\boldsymbol{w} \in C^2(0, T; [H^s(\Omega)]^d) \cap W^1_\infty(0, T; \boldsymbol{V})$$

and $(\boldsymbol{W}^n_h)^N_{n=0}$ *is a sequence of the fully discrete solution of* **(T)**. *Then we can observe optimal L_2 error as well as energy error estimates with fixed order accuracy in time. Therefore,*

$$\begin{aligned}\max_{0\leq n\leq N} \|\boldsymbol{w}^n - \boldsymbol{W}^n_h\|_{L_2(\Omega)} &= O(h^r + \Delta t^{2-\alpha}),\\ \max_{0\leq n\leq N} \|\boldsymbol{w}^n - \boldsymbol{W}^n_h\|_V &= O(h^{r-1} + \Delta t^{2-\alpha}),\end{aligned}$$

where $r = \min(s, k+1)$.

Proof. For any $n = 0, \ldots, N$, using triangular inequality, we have

$$\begin{aligned}\|\boldsymbol{w}^n - \boldsymbol{W}_h^n\|_{L_2(\Omega)} &= \|\boldsymbol{\theta}^n - \boldsymbol{\chi}^n\|_{L_2(\Omega)} \\ &\leq \|\boldsymbol{\theta}^n\|_{L_2(\Omega)} + \|\boldsymbol{\chi}^n\|_{L_2(\Omega)} .\end{aligned}$$

By (4.1.32) and Lemma 5.2, it is concluded that

$$\|\boldsymbol{w}^n - \boldsymbol{W}_h^n\|_{L_2(\Omega)} = O(h^r) + O(h^r + \Delta t^{2-\alpha}) = O(h^r + \Delta t^{2-\alpha}),$$

and so on account of arbitrary n,

$$\max_{0\leq n\leq N} \|\boldsymbol{w}^n - \boldsymbol{W}_h^n\|_{L_2(\Omega)} = O(h^r + \Delta t^{2-\alpha}).$$

In this manner, we can obtain

$$\|\boldsymbol{w}^n - \boldsymbol{W}_h^n\|_V \leq \|\boldsymbol{\theta}^n\|_V + \|\boldsymbol{\chi}^n\|_V ,$$

so that (4.1.31) and (1.4.11) lead us to have

$$\|\boldsymbol{w}^n - \boldsymbol{W}_h^n\|_V \leq \|\boldsymbol{\theta}^n\|_V + Ch^{-1} \|\boldsymbol{\chi}^n\|_{L_2(\Omega)} = O(h^{r-1} + \Delta t^{2-\alpha}).$$

Thus, we have

$$\max_{0\leq n\leq N} \|\boldsymbol{w}^n - \boldsymbol{W}_h^n\|_V = O(h^{r-1} + \Delta t^{2-\alpha}).$$

□

Corollary 5.1. *Under the same conditions in* Theorem 5.5*, suppose* (5.3.22) *holds. Then we can obtain optimal results of Crank-Nicolson scheme i.e.,*

$$\begin{aligned}\max_{0\leq n\leq N} \|\boldsymbol{w}^n - \boldsymbol{W}_h^n\|_{L_2(\Omega)} &= O(h^r + \Delta t^2), \\ \max_{0\leq n\leq N} \|\boldsymbol{w}^n - \boldsymbol{W}_h^n\|_V &= O(h^{r-1} + \Delta t^2),\end{aligned}$$

where $r = \min(s, k+1)$*.*

Proof. As shown in Theorem 5.5, triangular inequalities combined with (4.1.31), (4.1.32) and Lemma 5.2 with higher regularity complete the proof. □

Note that $\boldsymbol{w}(0) = \dot{\boldsymbol{w}}(0) = \mathbf{0}$ or $\boldsymbol{w} \in \ker(\mathcal{D})$ is a sufficient condition for (5.3.22).

We can solve the fractional order viscoelastic problem in a weak way, viz. using CGFEM and Crank-Nicolson method. The weak solution exhibits optimal spatial error estimates but generally lose a full advantage of the second order scheme. At least H^3 smoothness in time is the necessary condition for second order accuracy in time so that we may need to assume either $\boldsymbol{w}(0) = \dot{\boldsymbol{w}}(0) = \mathbf{0}$ or $\boldsymbol{w} \in \ker(\mathcal{D})$ for higher regularity.

5.3.2 DGFEM for Fractional Order Viscoelastic Problem

Turning back to (5.3.1)-(5.3.4), we now consider DG approximation, especially using SIPG, to the fractional order viscoelastic problem. In the first place, let us recall a broken Sobolev space $[H^s(\mathcal{E}_h)]^d$. Then we can derive a weak formulation in DG as we done before in Chapter 4.2. The weak form is given as $\forall \boldsymbol{v} \in [H^s(\mathcal{E}_h)]^d$,

$$(\rho\dot{\boldsymbol{w}}(t), \boldsymbol{v})_{L_2(\Omega)} + a_{-1}\left({}_0I_t^{1-\alpha}\boldsymbol{w}(t), \boldsymbol{v}\right) + J_0^{\alpha_0,\beta_0}(\boldsymbol{w}(t), \boldsymbol{v}) = F(t; \boldsymbol{v}),\ \forall t \in (0, T], \quad (5.3.28)$$

$$a_{-1}(\boldsymbol{w}(0), \boldsymbol{v}) = a_{-1}(\boldsymbol{w}_0, \boldsymbol{v}), \quad (5.3.29)$$

where $s > 3/2$ and

$$F(t; \boldsymbol{v}) = (\boldsymbol{f}(t), \boldsymbol{v})_{L_2(\Omega)} + \sum_{e\subset\Gamma_N} (\boldsymbol{g}_N(t), \boldsymbol{v})_{L_2(e)}.$$

If a strong solution and its fractional integration, which belong to $[H^s(\mathcal{E}_h)]^d$, are continuous on Ω for any $t \in [0, T]$, the strong solution satisfies (5.3.28) and (5.3.29) in a similar way with non-fractional order problems.

Let us consider finite dimensional spaces then we can formulate the semidiscrete form as following.

Remark Recall Theorems 4.5 and 4.6. For a sufficiently large α_0 and $\beta_0(d-1) \geq 1$, the DG bilinear form of SIPG is coercive and continuous on $[\mathcal{D}_k(\mathcal{E}_h)]^d$.

(U) Find $\boldsymbol{w}_h(t) \in [\mathcal{D}_k(\mathcal{E}_h)]^d$ for all $t \in [0, T]$ such that satisfying $\forall \boldsymbol{v} \in [\mathcal{D}_k(\mathcal{E}_h)]^d$,

$$(\rho\dot{\boldsymbol{w}}_h(t), \boldsymbol{v})_{L_2(\Omega)} + a_{-1}\left({}_0I_t^{1-\alpha}\boldsymbol{w}_h(t), \boldsymbol{v}\right) + J_0^{\alpha_0,\beta_0}(\boldsymbol{w}_h(t), \boldsymbol{v}) = F(t; \boldsymbol{v}), \quad (5.3.30)$$

$$a_{-1}(\boldsymbol{w}_h(0), \boldsymbol{v}) = a_{-1}(\boldsymbol{w}_0, \boldsymbol{v}). \quad (5.3.31)$$

Theorem 5.6 (Stability Analysis: semidiscrete formula)**.** *Suppose $\boldsymbol{w}_h$ is a solution of* (5.3.30)-(5.3.31) *and*

$$\boldsymbol{g}_N \in L_2(0, T; [L_2(\Gamma_N)]^d),$$
$$\boldsymbol{f} \in L_2(0, T; [L_2(\Omega)]^d),$$
$$\boldsymbol{w}_0 \in [H^1(\Omega)]^d \cap [H^s(\mathcal{E}_h)]^d.$$

If we assume $\beta_0(d-1) \geq 1$ and sufficiently large α_0, there exits a positive constant C such that

$$\begin{aligned}&\rho\,\|\boldsymbol{w}_h\|^2_{L_\infty(0,T;L_2(\Omega))} + \int_0^T J_0^{\alpha_0,\beta_0}(\boldsymbol{w}_h(t), \boldsymbol{w}_h(t))dt \\ \leq& C\left(\rho\,\|\boldsymbol{w}_0\|^2_{\mathcal{V}} + \|\boldsymbol{f}\|^2_{L_2(0,T;L_2(\Omega))} + h^{-1}\,\|\boldsymbol{g}_N\|^2_{L_2(0,T;L_2(\Gamma_N))}\right).\end{aligned}$$

Proof. Let $\boldsymbol{v} = \boldsymbol{w}_h(t)$ for $t \in (0, T]$. Put it into (5.3.30) then we have

$$\frac{\rho}{2}\frac{d}{dt}\|\boldsymbol{w}_h(t)\|^2_{L_2(\Omega)} + a_{-1}\left({}_0I_t^{1-\alpha}\boldsymbol{w}_h(t), \boldsymbol{w}_h(t)\right) + J_0^{\alpha_0,\beta_0}(\boldsymbol{w}_h(t), \boldsymbol{w}_h(t)) = F(\boldsymbol{w}_h(t)). \quad (5.3.32)$$

Taking into account the second term of the left hand side of (5.3.32), the definition of the fractional integral gives

$$a_{-1}\left({}_0I_t^{1-\alpha}\boldsymbol{w}_h(t),\boldsymbol{w}_h(t)\right)=\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-t')^{-\alpha}a_{-1}\left(\boldsymbol{w}_h(t'),\boldsymbol{w}_h(t)\right)dt', \qquad (5.3.33)$$

by Leibniz integral rule. By substitution of (5.3.33) into (5.3.32), integrating over time yields

$$\begin{aligned}\frac{\rho}{2}(\|\boldsymbol{w}_h(\tau)\|^2_{L_2(\Omega)}-\|\boldsymbol{w}_h(0)\|^2_{L_2(\Omega)})+\frac{1}{\Gamma(1-\alpha)}\int_0^\tau\int_0^t (t-t')^{-\alpha}a_{-1}\left(\boldsymbol{w}_h(t'),\boldsymbol{w}_h(t)\right)dt'dt\\ +\int_0^\tau J_0^{\alpha_0,\beta_0}(\boldsymbol{w}_h(t),\boldsymbol{w}_h(t))dt=\int_0^\tau F(\boldsymbol{w}_h(t))dt, \qquad (5.3.34)\end{aligned}$$

for $0<\tau\le T$. Consider

$$\frac{1}{\Gamma(1-\alpha)}\int_0^\tau\int_0^t (t-t')^{-\alpha}a_{-1}\left(\boldsymbol{w}_h(t'),\boldsymbol{w}_h(t)\right)dt'dt.$$

By (5.3.8), we can derive

$$\frac{1}{\Gamma(1-\alpha)}\int_0^\tau\int_0^t (t-t')^{-\alpha}a_{-1}\left(\boldsymbol{w}_h(t'),\boldsymbol{w}_h(t)\right)dt'dt\ge 0,$$

so that (5.3.34) can be written as

$$\begin{aligned}\frac{\rho}{2}\|\boldsymbol{w}_h(\tau)\|^2_{L_2(\Omega)}+\int_0^\tau J_0^{\alpha_0,\beta_0}(\boldsymbol{w}_h(t),\boldsymbol{w}_h(t))dt\le&\frac{\rho}{2}\|\boldsymbol{w}_h(0)\|^2_{L_2(\Omega)}+\int_0^\tau F(\boldsymbol{w}_h(t))dt\\ \le&\rho C\|\boldsymbol{w}_0\|^2_{\mathcal{V}}+\int_0^\tau F(\boldsymbol{w}_h(t))dt, \qquad (5.3.35)\end{aligned}$$

where C is a positive constant governed by piecewise Poincaré inequalities, and continuity and coercivity of SIPG, since, with (1.4.10),

$$\kappa\|\boldsymbol{w}_h(0)\|^2_{\mathcal{V}}\le a_{-1}(\boldsymbol{w}_h(0),\boldsymbol{w}_h(0))=a_{-1}(\boldsymbol{w}_0,\boldsymbol{w}_h(0))\le K\|\boldsymbol{w}_0\|_{\mathcal{V}}\|\boldsymbol{w}_h(0)\|_{\mathcal{V}}.$$

Then we shall show that the last term in (5.3.35) is bounded. Use of Cauchy-Schwarz inequalities and Young's inequality implies

$$\begin{aligned}\int_0^\tau(\boldsymbol{f}(t),\boldsymbol{w}_h(t))_{L_2(\Omega)}\le&\int_0^\tau\|\boldsymbol{f}(t)\|_{L_2(\Omega)}\|\boldsymbol{w}_h(t)\|_{L_2(\Omega)}dt\\ \le&\left(\int_0^\tau\|\boldsymbol{f}(t)\|^2_{L_2(\Omega)}\right)^{1/2}\left(\int_0^\tau\|\boldsymbol{w}_h(t)\|^2_{L_2(\Omega)}dt\right)^{1/2}\\ \le&\frac{\epsilon_a}{2}\|\boldsymbol{f}\|^2_{L_2(0,T;L_2(\Omega))}+\frac{1}{2\epsilon_a}\int_0^\tau\|\boldsymbol{w}_h(t)\|^2_{L_2(\Omega)}dt\\ \le&\frac{\epsilon_a}{2}\|\boldsymbol{f}\|^2_{L_2(0,T;L_2(\Omega))}+\frac{1}{2\epsilon_a}\int_0^\tau\|\boldsymbol{w}_h\|^2_{L_\infty(0,T;L_2(\Omega))}dt\end{aligned}$$

$$\leq \frac{\epsilon_a}{2} \|\boldsymbol{f}\|^2_{L_2(0,T;L_2(\Omega))} + \frac{T}{2\epsilon_a} \|\boldsymbol{w}_h\|^2_{L_\infty(0,T;L_2(\Omega))},$$

for any positive ϵ_a. On the other hand, for $e \subset \Gamma_N$, Cauchy-Schwarz inequality gives

$$(\boldsymbol{g}_N(t), \boldsymbol{w}_h(t))_{L_2(e)} \leq \|\boldsymbol{g}_N(t)\|_{L_2(e)} \|\boldsymbol{w}_h(t)\|_{L_2(e)},$$

hence Young's inequality and the inverse polynomial trace theorem yield

$$\begin{aligned}
&\int_0^\tau \sum_{e\subset\Gamma_N} (\boldsymbol{g}_N(t), \boldsymbol{w}_h(t))_{L_2(e)}\, dt \\
&\leq \int_0^\tau \sum_{e\subset\Gamma_N} \|\boldsymbol{g}_N(t)\|_{L_2(e)} \|\boldsymbol{w}_h(t)\|_{L_2(e)}\, dt \\
&\leq \int_0^\tau \frac{\epsilon_b}{2} \sum_{e\subset\Gamma_N} \|\boldsymbol{g}_N(t)\|^2_{L_2(e)} + \frac{1}{2\epsilon_b} \sum_{e\subset\Gamma_N} \|\boldsymbol{w}_h(t)\|^2_{L_2(e)}\, dt \\
&\leq \int_0^\tau \frac{\epsilon_b}{2} \sum_{e\subset\Gamma_N} \|\boldsymbol{g}_N(t)\|^2_{L_2(e)} + \frac{1}{2\epsilon_b} Ch^{-1} \sum_{E\in\mathcal{E}_h} \|\boldsymbol{w}_h(t)\|^2_{L_2(E)}\, dt \\
&= \frac{\epsilon_b}{2} \int_0^\tau \|\boldsymbol{g}_N(t)\|^2_{L_2(\Gamma_N)}\, dt + \frac{Ch^{-1}}{2\epsilon_b} \int_0^\tau \|\boldsymbol{w}_h(t)\|^2_{L_2(\Omega)}\, dt \\
&\leq \frac{\epsilon_b}{2} \|\boldsymbol{g}_N(t)\|^2_{L_2(0,T;L_2(\Gamma_N))} + \frac{Ch^{-1}}{2\epsilon_b} \int_0^T \|\boldsymbol{w}_h\|^2_{L_\infty(0,T;L_2(\Omega))}\, dt \\
&= \frac{\epsilon_b}{2} \|\boldsymbol{g}_N(t)\|^2_{L_2(0,T;L_2(\Gamma_N))} + \frac{Ch^{-1}T}{2\epsilon_b} \|\boldsymbol{w}_h\|^2_{L_\infty(0,T;L_2(\Omega))}
\end{aligned}$$

where C is a positive constant governed by Theorem 1.11 for any positive ϵ_b. Thus, we can obtain the bound

$$\begin{aligned}
\int_0^\tau F(\boldsymbol{w}_h(t))dt &= \int_0^\tau (\boldsymbol{f}(t), \boldsymbol{w}_h(t))_{L_2(\Omega)} + \sum_{e\subset\Gamma_N} (\boldsymbol{g}_N(t), \boldsymbol{w}_h(t))_{L_2(e)}\, dt \\
&\leq \frac{\epsilon_a}{2} \|\boldsymbol{f}\|^2_{L_2(0,T;L_2(\Omega))} + \frac{\epsilon_b}{2} \|\boldsymbol{g}_N\|^2_{L_2(0,T;L_2(\Gamma_N))} \\
&\quad + \left(\frac{T}{2\epsilon_a} + \frac{Ch^{-1}T}{2\epsilon_b}\right) \|\boldsymbol{w}_h\|^2_{L_\infty(0,T;L_2(\Omega))}. \qquad (5.3.36)
\end{aligned}$$

By substitution (5.3.36) into (5.3.35), we have

$$\begin{aligned}
&\frac{\rho}{2} \|\boldsymbol{w}_h(\tau)\|^2_{L_2(\Omega)} + \int_0^\tau J_0^{\alpha_0,\beta_0}(\boldsymbol{w}_h(t), \boldsymbol{w}_h(t))dt \\
&\leq \rho C \|\boldsymbol{w}_0\|^2_{\mathcal{V}} + \frac{\epsilon_a}{2} \|\boldsymbol{f}\|^2_{L_2(0,T;L_2(\Omega))} + \frac{\epsilon_b}{2} \|\boldsymbol{g}_N\|^2_{L_2(0,T;L_2(\Gamma_N))} \\
&\quad + \left(\frac{T}{2\epsilon_a} + \frac{Ch^{-1}T}{2\epsilon_b}\right) \|\boldsymbol{w}_h\|^2_{L_\infty(0,T;L_2(\Omega))},
\end{aligned}$$

since τ is arbitrary, taking into accounts supremum on the left hand side implies

$$
\begin{aligned}
&\frac{\rho}{2}\left\|\boldsymbol{w}_h\right\|^2_{L_\infty(0,T;L_2(\Omega))}+\int_0^T J_0^{\alpha_0,\beta_0}(\boldsymbol{w}_h(t),\boldsymbol{w}_h(t))dt\\
\leq&2\rho C\left\|\boldsymbol{w}_0\right\|^2_{\mathcal{V}}+\epsilon_a\left\|\boldsymbol{f}\right\|^2_{L_2(0,T;L_2(\Omega))}+\epsilon_b\left\|\boldsymbol{g}_N\right\|^2_{L_2(0,T;L_2(\Gamma_N))}\\
&+\left(\frac{T}{\epsilon_a}+\frac{Ch^{-1}T}{\epsilon_b}\right)\left\|\boldsymbol{w}_h\right\|^2_{L_\infty(0,T;L_2(\Omega))}.
\end{aligned}
$$

Consequently, the proper choice of ϵ_a and ϵ_b, for example $\epsilon_a=8T/\rho,\ \epsilon_b=8CTh^{-1}/\rho$, gives

$$
\begin{aligned}
&\frac{\rho}{4}\left\|\boldsymbol{w}_h\right\|^2_{L_\infty(0,T;L_2(\Omega))}+\int_0^T J_0^{\alpha_0,\beta_0}(\boldsymbol{w}_h(t),\boldsymbol{w}_h(t))dt\\
\leq&C\left(\rho\left\|\boldsymbol{w}_0\right\|^2_{\mathcal{V}}+\left\|\boldsymbol{f}\right\|^2_{L_2(0,T;L_2(\Omega))}+h^{-1}\left\|\boldsymbol{g}_N\right\|^2_{L_2(0,T;L_2(\Gamma_N))}\right)
\end{aligned}
$$

for some positive C that is independent of h and the solution. C is increasing in final time T but not exponentially due to avoiding use of Grönwall's inequality. □

In Theorem 5.6, the semidiscrete solution is bounded by data terms, which means the existence and uniqueness of the solution is shown. Here, h^{-1} is also observed in the bound because of the weakly imposing boundary condition, however it does not matter in a practical sense.

As seen in before, introducing SIPG DG elliptic operator $\boldsymbol{R}_{-1}$ leads us to use the approximation properties (4.2.22)-(4.2.24) so that we are able to derive error estimates. Let us define for $t\in[0,T]$

$$\boldsymbol{\theta}(t):=\boldsymbol{w}(t)-\boldsymbol{R}_{-1}\boldsymbol{w}(t),\qquad \boldsymbol{\chi}(t):=\boldsymbol{w}_h(t)-\boldsymbol{R}_{-1}\boldsymbol{w}(t).$$

Note that DG Galerkin orthogonality gives for any t

$$a_{-1}\left(\boldsymbol{\theta}(t),\boldsymbol{v}\right)=0,\ \forall\boldsymbol{v}\in[\mathcal{D}_k(\mathcal{E}_h)]^d,$$

and so it yields

$$
\begin{aligned}
a_{-1}\left({}_0I_t^{1-\alpha}\boldsymbol{\theta}(t),\boldsymbol{v}\right)=&\sum_{E\in\mathcal{E}_h}\int_E\underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}({}_0I_t^{1-\alpha}\boldsymbol{\theta}(t)):\underline{\boldsymbol{\varepsilon}}(\boldsymbol{v})\ dE\\
&-\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e\{\underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}({}_0I_t^{1-\alpha}\boldsymbol{\theta}(t))\cdot\boldsymbol{n}_e\}\cdot[\boldsymbol{v}]de\\
&-\sum_{e\subset\Gamma_h\cup\Gamma_D}\int_e\{\underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{v})\cdot\boldsymbol{n}_e\}\cdot[{}_0I_t^{1-\alpha}\boldsymbol{\theta}(t)]de+\boldsymbol{J}_0^{\alpha_0,\beta_0}({}_0I_t^{1-\alpha}\boldsymbol{\theta}(t),\boldsymbol{v})\\
=&\frac{1}{\Gamma(1-\alpha)}\int_0^t(t-t')^{-\alpha}\Bigg(\sum_{E\in\mathcal{E}_h}\int_E\underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{\theta}(t')):\underline{\boldsymbol{\varepsilon}}(\boldsymbol{v})\ dE
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{e\subset\Gamma_h\cup\Gamma_D} \int_e \{\underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{\theta}(t'))\cdot\boldsymbol{n}_e\}\cdot[\boldsymbol{v}]de \\
&- \sum_{e\subset\Gamma_h\cup\Gamma_D} \int_e \{\underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{v})\cdot\boldsymbol{n}_e\}\cdot[\boldsymbol{\theta}(t')]de + \boldsymbol{J}_0^{\alpha_0,\beta_0}(\boldsymbol{\theta}(t'),\boldsymbol{v})\Big)dt' \\
=&\frac{1}{\Gamma(1-\alpha)}\int_0^t (t-t')^{-\alpha}a_{-1}\left(\boldsymbol{\theta}(t'),\boldsymbol{v}\right)dt' \\
=&0.
\end{aligned}
$$

Lemma 5.3. *Let $\boldsymbol{w}\in H^1(0,T;[C^2(\Omega)]^d)\cap W^1_\infty(0,T;[H^s(\mathcal{E}_h)]^d)$ with convex Ω. When we suppose α_0 is sufficiently large and $\beta_0(d-1)\geq 1$, we have the following convergence order such that*

$$
\|\boldsymbol{\chi}\|_{L_\infty(0,T;L_2(\Omega))} + \left(\int_0^T J_0^{\alpha_0,\beta_0}(\boldsymbol{\chi}(t),\boldsymbol{\chi}(t))dt\right)^{1/2} = O(h^r)
$$

where $r=\min(s,k+1)$.

Proof. Consider subtraction of (5.3.30) from (5.3.28). It gives for $t\in(0,T]$, $\forall \boldsymbol{v}\in[\mathcal{D}_k(\mathcal{E}_h)]^d$

$$
\left(\rho\dot{\boldsymbol{\theta}}(t)-\dot{\boldsymbol{\chi}}(t),\boldsymbol{v}\right)_{L_2(\Omega)} + a_{-1}\left({}_0I_t^{1-\alpha}(\boldsymbol{\theta}-\boldsymbol{\chi})(t),\boldsymbol{v}\right) + J_0^{\alpha_0,\beta_0}(\boldsymbol{\theta}(t)-\boldsymbol{\chi}(t),\boldsymbol{v}) = 0.
$$

DG Galerkin orthogonality implies

$$
\begin{aligned}
&(\rho\dot{\boldsymbol{\chi}}(t),\boldsymbol{v})_{L_2(\Omega)} + a_{-1}\left({}_0I_t^{1-\alpha}\boldsymbol{\chi}(t),\boldsymbol{v}\right) + J_0^{\alpha_0,\beta_0}(\boldsymbol{\chi}(t),\boldsymbol{v}) \\
&= \left(\rho\dot{\boldsymbol{\theta}}(t),\boldsymbol{v}\right)_{L_2(\Omega)} + J_0^{\alpha_0,\beta_0}(\boldsymbol{\theta}(t),\boldsymbol{v}),
\end{aligned}
$$

and, since $[\boldsymbol{\theta}(t)]=0$ on $\Gamma_h\cup\Gamma_D$ for any t by the continuity of $\boldsymbol{w}$ and the homogeneous Dirichlet boundary condition,

$$
(\rho\dot{\boldsymbol{\chi}}(t),\boldsymbol{v})_{L_2(\Omega)} + a_{-1}\left({}_0I_t^{1-\alpha}\boldsymbol{\chi}(t),\boldsymbol{v}\right) + J_0^{\alpha_0,\beta_0}(\boldsymbol{\chi}(t),\boldsymbol{v}) = \left(\rho\dot{\boldsymbol{\theta}}(t),\boldsymbol{v}\right)_{L_2(\Omega)}. \tag{5.3.37}
$$

Put $\boldsymbol{v}=\boldsymbol{\chi}(t)$ into (5.3.37) then integrating with respect to time leads us to have

$$
\begin{aligned}
&\frac{\rho}{2}\|\boldsymbol{\chi}(\tau)\|^2_{L_2(\Omega)} + \frac{1}{\Gamma(1-\alpha)}\int_0^\tau\int_0^t (t-t')^{-\alpha}a_{-1}\left(\boldsymbol{\chi}(t'),\boldsymbol{\chi}(t)\right)dt' \\
&\quad + \int_0^\tau J_0^{\alpha_0,\beta_0}(\boldsymbol{\chi}(t),\boldsymbol{\chi}(t))dt = \frac{\rho}{2}\|\boldsymbol{\chi}(0)\|^2_{L_2(\Omega)} + \int_0^\tau \left(\rho\dot{\boldsymbol{\theta}}(t),\boldsymbol{\chi}(t)\right)_{L_2(\Omega)}dt,
\end{aligned}
$$

for $0<\tau\leq T$. By (5.3.8), we can obtain

$$
\frac{\rho}{2}\|\boldsymbol{\chi}(\tau)\|^2_{L_2(\Omega)} + \int_0^\tau J_0^{\alpha_0,\beta_0}(\boldsymbol{\chi}(t),\boldsymbol{\chi}(t))dt \leq \frac{\rho}{2}\|\boldsymbol{\chi}(0)\|^2_{L_2(\Omega)} + \int_0^\tau \left(\rho\dot{\boldsymbol{\theta}}(t),\boldsymbol{\chi}(t)\right)_{L_2(\Omega)}dt. \tag{5.3.38}
$$

It is easy to show that $\|\boldsymbol{\chi}(0)\|_{L_2(\Omega)} = 0$ by using (5.3.31) and a broken Sobolev analogue of Poincaré's inequality. For any $\boldsymbol{v} \in [\mathcal{D}_k(\mathcal{E}_h)]^d$,

$$0 = a_{-1}(\boldsymbol{w}_0 - \boldsymbol{w}_h(0), \boldsymbol{v}) = a_{-1}(\boldsymbol{\theta}(0) - \boldsymbol{\chi}(0), \boldsymbol{v}) = -a_{-1}(\boldsymbol{\chi}(0), \boldsymbol{v})$$

by DG Galerkin orthogonality and hence $\kappa \|\boldsymbol{\chi}(0)\|_{\mathcal{V}}^2 \leq 0$ where $\boldsymbol{v} = \boldsymbol{\chi}(0)$. Thus, $\|\boldsymbol{\chi}(0)\|_{\mathcal{V}} = 0$ with (1.4.10) implies that $\|\boldsymbol{\chi}(0)\|_{L_2(\Omega)} = 0$.

On the other hand, use of Cauchy-Schwarz inequality and Young's inequality makes

$$\begin{aligned}\int_0^\tau \left(\rho\dot{\boldsymbol{\theta}}(t), \boldsymbol{\chi}(t)\right)_{L_2(\Omega)} dt &\leq \int_0^\tau \left\|\rho\dot{\boldsymbol{\theta}}(t)\right\|_{L_2(\Omega)} \|\boldsymbol{\chi}(t)\|_{L_2(\Omega)}\, dt \\ &\leq \left(\int_0^\tau \left\|\rho\dot{\boldsymbol{\theta}}(t)\right\|_{L_2(\Omega)}^2 dt\right)^{1/2} \left(\int_0^\tau \|\boldsymbol{\chi}(t)\|_{L_2(\Omega)}^2\, dt\right)^{1/2} \\ &\leq \left\|\rho\dot{\boldsymbol{\theta}}\right\|_{L_2(0,T;L_2(\Omega))} \sqrt{T}\, \|\boldsymbol{\chi}\|_{L_\infty(0,T;L_2(\Omega))} \\ &\leq \frac{\epsilon}{2} \left\|\rho\dot{\boldsymbol{\theta}}\right\|_{L_2(0,T;L_2(\Omega))}^2 + \frac{T}{2\epsilon} \|\boldsymbol{\chi}\|_{L_\infty(0,T;L_2(\Omega))}^2\end{aligned}$$

for any positive ϵ. Therefore, the left hand side of (5.3.38) is bounded by

$$\begin{aligned}&\frac{\rho}{2} \|\boldsymbol{\chi}(\tau)\|_{L_2(\Omega)}^2 + \int_0^\tau J_0^{\alpha_0,\beta_0}(\boldsymbol{\chi}(t), \boldsymbol{\chi}(t))dt \\ \leq &\frac{\epsilon}{2} \left\|\rho\dot{\boldsymbol{\theta}}\right\|_{L_2(0,T;L_2(\Omega))}^2 + \frac{T}{2\epsilon} \|\boldsymbol{\chi}\|_{L_\infty(0,T;L_2(\Omega))}^2,\end{aligned} \tag{5.3.39}$$

and since τ is arbitrary

$$\begin{aligned}&\frac{\rho}{2} \|\boldsymbol{\chi}\|_{L_\infty(0,T;L_2(\Omega))}^2 + \int_0^T J_0^{\alpha_0,\beta_0}(\boldsymbol{\chi}(t), \boldsymbol{\chi}(t))dt \\ \leq &\epsilon \left\|\rho\dot{\boldsymbol{\theta}}\right\|_{L_2(0,T;L_2(\Omega))}^2 + \frac{T}{\epsilon} \|\boldsymbol{\chi}\|_{L_\infty(0,T;L_2(\Omega))}^2.\end{aligned} \tag{5.3.40}$$

Choosing $\epsilon = T\rho/4$ and applying (4.2.24) impose that

$$\frac{\rho}{4} \|\boldsymbol{\chi}\|_{L_\infty(0,T;L_2(\Omega))}^2 + \int_0^T J_0^{\alpha_0,\beta_0}(\boldsymbol{\chi}(t), \boldsymbol{\chi}(t))dt \leq Ch^{2r},$$

where $r = \min(s, k+1)$ and C is a non-exponentially increasing positive constant independent of h. Consequently, we have

$$\|\boldsymbol{\chi}\|_{L_\infty(0,T;L_2(\Omega))} + \left(\int_0^T J_0^{\alpha_0,\beta_0}(\boldsymbol{\chi}(t), \boldsymbol{\chi}(t))dt\right)^{1/2} = O(h^r).$$

□

Now, we can derive semidiscrete error estimates by using Lemma 5.3 and the approximation properties (4.2.22)-(4.2.24).

Theorem 5.7 (Error Analysis: semidiscrete formula)**.** *Suppose* $\boldsymbol{w}$ *satisfies* Lemma 5.3. *With large enough* α_0 *and* $\beta(d-1) \geq 1$*, optimal* L_2 *error estimates are given. In addition, we can also observe optimal DG energy norm of the error. Therefore,*

$$\|\boldsymbol{w} - \boldsymbol{w}_h\|_{L_\infty(0,T;L_2(\Omega))} = O(h^r), \text{ and } \|\boldsymbol{w} - \boldsymbol{w}_h\|_{L_2(0,T;\mathcal{V})} = O(h^{r-1}),$$

where $r = \min(s, k+1)$*. In addition, it holds* $\|\boldsymbol{w} - \boldsymbol{w}_h\|_{L_\infty(0,T;\mathcal{V})} = O(h^{r-1})$*.*

Proof. Clearly, triangular inequality gives

$$\begin{aligned}\|\boldsymbol{w} - \boldsymbol{w}_h\|_{L_\infty(0,T;L_2(\Omega))} &= \|\boldsymbol{w} - \boldsymbol{R}_{-1}\boldsymbol{w} - (\boldsymbol{w}_h - \boldsymbol{R}_{-1}\boldsymbol{w})\|_{L_\infty(0,T;L_2(\Omega))} \\ &= \|\boldsymbol{\theta} - \boldsymbol{\chi}\|_{L_\infty(0,T;L_2(\Omega))} \\ &\leq \|\boldsymbol{\theta}\|_{L_\infty(0,T;L_2(\Omega))} + \|\boldsymbol{\chi}\|_{L_\infty(0,T;L_2(\Omega))}.\end{aligned}$$

By (4.2.23), (4.2.24) and Lemma 5.3, it is concluded that

$$\|\boldsymbol{w} - \boldsymbol{w}_h\|_{L_\infty(0,T;L_2(\Omega))} \leq Ch^r,$$

where C is a positive constant independent of h.

Whereas L_2 error estimates with respect to the spatial domain are seen directly by triangular inequalities and Lemma 5.3, DG energy estimates are not clear to show in the same way. In order to show that, it is necessary to introduce the vector-valued analogue of inverse inequality (1.4.11) and (4.2.5). By the definition of DG energy norm, for any $\boldsymbol{v} \in [\mathcal{D}_k(\mathcal{E}_h)]^d$,

$$\begin{aligned}\|\boldsymbol{v}\|_{\mathcal{V}}^2 &= \sum_{E\in\mathcal{E}_h} \int_E \underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{v}) : \underline{\boldsymbol{\varepsilon}}(\boldsymbol{v}) dE + J_0^{\alpha_0,\beta_0}(\boldsymbol{v},\boldsymbol{v}) \\ &\leq C \sum_{E\in\mathcal{E}_h} |\boldsymbol{v}|_{H^1(E)}^2 + J_0^{\alpha_0,\beta_0}(\boldsymbol{v},\boldsymbol{v}) \qquad \text{by (4.2.5) for some positive } C, \\ &\leq Ch^{-2}\|\boldsymbol{v}\|_{L_2(\Omega)}^2 + J_0^{\alpha_0,\beta_0}(\boldsymbol{v},\boldsymbol{v}) \qquad \text{by (1.4.11)}.\end{aligned}$$

Since $\boldsymbol{\chi}(t) \in [\mathcal{D}_k(\mathcal{E}_h)]^d$ for any $t \in [0,T]$,

$$\|\boldsymbol{\chi}(t)\|_{\mathcal{V}}^2 \leq Ch^{-2}\|\boldsymbol{\chi}(t)\|_{L_2(\Omega)}^2 + J_0^{\alpha_0,\beta_0}(\boldsymbol{\chi}(t),\boldsymbol{\chi}(t)),$$

hence we have

$$\begin{aligned}&\|\boldsymbol{w} - \boldsymbol{w}_h\|_{L_2(0,T;\mathcal{V})}^2 \\ =&\|\boldsymbol{\theta} - \boldsymbol{\chi}\|_{L_2(0,T;\mathcal{V})}^2 \\ \leq&2\|\boldsymbol{\theta}\|_{L_2(0,T;\mathcal{V})}^2 + 2\|\boldsymbol{\chi}\|_{L_2(0,T;\mathcal{V})}^2 \\ \leq&2\|\boldsymbol{\theta}\|_{L_2(0,T;\mathcal{V})}^2 + 2\left(Ch^{-2}\|\boldsymbol{\chi}\|_{L_2(0,T;L_2(\Omega))}^2 + \int_0^T J_0^{\alpha_0,\beta_0}(\boldsymbol{\chi}(t),\boldsymbol{\chi}(t))dt\right) \\ \leq&C(h^{2(r-1)} + h^{2(r-1)} + h^{2r}),\end{aligned}$$

for some positive C by (4.2.22) and Lemma 5.3 with Sobolev embedding theorem. As a consequence we can obtain

$$\|\boldsymbol{w}-\boldsymbol{w}_h\|_{L_2(0,T;\mathcal{V})} \leq Ch^{r-1}.$$

Furthermore, $\|\boldsymbol{w}(t)-\boldsymbol{w}_h(t)\|_{\mathcal{V}}$ is, indeed, uniformly bounded in time so that we can conclude that $\|\boldsymbol{w}-\boldsymbol{w}_h\|_{L_\infty(0,T;\mathcal{V})} \leq Ch^{r-1}$. □

Next, we shall consider time discretisation. To derive a fully discrete formulation, we use Crank-Nicolson method as well as numerical approach of fractional order integration by linear interpolation. Our fully discrete problem is given as

(U) Find $\boldsymbol{W}_h^n \in [\mathcal{D}_k(\mathcal{E}_h)]^d$ for $n=0,\dots,N$ such that satisfying for any $\boldsymbol{v} \in [\mathcal{D}_k(\mathcal{E}_h)]^d$,

$$\begin{aligned}&\left(\rho\frac{\boldsymbol{W}_h^{n+1}-\boldsymbol{W}_h^n}{\Delta t},\boldsymbol{v}\right)_{L_2(\Omega)} + a_{-1}\left(\frac{\boldsymbol{q}_{n+1}(\boldsymbol{W}_h)+\boldsymbol{q}_n(\boldsymbol{W}_h)}{2},\boldsymbol{v}\right)\\ &+ J_0^{\alpha_0,\beta_0}\left(\frac{\boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n}{2},\boldsymbol{v}\right) = \bar{F}^n(\boldsymbol{v}),\end{aligned} \tag{5.3.41}$$

for $n=0,\dots,N-1$, and

$$a_{-1}\left(\boldsymbol{W}_h^0,\boldsymbol{v}\right) = a_{-1}\left(\boldsymbol{w}_0,\boldsymbol{v}\right). \tag{5.3.42}$$

In the fully discrete solution, the fractional order integration is replaced by Theorem 5.1 and so we now have to introduce a discrete case of (5.3.8) in order to analyse a discrete stability bound and error bounds.

Theorem 5.8 (Stability Analysis: fully discrete formula)**.** *Let $\boldsymbol{W}_h^n$ be a fully discrete solution to* **(U)** *for $n=0,\dots,N$. Suppose data terms are sufficiently smooth and penalty parameters in SIPG are large enough. Then there exists a positive constant C such that*

$$\begin{aligned}&\max_{0\leq n\leq N}\|\boldsymbol{W}_h^n\|_{L_2(\Omega)}^2 + \frac{\Delta t^{2-\alpha}}{\Gamma(3-\alpha)}\sum_{n=0}^{N-1}\|\boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n\|_{\mathcal{V}}^2\\ &+\Delta t\sum_{n=0}^{N-1}J_0^{\alpha_0,\beta_0}(\boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n,\boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n)\\ &\leq C\left(\|\boldsymbol{w}_0\|_{\mathcal{V}}^2+\Delta t\sum_{n=0}^{N}\|\boldsymbol{f}(t_n)\|_{L_2(\Omega)}^2+\Delta t\sum_{n=0}^{N}h^{-1}\|\boldsymbol{g}_N(t_n)\|_{L_2(\Gamma_N)}^2\right)\\ &\leq C\left(\|\boldsymbol{w}_0\|_{\mathcal{V}}^2+\|\boldsymbol{f}\|_{L_\infty(0,T;L_2(\Omega))}^2+h^{-1}\|\boldsymbol{g}_N\|_{L_\infty(0,T;L_2(\Gamma_N))}^2\right).\end{aligned}$$

Proof. Let $1\leq m\leq N$ in $\mathbb{N}$. When we choose $\boldsymbol{v}=2\Delta t(\boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n)$ in (5.3.41), summing from $n=0$ to $n=m-1$ yields

$$2\rho\left(\|\boldsymbol{W}_h^m\|_{L_2(\Omega)}^2-\|\boldsymbol{W}_h^0\|_{L_2(\Omega)}^2\right)+\Delta t\sum_{n=0}^{m-1}a_{-1}\left(\boldsymbol{q}_{n+1}(\boldsymbol{W}_h)+\boldsymbol{q}_n(\boldsymbol{W}_h),\boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n\right)$$

$$+\Delta t \sum_{n=0}^{m-1} J_0^{\alpha_0,\beta_0}(\boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n, \boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n)$$

$$=\Delta t \sum_{n=0}^{m-1} \left(\boldsymbol{f}(t_{n+1}) + \boldsymbol{f}(t_n), \boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n\right)_{L_2(\Omega)}$$

$$+ \Delta t \sum_{n=0}^{m-1} \left(\boldsymbol{g}_N(t_{n+1}) + \boldsymbol{g}_N(t_n), \boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n\right)_{L_2(\Gamma_N)} \tag{5.3.43}$$

The second terms of (5.3.43) can be written as

$$\frac{\Delta t^{2-\alpha}}{\Gamma(3-\alpha)} \sum_{n=0}^{m-1} a_{-1} \left(\sum_{i=0}^{n+1} B_{n+1,i} \boldsymbol{W}_h^i + \sum_{i=0}^{n} B_{n,i} \boldsymbol{W}_h^i, \boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n \right).$$

As we concerned before, we shall consider a bound for bilinear form terms. Since $B_{n,n} = 1$, we have

$$\sum_{n=0}^{m-1} a_{-1} \left(\sum_{i=0}^{n+1} B_{n+1,i} \boldsymbol{W}_h^i + \sum_{i=0}^{n} B_{n,i} \boldsymbol{W}_h^i, \boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n \right)$$

$$= \sum_{n=0}^{m-1} a_{-1} \left(\sum_{i=0}^{n} B_{n+1,i} \boldsymbol{W}_h^i + \sum_{i=0}^{n-1} B_{n,i} \boldsymbol{W}_h^i, \boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n \right)$$

$$+ \sum_{n=0}^{m-1} a_{-1} \left(\boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n, \boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n \right)$$

$$\geq \kappa \sum_{n=0}^{m-1} \left\| \boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n \right\|_{\mathcal{V}}^2 + \sum_{n=0}^{m-1} a_{-1} \left(\sum_{i=0}^{n} B_{n+1,i} \boldsymbol{W}_h^i + \sum_{i=0}^{n-1} B_{n,i} \boldsymbol{W}_h^i, \boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n \right)$$

by coercivity. Thus, (5.3.43) becomes

$$2\rho \left\| \boldsymbol{W}_h^m \right\|_{L_2(\Omega)}^2 + \frac{\kappa \Delta t^{2-\alpha}}{\Gamma(3-\alpha)} \sum_{n=0}^{m-1} \left\| \boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n \right\|_{\mathcal{V}}^2$$

$$+\Delta t \sum_{n=0}^{m-1} J_0^{\alpha_0,\beta_0}(\boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n, \boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n)$$

$$\leq 2\rho \left\| \boldsymbol{W}_h^0 \right\|_{L_2(\Omega)}^2 + \Delta t \sum_{n=0}^{m-1} \left(\boldsymbol{f}(t_{n+1}) + \boldsymbol{f}(t_n), \boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n\right)_{L_2(\Omega)}$$

$$+ \Delta t \sum_{n=0}^{m-1} \left(\boldsymbol{g}_N(t_{n+1}) + \boldsymbol{g}_N(t_n), \boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n\right)_{L_2(\Gamma_N)}$$

$$- \frac{\Delta t^{2-\alpha}}{\Gamma(3-\alpha)} \sum_{n=0}^{m-1} a_{-1} \left(\sum_{i=0}^{n} B_{n+1,i} \boldsymbol{W}_h^i + \sum_{i=0}^{n-1} B_{n,i} \boldsymbol{W}_h^i, \boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n \right). \tag{5.3.44}$$

As seen in the previous manner, let us consider the bounds for the right hand side of (5.3.44).

- $\left\|\boldsymbol{W}_h^0\right\|_{L_2(\Omega)}^2$
  A choice of $\boldsymbol{v} = \boldsymbol{W}_h^0$ in (5.3.42) leads us to have

$$\kappa \left\|\boldsymbol{W}_h^0\right\|_{\mathcal{V}}^2 \leq a_{-1}\left(\boldsymbol{W}_h^0, \boldsymbol{W}_h^0\right) = a_{-1}\left(\boldsymbol{w}_0, \boldsymbol{W}_h^0\right) \leq K \left\|\boldsymbol{w}_0\right\|_{\mathcal{V}} \left\|\boldsymbol{W}_h^0\right\|_{\mathcal{V}}$$

  and hence (1.4.10) implies

$$\left\|\boldsymbol{W}_h^0\right\|_{L_2(\Omega)}^2 \leq C \left\|\boldsymbol{w}_0\right\|_{\mathcal{V}}^2$$

  for some positive C.

- $\Delta t \sum_{n=0}^{m-1} \left(\boldsymbol{f}(t_{n+1}) + \boldsymbol{f}(t_n), \boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n\right)_{L_2(\Omega)}$
  Use of Cauchy-Schwarz inequalities and Young's inequalities gives

$$\begin{aligned}
&\Delta t \sum_{n=0}^{m-1} \left(\boldsymbol{f}(t_{n+1}) + \boldsymbol{f}(t_n), \boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n\right)_{L_2(\Omega)} \\
\leq &\Delta t \sum_{n=0}^{m} \left(2\epsilon_a \left\|\boldsymbol{f}(t_n)\right\|_{L_2(\Omega)}^2 + \frac{2}{\epsilon_a} \left\|\boldsymbol{W}_h^n\right\|_{L_2(\Omega)}^2\right) \\
\leq &\Delta t \sum_{n=0}^{N} 2\epsilon_a \left\|\boldsymbol{f}(t_n)\right\|_{L_2(\Omega)}^2 + \frac{2(T+\Delta t)}{\epsilon_a} \max_{0\leq n\leq N} \left\|\boldsymbol{W}_h^n\right\|_{L_2(\Omega)}^2
\end{aligned}$$

  for any positive ϵ_a.

- $\Delta t \sum_{n=0}^{m-1} \left(\boldsymbol{g}_N(t_{n+1}) + \boldsymbol{g}_N(t_n), \boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n\right)_{L_2(\Gamma_N)}$
  In the same way with $\Delta t \sum_{n=0}^{m-1} \left(\boldsymbol{f}(t_{n+1}) + \boldsymbol{f}(t_n), \boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n\right)_{L_2(\Omega)}$, we can also derive

$$\begin{aligned}
&\Delta t \sum_{n=0}^{m-1} \left(\boldsymbol{g}_N(t_{n+1}) + \boldsymbol{g}_N(t_n), \boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n\right)_{L_2(\Gamma_N)} \\
\leq &\Delta t \sum_{n=0}^{m} 2\epsilon_b \left\|\boldsymbol{g}_N(t_n)\right\|_{L_2(\Gamma_N)}^2 + \Delta t \sum_{n=0}^{m} \frac{2}{\epsilon_b} \sum_{e\subset\Gamma_N} \left\|\boldsymbol{W}_h^n\right\|_{L_2(e)}^2 \\
\leq &\Delta t \sum_{n=0}^{m} 2\epsilon_b \left\|\boldsymbol{g}_N(t_n)\right\|_{L_2(\Gamma_N)}^2 + \Delta t \sum_{n=0}^{m} \frac{2}{\epsilon_b} C h^{-1} \sum_{E\in\mathcal{E}_h} \left\|\boldsymbol{W}_h^n\right\|_{L_2(E)}^2 \\
\leq &\Delta t \sum_{n=0}^{N} 2\epsilon_b \left\|\boldsymbol{g}_N(t_n)\right\|_{L_2(\Gamma_N)}^2 + \frac{C h^{-1} 2(T+\Delta t)}{\epsilon_b} \max_{0\leq n\leq N} \left\|\boldsymbol{W}_h^n\right\|_{L_2(\Omega)}^2
\end{aligned}$$

  by the inverse polynomial trace theorem, for any positive ϵ_b.

As following the above bounds, (5.3.44) is bounded by

$$
\begin{aligned}
&2\rho\left\|\boldsymbol{W}_h^m\right\|_{L_2(\Omega)}^2+\frac{\kappa\Delta t^{2-\alpha}}{\Gamma(3-\alpha)}\sum_{n=0}^{m-1}\left\|\boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n\right\|_{\mathcal{V}}^2\\
&+\Delta t\sum_{n=0}^{m-1}J_0^{\alpha_0,\beta_0}(\boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n,\boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n)\\
&\le\rho C\left\|\boldsymbol{w}_0\right\|_{\mathcal{V}}^2+\Delta t\sum_{n=0}^{N}2\epsilon_a\left\|\boldsymbol{f}(t_n)\right\|_{L_2(\Omega)}^2+\frac{2(T+\Delta t)}{\epsilon_a}\max_{0\le n\le N}\left\|\boldsymbol{W}_h^n\right\|_{L_2(\Omega)}^2\\
&\quad+\Delta t\sum_{n=0}^{N}2\epsilon_b\left\|\boldsymbol{g}_N(t_n)\right\|_{L_2(\Gamma_N)}^2+\frac{Ch^{-1}2(T+\Delta t)}{\epsilon_b}\max_{0\le n\le N}\left\|\boldsymbol{W}_h^n\right\|_{L_2(\Omega)}^2\\
&\quad-\frac{\Delta t^{2-\alpha}}{\Gamma(3-\alpha)}\sum_{n=0}^{m-1}a_{-1}\left(\sum_{i=0}^{n}B_{n+1,i}\boldsymbol{W}_h^i+\sum_{i=0}^{n-1}B_{n,i}\boldsymbol{W}_h^i,\boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n\right).
\end{aligned}
\tag{5.3.45}
$$

Now, the last term of (5.3.45) remains to show its boundedness. Note that the right hand side terms of (5.3.45) are independent of m except the last one hence let us denote them by $\mathcal{R}$ then we can rewrite (5.3.45) as

$$
\begin{aligned}
&2\rho\left\|\boldsymbol{W}_h^m\right\|_{L_2(\Omega)}^2+\frac{\kappa\Delta t^{2-\alpha}}{\Gamma(3-\alpha)}\sum_{n=0}^{m-1}\left\|\boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n\right\|_{\mathcal{V}}^2\\
&+\Delta t\sum_{n=0}^{m-1}J_0^{\alpha_0,\beta_0}(\boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n,\boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n)\\
&\le\mathcal{R}-\frac{\Delta t^{2-\alpha}}{\Gamma(3-\alpha)}\sum_{n=0}^{m-1}a_{-1}\left(\sum_{i=0}^{n}B_{n+1,i}\boldsymbol{W}_h^i+\sum_{i=0}^{n-1}B_{n,i}\boldsymbol{W}_h^i,\boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n\right).
\end{aligned}
\tag{5.3.46}
$$

For $m=1$ in the last term of (5.3.46), we have

$$
-a_{-1}\left(B_{1,0}\boldsymbol{W}_h^0,\boldsymbol{W}_h^1+\boldsymbol{W}_h^0\right)\le\frac{K^2B_{1,0}^2\epsilon}{2}\left\|\boldsymbol{W}_h^0\right\|_{\mathcal{V}}^2+\frac{1}{2\epsilon}\left\|\boldsymbol{W}_h^1+\boldsymbol{W}_h^0\right\|_{\mathcal{V}}^2
$$

by the continuity of SIPG and Young's inequality with any positive ϵ. Hence, proper ϵ, for example $\epsilon=1/\kappa$, allows us to have

$$
\begin{aligned}
&2\rho\left\|\boldsymbol{W}_h^1\right\|_{L_2(\Omega)}^2+\frac{\kappa\Delta t^{2-\alpha}}{2\Gamma(3-\alpha)}\left\|\boldsymbol{W}_h^1+\boldsymbol{W}_h^0\right\|_{\mathcal{V}}^2+\Delta tJ_0^{\alpha_0,\beta_0}(\boldsymbol{W}_h^1+\boldsymbol{W}_h^0,\boldsymbol{W}_h^1+\boldsymbol{W}_h^0)\\
&\le\mathcal{R}+\frac{K^2B_{1,0}^2}{2\kappa}\frac{\Delta t^{2-\alpha}}{\Gamma(3-\alpha)}\left\|\boldsymbol{W}_h^0\right\|_{\mathcal{V}}^2.
\end{aligned}
\tag{5.3.47}
$$

In case of $m=2$, (5.3.46) gives

$$
2\rho\left\|\boldsymbol{W}_h^2\right\|_{L_2(\Omega)}^2+\frac{\kappa\Delta t^{2-\alpha}}{\Gamma(3-\alpha)}(\left\|\boldsymbol{W}_h^2+\boldsymbol{W}_h^1\right\|_{\mathcal{V}}^2+\left\|\boldsymbol{W}_h^1+\boldsymbol{W}_h^0\right\|_{\mathcal{V}}^2)
$$

$$
\begin{aligned}
&+\Delta t\sum_{n=0}^{1} J_0^{\alpha_0,\beta_0}(\boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n,\boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n)\\
\le&\mathcal{R}-\frac{\Delta t^{2-\alpha}}{\Gamma(3-\alpha)}a_{-1}\left(B_{2,1}\boldsymbol{W}_h^1+B_{2,0}\boldsymbol{W}_h^0+B_{1,0}\boldsymbol{W}_h^0,\boldsymbol{W}_h^2+\boldsymbol{W}_h^1\right)\\
&-\frac{\Delta t^{2-\alpha}}{\Gamma(3-\alpha)}a_{-1}\left(B_{1,0}\boldsymbol{W}_h^0,\boldsymbol{W}_h^1+\boldsymbol{W}_h^0\right).
\end{aligned}
$$

Using the continuity and Young's inequalities, we can have

$$
\begin{aligned}
&-a_{-1}\left(B_{2,1}\boldsymbol{W}_h^1+B_{2,0}\boldsymbol{W}_h^0+B_{1,0}\boldsymbol{W}_h^0,\boldsymbol{W}_h^2+\boldsymbol{W}_h^1\right)\\
\le&\frac{K^2(\max(B_{2,1},B_{2,0}+B_{1,0}))^2}{2\epsilon}\left\|\boldsymbol{W}_h^1+\boldsymbol{W}_h^0\right\|_{\mathcal{V}}^2+\frac{\epsilon}{2}\left\|\boldsymbol{W}_h^2+\boldsymbol{W}_h^1\right\|_{\mathcal{V}}^2,
\end{aligned}
$$

and

$$
-a_{-1}\left(B_{1,0}\boldsymbol{W}_h^0,\boldsymbol{W}_h^1+\boldsymbol{W}_h^0\right)\le\frac{K^2B_{1,0}^2}{2\epsilon}\left\|\boldsymbol{W}_h^0\right\|_{\mathcal{V}}^2+\frac{\epsilon}{2}\left\|\boldsymbol{W}_h^1+\boldsymbol{W}_h^0\right\|_{\mathcal{V}}^2
$$

for any positive ϵ. Hence coupling with (5.3.47) and choosing $\epsilon=\kappa$, (5.3.46) for $m=2$ becomes

$$
\begin{aligned}
&2\rho\left\|\boldsymbol{W}_h^2\right\|_{L_2(\Omega)}^2+\frac{\kappa\Delta t^{2-\alpha}}{2\Gamma(3-\alpha)}\sum_{n=0}^{1}\left\|\boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n\right\|_{\mathcal{V}}^2\\
&+\Delta t\sum_{n=0}^{1}J_0^{\alpha_0,\beta_0}(\boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n,\boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n)\\
\le&C(\mathcal{R}+\Delta t^{2-\alpha}\left\|\boldsymbol{W}_h^0\right\|_{\mathcal{V}}^2),
\end{aligned}
$$

for some positive C. As following induction method, let us assume that it holds for $m=j<N\in\mathbb{N}$ such that

$$
\begin{aligned}
&2\rho\left\|\boldsymbol{W}_h^j\right\|_{L_2(\Omega)}^2+\frac{\kappa\Delta t^{2-\alpha}}{2\Gamma(3-\alpha)}\sum_{n=0}^{j-1}\left\|\boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n\right\|_{\mathcal{V}}^2\\
&+\Delta t\sum_{n=0}^{j-1}J_0^{\alpha_0,\beta_0}(\boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n,\boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n)\\
\le&C(\mathcal{R}+\Delta t^{2-\alpha}\left\|\boldsymbol{W}_h^0\right\|_{\mathcal{V}}^2).
\end{aligned}
$$

Taking into account the case of $m=j+1$, we have

$$
\begin{aligned}
&\sum_{n=0}^{j}a_{-1}\left(\sum_{i=0}^{n}B_{n+1,i}\boldsymbol{W}_h^i+\sum_{i=0}^{n-1}B_{n,i}\boldsymbol{W}_h^i,\boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n\right)\\
=&\sum_{n=0}^{j}a_{-1}\left(\sum_{i=1}^{n-1}B_{n,i}(\boldsymbol{W}_h^{i+1}+\boldsymbol{W}_h^i),\boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{n=0}^{j} a_{-1}\left(B_{n,0}\boldsymbol{W}_h^0 + B_{n+1,0}\boldsymbol{W}_h^0 + B_{n+1,1}\boldsymbol{W}_h^1, \boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n\right)\\
=&\sum_{n=0}^{j}\sum_{i=1}^{n-1} B_{n,i}a_{-1}\left(\boldsymbol{W}_h^{i+1} + \boldsymbol{W}_h^i, \boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n\right)\\
&+ \sum_{n=0}^{j} a_{-1}\left((B_{n,0} + B_{n+1,0} - B_{n+1,1})\boldsymbol{W}_h^0, \boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n\right)\\
&+ \sum_{n=0}^{j} a_{-1}\left(B_{n+1,1}(\boldsymbol{W}_h^1 + \boldsymbol{W}_h^0), \boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n\right)\\
\leq&\sum_{n=0}^{j}\sum_{i=1}^{n-1}\left(\frac{K^2G^2\epsilon}{2}\left\|\boldsymbol{W}_h^{i+1} + \boldsymbol{W}_h^i\right\|_{\mathcal{V}}^2 + \frac{1}{2\epsilon}\left\|\boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n\right\|_{\mathcal{V}}^2\right)\\
&+ \sum_{n=0}^{j}\left(\frac{K^2(3G)^2\tilde{\epsilon}}{2}\left\|\boldsymbol{W}_h^0\right\|_{\mathcal{V}}^2 + \frac{1}{2\tilde{\epsilon}}\left\|\boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n\right\|_{\mathcal{V}}^2\right)\\
&+ \sum_{n=0}^{j}\left(\frac{K^2G^2\check{\epsilon}}{2}\left\|\boldsymbol{W}_h^1 + \boldsymbol{W}_h^0\right\|_{\mathcal{V}}^2 + \frac{1}{2\check{\epsilon}}\left\|\boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n\right\|_{\mathcal{V}}^2\right)
\end{aligned}
$$

where $0 < G = \max_{0\leq i\leq n\leq N} B_{n,i} < 2$ by Lemma 5.1, for any positive $\epsilon, \tilde{\epsilon}, \check{\epsilon}$. Thus, we can obtain the boundedness of $\sum_{n=0}^{j}\sum_{i=1}^{n-1}\left\|\boldsymbol{W}_h^{i+1} + \boldsymbol{W}_h^i\right\|_{\mathcal{V}}^2$ since $\sum_{i=1}^{n-1}\left\|\boldsymbol{W}_h^{i+1} + \boldsymbol{W}_h^i\right\|_{\mathcal{V}}^2$ is bounded for $0 \leq i \leq j$ by the induction assumption. Consequently, appropriate choice of $(\epsilon, \tilde{\epsilon}, \check{\epsilon})$ yields

$$
\begin{aligned}
&2\rho\left\|\boldsymbol{W}_h^{j+1}\right\|_{L_2(\Omega)}^2 + \frac{\kappa\Delta t^{2-\alpha}}{2\Gamma(3-\alpha)}\sum_{n=0}^{j}\left\|\boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n\right\|_{\mathcal{V}}^2\\
&\quad + \Delta t\sum_{n=0}^{j} J_0^{\alpha_0,\beta_0}(\boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n, \boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n)\\
&\leq C(\mathcal{R} + \Delta t^{2-\alpha}\left\|\boldsymbol{W}_h^0\right\|_{\mathcal{V}}^2).
\end{aligned} \tag{5.3.48}
$$

When we consider maximum in (5.3.48), since $\left\|\boldsymbol{W}_h^0\right\|_{\mathcal{V}} \leq \frac{K}{\kappa}\left\|\boldsymbol{w}_0\right\|_{\mathcal{V}}$ as well as j is arbitrary, (5.3.48) can be written as

$$
\begin{aligned}
&2\rho\max_{0\leq n\leq N}\left\|\boldsymbol{W}_h^n\right\|_{L_2(\Omega)}^2 + \frac{\kappa\Delta t^{2-\alpha}}{2\Gamma(3-\alpha)}\sum_{n=0}^{N-1}\left\|\boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n\right\|_{\mathcal{V}}^2\\
&+\Delta t\sum_{n=0}^{N-1} J_0^{\alpha_0,\beta_0}(\boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n, \boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n)\\
&\leq 3C\Bigg(\left\|\boldsymbol{w}_0\right\|_{\mathcal{V}}^2 + \Delta t\sum_{n=0}^{N} 2\epsilon_a\left\|\boldsymbol{f}(t_n)\right\|_{L_2(\Omega)}^2 + \frac{2(T+\Delta t)}{\epsilon_a}\max_{0\leq n\leq N}\left\|\boldsymbol{W}_h^n\right\|_{L_2(\Omega)}^2
\end{aligned}
$$

$$+\Delta t\sum_{n=0}^{N}2\epsilon_b\,\|\boldsymbol{g}_N(t_n)\|^2_{L_2(\Gamma_N)}+\frac{h^{-1}2(T+\Delta t)}{\epsilon_b}\max_{0\le n\le N}\|\boldsymbol{W}_h^n\|^2_{L_2(\Omega)}\Bigg).$$

Therefore, choosing $\epsilon_a = 12C(T+\Delta t)/\rho$ and $\epsilon_b = 12Ch^{-1}(T+\Delta t)/\rho$ gives

$$\begin{aligned}
&\rho\max_{0\le n\le N}\|\boldsymbol{W}_h^n\|^2_{L_2(\Omega)}+\frac{\kappa\Delta t^{2-\alpha}}{2\Gamma(3-\alpha)}\sum_{n=0}^{N-1}\|\boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n\|^2_{\mathcal{V}}\\
&+\Delta t\sum_{n=0}^{N-1}J_0^{\alpha_0,\beta_0}(\boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n,\boldsymbol{W}_h^{n+1}+\boldsymbol{W}_h^n)\\
&\le C\Bigg(\|\boldsymbol{w}_0\|^2_{\mathcal{V}}+\Delta t\sum_{n=0}^{N}\|\boldsymbol{f}(t_n)\|^2_{L_2(\Omega)}+\Delta t\sum_{n=0}^{N}h^{-1}\,\|\boldsymbol{g}_N(t_n)\|^2_{L_2(\Gamma_N)}\Bigg)\\
&\le C\Bigg(\|\boldsymbol{w}_0\|^2_{\mathcal{V}}+\|\boldsymbol{f}\|^2_{L_\infty(0,T;L_2(\Omega))}+h^{-1}\,\|\boldsymbol{g}_N\|^2_{L_\infty(0,T;L_2(\Gamma_N))}\Bigg)
\end{aligned}$$

for some positive C. □

Remark In the discrete stability bound, h^{-1} is seen but it does not matter in a practical sense, which means only weakly imposed boundary conditions. Here, introducing maximum, rather than using discrete Grönwall's inequality, leads us to have non-exponentially increasing C in time. Furthermore, the bound constant C is independent of h and the solution. As a result, the fully discrete stability bound implies the existence and uniqueness of the fully discrete solution.

In a similar way in the semidiscrete case, we can analyse error estimates. However, since the fractional integration replaced by numerical integration, we should concern carefully the difference error between the exact and discrete solution. When we recall SIPG elliptic projection, let us define for $n = 0,\dots,N$

$$\boldsymbol{\theta}(t) := \boldsymbol{w}(t)-\boldsymbol{R}_{-1}\boldsymbol{w}(t)\text{ for }t\in[0,T],\qquad \boldsymbol{\chi}^n := \boldsymbol{W}_h^n-\boldsymbol{R}_{-1}\boldsymbol{w}(t_n).$$

Lemma 5.4. *Assume that α_0 is large enough and $\beta_0(d-1)\ge 1$. Suppose*

$$\boldsymbol{w}\in C^2(0,T;[C^2(\Omega)]^d)\cap W^1_\infty(0,T;[H^s(\mathcal{E}_h)]^d)\cap W^2_1(0,T;[H^s(\mathcal{E}_h)]^d)$$

and $(\boldsymbol{W}_h^n)_{n=0}^N$ satisfies the fully discrete formula **(U)** *for $s>3/2$, $N\in\mathbb{N}$. Then we have*

$$\begin{aligned}
&\max_{0\le n\le N}\|\boldsymbol{\chi}^n\|_{L_2(\Omega)}+\Bigg(\Delta t^{2-\alpha}\sum_{n=0}^{N-1}\|\boldsymbol{\chi}^{n+1}+\boldsymbol{\chi}^n\|^2_{\mathcal{V}}\Bigg)^{1/2}\\
&+\Bigg(\Delta t\sum_{n=0}^{N-1}J_0^{\alpha_0,\beta_0}(\boldsymbol{\chi}^{n+1}+\boldsymbol{\chi}^n,\boldsymbol{\chi}^{n+1}+\boldsymbol{\chi}^n)\Bigg)^{1/2}=O(h^r+\Delta t^{2-\alpha}),
\end{aligned}$$

where $r=\min(s,k+1)$.

Moreover, if $\boldsymbol{w}(0) = \dot{\boldsymbol{w}}(0) = \mathbf{0}$ *or* $\boldsymbol{w} \in \ker(\mathcal{D})$*, then we have*

$$\max_{0\le n\le N} \|\boldsymbol{\chi}^n\|_{L_2(\Omega)} + \left(\Delta t^{2-\alpha} \sum_{n=0}^{N-1} \|\boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^n\|_{\mathcal{V}}^2\right)^{1/2}$$
$$+ \left(\Delta t \sum_{n=0}^{N-1} J_0^{\alpha_0,\beta_0}(\boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^n, \boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^n)\right)^{1/2} = O(h^r + \Delta t^2),$$

where $r = \min(s, k+1)$*.*

Proof. Let $m \in \{1, \dots, N\}$. Consider the subtraction of (5.3.28) for average between $t = t_{n+1}$ and $t = t_n$, from (5.3.41) where $0 \le n \le m-1$. Then we have

$$\rho\left(\frac{\boldsymbol{W}_h^{n+1} - \boldsymbol{W}_h^n}{\Delta t} - \frac{\dot{\boldsymbol{w}}^{n+1} + \dot{\boldsymbol{w}}^n}{2}, \boldsymbol{v}\right)_{L_2(\Omega)}$$
$$+ a_{-1}\left(\frac{\boldsymbol{q}_{n+1}(\boldsymbol{W}_h) + \boldsymbol{q}_n(\boldsymbol{W}_h)}{2} - \frac{{}_0I_{t_{n+1}}^{1-\alpha}\boldsymbol{w} + {}_0I_{t_n}^{1-\alpha}\boldsymbol{w}}{2}, \boldsymbol{v}\right)$$
$$+ J_0^{\alpha_0,\beta_0}\left(\frac{\boldsymbol{W}_h^{n+1} + \boldsymbol{W}_h^n}{2} - \frac{\boldsymbol{w}^{n+1} + \boldsymbol{w}^n}{2}, \boldsymbol{v}\right) = 0$$

for any $\boldsymbol{v} \in [\mathcal{D}_k(\mathcal{E}_h)]^d$. With use of $\boldsymbol{\theta}$ and $\boldsymbol{\chi}$, when applying linear interpolation approach to the fractional integral of $\boldsymbol{w}$, we can rewrite it by

$$\begin{aligned}
&\frac{\rho}{\Delta t}\left(\boldsymbol{\chi}^{n+1} - \boldsymbol{\chi}^n, \boldsymbol{v}\right)_{L_2(\Omega)} + \frac{1}{2}a_{-1}\left(\boldsymbol{q}_{n+1}(\boldsymbol{\chi}) + \boldsymbol{q}_n(\boldsymbol{\chi}), \boldsymbol{v}\right) + \frac{1}{2}J_0^{\alpha_0,\beta_0}(\boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^n, \boldsymbol{v}) \\
=&\frac{\rho}{\Delta t}\left(\boldsymbol{\theta}^{n+1} - \boldsymbol{\theta}^n, \boldsymbol{v}\right)_{L_2(\Omega)} + \frac{1}{2}a_{-1}\left(\boldsymbol{q}_{n+1}(\boldsymbol{\theta}) + \boldsymbol{q}_n(\boldsymbol{\theta}), \boldsymbol{v}\right) + \frac{1}{2}a_{-1}\left(\boldsymbol{e}^{n+1} + \boldsymbol{e}^n, \boldsymbol{v}\right) \\
&+ \frac{1}{2}J_0^{\alpha_0,\beta_0}(\boldsymbol{\theta}^{n+1} + \boldsymbol{\theta}^n, \boldsymbol{v}) + \rho\left(\boldsymbol{\mathcal{E}}^n, \boldsymbol{v}\right)_{L_2(\Omega)},
\end{aligned} \tag{5.3.49}$$

where $\boldsymbol{e}^n := \boldsymbol{q}_n(\boldsymbol{w}) - {}_0I_{t_n}^{1-\alpha}\boldsymbol{w}$ and $\boldsymbol{\mathcal{E}}(t) := \frac{\dot{\boldsymbol{w}}(t+\Delta t)+\dot{\boldsymbol{w}}(t)}{2} - \frac{\boldsymbol{w}(t+\Delta t)-\boldsymbol{w}(t)}{\Delta t}$ for $t \in [0, T-\Delta t]$. Note that $\boldsymbol{e}^n$ is of C^2 with respect to the spatial domain since $\boldsymbol{w}(t) \in C^2(\Omega)$. Moreover, it is easily to see that $\boldsymbol{e}^n(\boldsymbol{x}) = \mathbf{0}$ on $\partial\Omega$. Hence we can write

$$a_{-1}\left(\boldsymbol{e}^{n+1} + \boldsymbol{e}^n, \boldsymbol{v}\right) = \left(-\nabla\cdot\underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{e}^{n+1} + \boldsymbol{e}^n), \boldsymbol{v}\right)_{L_2(\Omega)}, \tag{5.3.50}$$

by the integration by parts and continuity over the space. Due to DG Galerkin orthogonality, continuity of the strong solution, the homogeneous Dirichlet boundary condition and (5.3.50), we can obtain

$$\begin{aligned}
&\frac{\rho}{\Delta t}\left(\boldsymbol{\chi}^{n+1} - \boldsymbol{\chi}^n, \boldsymbol{v}\right)_{L_2(\Omega)} + \frac{1}{2}a_{-1}\left(\boldsymbol{q}_{n+1}(\boldsymbol{\chi}) + \boldsymbol{q}_n(\boldsymbol{\chi}), \boldsymbol{v}\right) + \frac{1}{2}J_0^{\alpha_0,\beta_0}(\boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^n, \boldsymbol{v}) \\
=&\frac{\rho}{\Delta t}\left(\boldsymbol{\theta}^{n+1} - \boldsymbol{\theta}^n, \boldsymbol{v}\right)_{L_2(\Omega)} + \frac{1}{2}\left(-\nabla\cdot\underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{e}^{n+1} + \boldsymbol{e}^n), \boldsymbol{v}\right)_{L_2(\Omega)} + \rho\left(\boldsymbol{\mathcal{E}}^n, \boldsymbol{v}\right)_{L_2(\Omega)}.
\end{aligned} \tag{5.3.51}$$

By substitution of $\boldsymbol{v} = 2\Delta t(\boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^n)$ into (5.3.51), adding together from $n = 0$ to $n = m-1$ gives

$$
\begin{aligned}
&2\rho \|\boldsymbol{\chi}^m\|^2_{L_2(\Omega)} + \Delta t \sum_{n=0}^{m-1} a_{-1}\left(\boldsymbol{q}_{n+1}(\boldsymbol{\chi}) + \boldsymbol{q}_n(\boldsymbol{\chi}), \boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^n\right) \\
&+\Delta t \sum_{n=0}^{m-1} J_0^{\alpha_0,\beta_0}(\boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^n, \boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^n) \\
&=2\rho \left\|\boldsymbol{\chi}^0\right\|^2_{L_2(\Omega)} + 2\rho \sum_{n=0}^{m-1} \left(\boldsymbol{\theta}^{n+1} - \boldsymbol{\theta}^n, \boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^n\right)_{L_2(\Omega)} \\
&\quad + \Delta t \sum_{n=0}^{m-1} \left(-\nabla \cdot \underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{e}^{n+1} + \boldsymbol{e}^n), \boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^n\right)_{L_2(\Omega)} + 2\rho\Delta t \sum_{n=0}^{m-1} \left(\boldsymbol{\mathcal{E}}^n, \boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^n\right)_{L_2(\Omega)}.
\end{aligned}
\tag{5.3.52}
$$

Use of coercivity and the definition of $\boldsymbol{q}$ implies

$$
\begin{aligned}
&2\rho \|\boldsymbol{\chi}^m\|^2_{L_2(\Omega)} + \frac{\kappa\Delta t^{2-\alpha}}{\Gamma(3-\alpha)} \sum_{n=0}^{m-1} \left\|\boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^n\right\|^2_{\mathcal{V}} + \Delta t \sum_{n=0}^{m-1} J_0^{\alpha_0,\beta_0}(\boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^n, \boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^n) \\
&\leq 2\rho \left\|\boldsymbol{\chi}^0\right\|^2_{L_2(\Omega)} + 2\rho \sum_{n=0}^{m-1} \left(\boldsymbol{\theta}^{n+1} - \boldsymbol{\theta}^n, \boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^n\right)_{L_2(\Omega)} \\
&\quad + \Delta t \sum_{n=0}^{m-1} \left(-\nabla \cdot \underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{e}^{n+1} + \boldsymbol{e}^n), \boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^n\right)_{L_2(\Omega)} + 2\rho\Delta t \sum_{n=0}^{m-1} \left(\boldsymbol{\mathcal{E}}^n, \boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^n\right)_{L_2(\Omega)} \\
&\quad - \frac{\Delta t^{2-\alpha}}{\Gamma(3-\alpha)} \sum_{n=0}^{m-1} a_{-1}\left(\sum_{i=0}^{n} B_{n+1,i}\boldsymbol{\chi}^i + \sum_{i=0}^{n-1} B_{n,i}\boldsymbol{\chi}^i, \boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^n\right).
\end{aligned}
\tag{5.3.53}
$$

Now, we shall consider the bounds for the right hand side of (5.3.53) as following.

- $\left\|\boldsymbol{\chi}^0\right\|^2_{L_2(\Omega)}$
  By (1.4.10), $\left\|\boldsymbol{\chi}^0\right\|^2_{L_2(\Omega)} \leq C \left\|\boldsymbol{\chi}^0\right\|^2_{\mathcal{V}}$ for some positive C. Also, (5.3.42) and the DG Galerkin orthogonality lead us to have

$$
\begin{aligned}
a_{-1}\left(\boldsymbol{\chi}^0, \boldsymbol{v}\right) &= a_{-1}\left(\boldsymbol{W}_h^0 - \boldsymbol{R}_{-1}\boldsymbol{w}_0, \boldsymbol{v}\right) \\
&= a_{-1}\left(\boldsymbol{W}_h^0 - \boldsymbol{w}_0 + \boldsymbol{w}_0 - \boldsymbol{R}_{-1}\boldsymbol{w}_0, \boldsymbol{v}\right) \\
&= a_{-1}\left(\boldsymbol{W}_h^0 - \boldsymbol{w}_0, \boldsymbol{v}\right) + a_{-1}\left(\boldsymbol{w}_0 - \boldsymbol{R}_{-1}\boldsymbol{w}_0, \boldsymbol{v}\right) \\
&= 0
\end{aligned}
$$

  for any $\boldsymbol{v} \in [\mathcal{D}_k(\mathcal{E}_h)]^d$. It implies that

$$
\left\|\boldsymbol{\chi}^0\right\|^2_{L_2(\Omega)} \leq C \left\|\boldsymbol{\chi}^0\right\|^2_{\mathcal{V}} = C a_{-1}\left(\boldsymbol{\chi}^0, \boldsymbol{\chi}^0\right) = 0.
$$

- $\sum_{n=0}^{m-1} \left(\boldsymbol{\theta}^{n+1} - \boldsymbol{\theta}^{n}, \boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^{n}\right)_{L_2(\Omega)}$
  Since $\boldsymbol{w}$ belongs to H^1 in time, we can write

$$
\begin{aligned}
&\sum_{n=0}^{m-1} \left(\boldsymbol{\theta}^{n+1} - \boldsymbol{\theta}^{n}, \boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^{n}\right)_{L_2(\Omega)} \\
&\quad = \sum_{n=0}^{m-1} \int_{t_n}^{t_{n+1}} \left(\dot{\boldsymbol{\theta}}(t'), \boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^{n}\right)_{L_2(\Omega)} dt' \\
&\quad \leq \sum_{n=0}^{m-1} \int_{t_n}^{t_{n+1}} \left\|\dot{\boldsymbol{\theta}}(t')\right\|_{L_2(\Omega)} \left\|\boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^{n}\right\|_{L_2(\Omega)} dt' \\
&\quad \leq \frac{\epsilon_a}{2} \int_0^{t_m} \left\|\dot{\boldsymbol{\theta}}(t')\right\|_{L_2(\Omega)}^2 dt' + \frac{\Delta t}{2\epsilon_a} \sum_{n=0}^{m-1} \left\|\boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^{n}\right\|_{L_2(\Omega)}^2 \\
&\quad \leq \frac{\epsilon_a}{2} \left\|\dot{\boldsymbol{\theta}}\right\|_{L_2(0,T;L_2(\Omega))}^2 + \frac{\Delta t}{2\epsilon_a} 4N \max_{0 \leq n \leq N} \left\|\boldsymbol{\chi}^{n}\right\|_{L_2(\Omega)}^2 \\
&\quad = \frac{\epsilon_a}{2} \left\|\dot{\boldsymbol{\theta}}\right\|_{L_2(0,T;L_2(\Omega))}^2 + \frac{2T}{\epsilon_a} \max_{0 \leq n \leq N} \left\|\boldsymbol{\chi}^{n}\right\|_{L_2(\Omega)}^2 \\
&\quad \leq \frac{\epsilon_a}{2} O(h^{2r}) + \frac{2T}{\epsilon_a} \max_{0 \leq n \leq N} \left\|\boldsymbol{\chi}^{n}\right\|_{L_2(\Omega)}^2
\end{aligned}
$$

  by Cauchy-Schwarz inequalities, Young's inequality and (4.2.24) for any positive ϵ_a where $r = \min(k+1, s)$.

- $\Delta t \sum_{n=0}^{m-1} \left(-\nabla \cdot \underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{e}^{n+1} + \boldsymbol{e}^{n}), \boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^{n}\right)_{L_2(\Omega)}$
  Using Cauchy-Schwarz inequalities and Young's inequality, we can obtain

$$
\begin{aligned}
&\Delta t \sum_{n=0}^{m-1} \left(-\nabla \cdot \underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{e}^{n+1} + \boldsymbol{e}^{n}), \boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^{n}\right)_{L_2(\Omega)} \\
&\leq \Delta t \sum_{n=0}^{m-1} \left\|\nabla \cdot \underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{e}^{n+1} + \boldsymbol{e}^{n})\right\|_{L_2(\Omega)} \left\|\boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^{n}\right\|_{L_2(\Omega)} \\
&\leq \Delta t \sum_{n=0}^{m-1} \frac{\epsilon_b}{2} \left\|\nabla \cdot \underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{e}^{n+1} + \boldsymbol{e}^{n})\right\|_{L_2(\Omega)}^2 + \Delta t \sum_{n=0}^{m-1} \frac{1}{2\epsilon_b} \left\|\boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^{n}\right\|_{L_2(\Omega)}^2 \\
&\leq \Delta t \sum_{n=0}^{N-1} \frac{\epsilon_b}{2} \left\|\nabla \cdot \underline{\boldsymbol{D}}\underline{\boldsymbol{\varepsilon}}(\boldsymbol{e}^{n+1} + \boldsymbol{e}^{n})\right\|_{L_2(\Omega)}^2 + \frac{2T}{\epsilon_b} \max_{0 \leq n \leq N} \left\|\boldsymbol{\chi}^{n}\right\|_{L_2(\Omega)}^2
\end{aligned}
$$

  for any positive ϵ_b, since $m\Delta t \leq N\Delta t = T$. Recall the numerical integration error then we have $\boldsymbol{e}^n = \boldsymbol{O}(\Delta t^2)$. Note that each component of $\boldsymbol{O}(\Delta t^2)$ is an order of Δt^2 and our domain is bounded and so any norm of $\boldsymbol{O}(\Delta t^2)$ with respect to the spatial domain is bounded by $C\Delta t^2$ for some positive C. Therefore, we have

$$\begin{aligned}
&\Delta t \sum_{n=0}^{m-1} \left(-\nabla \cdot \underline{\boldsymbol{D}}\boldsymbol{\varepsilon}(\boldsymbol{e}^{n+1}+\boldsymbol{e}^n), \boldsymbol{\chi}^{n+1}+\boldsymbol{\chi}^n\right)_{L_2(\Omega)} \\
\leq &\Delta t \sum_{n=0}^{N-1} \frac{\epsilon_b}{2} O(\Delta t^4) + \frac{2T}{\epsilon_b} \max_{0\leq n\leq N} \|\boldsymbol{\chi}^n\|^2_{L_2(\Omega)} \\
= &\epsilon_b T O(\Delta t^4) + \frac{2T}{\epsilon_b} \max_{0\leq n\leq N} \|\boldsymbol{\chi}^n\|^2_{L_2(\Omega)}
\end{aligned}$$

- $\Delta t \sum\limits_{n=0}^{m-1} \left(\boldsymbol{\mathcal{E}}^n, \boldsymbol{\chi}^{n+1}+\boldsymbol{\chi}^n\right)_{L_2(\Omega)}$
  Note that (5.3.21) implies $\|\boldsymbol{\mathcal{E}}^n\|_{L_2(\Omega)} = O(\Delta t^{2-\alpha})$ for any $n = 0, \ldots, N-1$. In this manner, Cauchy-Schwarz inequalities and Young's inequality yield

$$\begin{aligned}
&\Delta t \sum_{n=0}^{m-1} \left(\boldsymbol{\mathcal{E}}^n, \boldsymbol{\chi}^{n+1}+\boldsymbol{\chi}^n\right)_{L_2(\Omega)} \\
\leq &\Delta t \sum_{n=0}^{m-1} \|\boldsymbol{\mathcal{E}}^n\|_{L_2(\Omega)} \left\|\boldsymbol{\chi}^{n+1}+\boldsymbol{\chi}^n\right\|_{L_2(\Omega)} \\
\leq &\Delta t \sum_{n=0}^{m-1} \frac{\epsilon_c}{2} \|\boldsymbol{\mathcal{E}}^n\|^2_{L_2(\Omega)} + \Delta t \sum_{n=0}^{m-1} \frac{1}{2\epsilon_c} \left\|\boldsymbol{\chi}^{n+1}+\boldsymbol{\chi}^n\right\|^2_{L_2(\Omega)} \\
\leq &\Delta t \sum_{n=0}^{N-1} \frac{\epsilon_c}{2} \|\boldsymbol{\mathcal{E}}^n\|^2_{L_2(\Omega)} + \Delta t \sum_{n=0}^{N-1} \frac{2}{\epsilon_c} \max_{0\leq n\leq N} \|\boldsymbol{\chi}^n\|^2_{L_2(\Omega)} \\
\leq &C\frac{T\epsilon_c}{2} O(\Delta t^{4-2\alpha}) + \frac{2T}{\epsilon_c} \max_{0\leq n\leq N} \|\boldsymbol{\chi}^n\|^2_{L_2(\Omega)}
\end{aligned}$$

  for any positive ϵ_c.

Tidy up the above results then (5.3.53) becomes

$$\begin{aligned}
&2\rho \|\boldsymbol{\chi}^m\|^2_{L_2(\Omega)} + \frac{\kappa \Delta t^{2-\alpha}}{\Gamma(3-\alpha)} \sum_{n=0}^{m-1} \left\|\boldsymbol{\chi}^{n+1}+\boldsymbol{\chi}^n\right\|^2_{\mathcal{V}} + \Delta t \sum_{n=0}^{m-1} J_0^{\alpha_0,\beta_0}(\boldsymbol{\chi}^{n+1}+\boldsymbol{\chi}^n, \boldsymbol{\chi}^{n+1}+\boldsymbol{\chi}^n) \\
\leq &\rho\epsilon_a O(h^{2r}) + \frac{4\rho T}{\epsilon_a} \max_{0\leq n\leq N} \|\boldsymbol{\chi}^n\|^2_{L_2(\Omega)} + \epsilon_b O(\Delta t^4) + \frac{2T}{\epsilon_b} \max_{0\leq n\leq N} \|\boldsymbol{\chi}^n\|^2_{L_2(\Omega)} \\
&+ \epsilon_c O(\Delta t^{4-2\alpha}) + \frac{4\rho T}{\epsilon_c} \max_{0\leq n\leq N} \|\boldsymbol{\chi}^n\|^2_{L_2(\Omega)} \\
&- \frac{\Delta t^{2-\alpha}}{\Gamma(3-\alpha)} \sum_{n=0}^{m-1} a_{-1} \left(\sum_{i=0}^{n} B_{n+1,i}\boldsymbol{\chi}^i + \sum_{i=0}^{n-1} B_{n,i}\boldsymbol{\chi}^i, \boldsymbol{\chi}^{n+1}+\boldsymbol{\chi}^n\right). \qquad (5.3.54)
\end{aligned}$$

As seen in the proof of Theorem 5.8, the boundedness of the last term of (5.3.54) can be shown by mathematical induction. In addition, since $\left\|\boldsymbol{\chi}^0\right\|_{\mathcal{V}} = 0$, we can obtain

$$2\rho \left\|\boldsymbol{\chi}^m\right\|_{L_2(\Omega)}^2 + \frac{\kappa\Delta t^{2-\alpha}}{2\Gamma(3-\alpha)} \sum_{n=0}^{m-1} \left\|\boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^n\right\|_{\mathcal{V}}^2 + \Delta t \sum_{n=0}^{m-1} J_0^{\alpha_0,\beta_0}(\boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^n, \boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^n)$$
$$\leq C \left(\rho\epsilon_a O(h^{2r}) + \epsilon_b O(\Delta t^4) + \epsilon_c O(\Delta t^{4-2\alpha}) + \left(\frac{4\rho T}{\epsilon_a} + \frac{2T}{\epsilon_b} + \frac{4\rho T}{\epsilon_c} \right) \max_{0\leq n\leq N} \left\|\boldsymbol{\chi}^n\right\|_{L_2(\Omega)}^2 \right) \tag{5.3.55}$$

for some positive C. Whence we consider maximum on (5.3.55), we have

$$2\rho \max_{0\leq n\leq N} \left\|\boldsymbol{\chi}^n\right\|_{L_2(\Omega)}^2 + \frac{\kappa\Delta t^{2-\alpha}}{2\Gamma(3-\alpha)} \sum_{n=0}^{N-1} \left\|\boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^n\right\|_{\mathcal{V}}^2$$
$$+ \Delta t \sum_{n=0}^{N-1} J_0^{\alpha_0,\beta_0}(\boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^n, \boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^n)$$
$$\leq 3C \Bigg(\rho\epsilon_a O(h^{2r}) + \epsilon_b O(\Delta t^4) + \epsilon_c O(\Delta t^{4-2\alpha})$$
$$+ \left(\frac{4\rho T}{\epsilon_a} + \frac{2T}{\epsilon_b} + \frac{4\rho T}{\epsilon_c} \right) \max_{0\leq n\leq N} \left\|\boldsymbol{\chi}^n\right\|_{L_2(\Omega)}^2 \Bigg),$$

therefore choosing $\epsilon_a, \epsilon_c = 48CT$ and $\epsilon_b = 12CT/\rho$ implies

$$\rho \max_{0\leq n\leq N} \left\|\boldsymbol{\chi}^n\right\|_{L_2(\Omega)}^2 + \frac{\kappa\Delta t^{2-\alpha}}{2\Gamma(3-\alpha)} \sum_{n=0}^{N-1} \left\|\boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^n\right\|_{\mathcal{V}}^2$$
$$+ \Delta t \sum_{n=0}^{N-1} J_0^{\alpha_0,\beta_0}(\boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^n, \boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^n) \leq O(h^{2r}) + O(\Delta t^4) + O(\Delta t^{4-2\alpha}).$$

As a consequence, we can conclude that

$$\max_{0\leq n\leq N} \left\|\boldsymbol{\chi}^n\right\|_{L_2(\Omega)} + \left(\Delta t^{2-\alpha} \sum_{n=0}^{N-1} \left\|\boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^n\right\|_{\mathcal{V}}^2 \right)^{1/2}$$
$$+ \left(\Delta t \sum_{n=0}^{N-1} J_0^{\alpha_0,\beta_0}(\boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^n, \boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^n) \right)^{1/2} = O(h^r + \Delta t^{2-\alpha}).$$

On the other hand, if the strong solution has more smoothness in time such that (5.3.22) is fulfilled, then we could take second order accuracy in time. To be specific, when we suppose $\boldsymbol{w}(0) = \dot{\boldsymbol{w}}(0) = \boldsymbol{0}$ or $\boldsymbol{w} \in \ker(\mathcal{D})$, (5.3.22) implies $\left\|\boldsymbol{\mathcal{E}}^n\right\|_{L_2(\Omega)} = O(\Delta t^2)$. Therefore, $O(\Delta t^{2-\alpha})$ can be replaced by $O(\Delta t^2)$ so that we have

$$\max_{0\leq n\leq N} \left\|\boldsymbol{\chi}^n\right\|_{L_2(\Omega)} + \left(\Delta t^{2-\alpha} \sum_{n=0}^{N-1} \left\|\boldsymbol{\chi}^{n+1} + \boldsymbol{\chi}^n\right\|_{\mathcal{V}}^2 \right)^{1/2}$$

$$+\left(\Delta t\sum_{n=0}^{N-1} J_0^{\alpha_0,\beta_0}(\boldsymbol{\chi}^{n+1}+\boldsymbol{\chi}^n,\boldsymbol{\chi}^{n+1}+\boldsymbol{\chi}^n)\right)^{1/2} = O(h^r+\Delta t^2).$$

□

Theorem 5.9 (Error Analysis: fully discrete formula **(U)**)**.** *Assume that*

$$\boldsymbol{w}\in C^2(0,T;[C^2(\Omega)]^d)\cap W^1_\infty(0,T;[H^s(\mathcal{E}_h)]^d)\cap W^2_1(0,T;[H^s(\mathcal{E}_h)]^d)$$

and $(\boldsymbol{W}_h^n)_{n=0}^N$ *satisfies the fully discrete formula* **(U)** *for* $s>3/2$, $N\in\mathbb{N}$ *with large enough* α_0 *and* $\beta_0(d-1)\geq 1$. *Then we can observe optimal* L_2 *error estimates as well as optimal DG energy error estimates with fixed order accuracy in time. Therefore,*

$$\max_{0\leq n\leq N}\|\boldsymbol{w}^n-\boldsymbol{W}_h^n\|_{L_2(\Omega)} = O(h^r+\Delta t^{2-\alpha}),$$
$$\max_{0\leq n\leq N}\|\boldsymbol{w}^n-\boldsymbol{W}_h^n\|_{\mathcal{V}} = O(h^{r-1}+\Delta t^{2-\alpha}),$$

where $r=\min(s,k+1)$.

Proof. For any $n=0,\ldots,N$, we have

$$\begin{aligned}\|\boldsymbol{w}^n-\boldsymbol{W}_h^n\|_{L_2(\Omega)} &= \|\boldsymbol{w}^n-\boldsymbol{R}_{-1}\boldsymbol{w}^n-(\boldsymbol{W}_h^n-\boldsymbol{R}_{-1}\boldsymbol{w}^n)\|_{L_2(\Omega)}\\ &= \|\boldsymbol{\theta}^n-\boldsymbol{\chi}^n\|_{L_2(\Omega)}\\ &\leq \|\boldsymbol{\theta}^n\|_{L_2(\Omega)}+\|\boldsymbol{\chi}^n\|_{L_2(\Omega)}.\end{aligned}$$

By (4.2.24) and Lemma 5.4, it is concluded that

$$\|\boldsymbol{w}^n-\boldsymbol{W}_h^n\|_{L_2(\Omega)} = O(h^r)+O(h^r+\Delta t^{2-\alpha}) = O(h^r+\Delta t^{2-\alpha}),$$

and so since n is arbitrary,

$$\max_{0\leq n\leq N}\|\boldsymbol{w}^n-\boldsymbol{W}_h^n\|_{L_2(\Omega)} = O(h^r+\Delta t^{2-\alpha}).$$

As shown in the semidiscrete case, DG energy estimates can be given by the vector-valued analogue of inverse inequality (1.4.11), (4.2.5) and Lemma 5.4. Thus, we have

$$\max_{0\leq n\leq N}\|\boldsymbol{w}^n-\boldsymbol{W}_h^n\|_{\mathcal{V}} = O(h^{r-1}+\Delta t^{2-\alpha}).$$

□

Corollary 5.2. *Suppose the condition of* Theorem 5.9 *holds. In addition, we assume that* $\boldsymbol{w}(0)=\dot{\boldsymbol{w}}(0)=\boldsymbol{0}$ *or* $\boldsymbol{w}\in\ker(\mathcal{D})$. *Then the error estimates can be given by*

$$\max_{0\leq n\leq N}\|\boldsymbol{w}^n-\boldsymbol{W}_h^n\|_{L_2(\Omega)} = O(h^r+\Delta t^2),$$
$$\max_{0\leq n\leq N}\|\boldsymbol{w}^n-\boldsymbol{W}_h^n\|_{\mathcal{V}} = O(h^{r-1}+\Delta t^2),$$

regardless of α, *where* $r=\min(s,k+1)$.

Proof. A proof parallels to the proof of Theorem 5.9 but we can take full advantage of the second order scheme, since we have the regularity of solution in time. It improves the time order accuracy to Δt^2. Hence our the above statement is true. □

5.3.3 Numerical Experiments

We solve (5.3.1)-(5.3.4) in two ways (CG/DG). We consider two examples; one is an example that is not of class H^3 in time but the other is a smoother case. We set our spatial domain as the unit square, $T = 1$ and $\alpha = 1/2$.

Example 5.1.

Let us define

$$\boldsymbol{w}(x, y, t) = (t + t^{1.5}) \begin{bmatrix} \sin(\pi x)\sin(\pi y) \\ xy(1-x)(1-y). \end{bmatrix}$$

Then $\boldsymbol{w} \in C^2(0, T; [C^\infty(\Omega)]^2) \cap W_1^2(0, T; [C^\infty(\Omega)]^2)$ with homogeneous Dirichlet boundary. Also, we can derive data terms which satisfy (5.3.1) by using (5.1.6). Note that $\boldsymbol{w}^{(3)}(t)$ is not bounded and not integrable in time so that we cannot fully take an advantage of second order schemes thus we may observe sub-optimal results with respect to time.

Let $\boldsymbol{W}_{\text{CG}}^n$ and $\boldsymbol{W}_{\text{DG}}^n$ be approximation solutions of **(T)** and **(U)**, respectively. We define numerical errors by $\boldsymbol{e}_{\text{CG}}^n = \boldsymbol{w}(t_n) - \boldsymbol{W}_{\text{CG}}^n$ and $\boldsymbol{e}_{\text{DG}}^n = \boldsymbol{w}(t_n) - \boldsymbol{W}_{\text{DG}}^n$. By error estimates theorems for both solutions, each energy norm of numerical error is of order 1.5 in time and k with respect to spatial meshes, where k is a degree of polynomials. Similarly, in case of L_2 norm,

$$\|\boldsymbol{e}_{\text{CG}}^n\|_{L_2(\Omega)} = O(h^{k+1} + \Delta t^{1.5}), \text{ and } \|\boldsymbol{e}_{\text{DG}}^n\|_{L_2(\Omega)} = O(h^{k+1} + \Delta t^{1.5}).$$

To compare both numerical errors in energy norm, we have to replace and match the norms. We consider H^1 norm (broken Sobolev H^1 norm for DG solutions) of numerical errors. Note that we consider SIPG for DG approximation so that we no longer use the super-penalisation. Hence we set the penalty parameters as $\alpha_0 = 50$ and $\beta_0 = 1$.

As a result, numerical simulations give us the following error tables with respect to spatial approximation methods and degrees of polynomial basis. The results in Table 5.1 indicate that the errors are $O(h + \Delta t^{1.5})$ and $O(h^2 + \Delta t^{1.5})$ in H^1 norm and L_2 norm, respectively for both CG and DG. On the other hand, as the degree of polynomials increasing, the order of accuracy in terms of h is also growing but the convergent order in time is fixed by 1.5. Thus, we can observe that H^1 norm and L_2 norms of errors are $O(h^2 + \Delta t^{1.5})$ and $O(h^3 + \Delta t^{1.5})$, respectively.

Let us consider $\Delta t \approx h$. Then, regardless of the degree of polynomial basis, the convergent order of L_2 norm is fixed by $O(h^{1.5})$. In the same sense, H^1 errors are given by $O(h)$ if $k = 1$ or $O(h^{1.5})$, otherwise. Figure 5.1 illustrates the comparison between CG and DG. Both finite element approximations have same convergent orders. DG solutions, however, encounter loss of accuracy for fine meshes. As we concerned before, the global matrix becomes ill-conditioned so that it may deteriorate solving the linear systems. Main reason is that DGFEM requires much more degrees of freedom than CGFEM.

H^1 error of CG

h \ Δt	1/8	1/16	1/32	1/64	1/128	1/256	1/512
1/2	2.768e-01	2.842e-01	2.869e-01	2.877e-01	2.879e-01	2.880e-01	2.880e-01
1/4	1.447e-01	1.476e-01	1.490e-01	1.495e-01	1.497e-01	1.497e-01	1.498e-01
1/8	7.438e-02	7.353e-02	7.396e-02	7.419e-02	7.428e-02	7.431e-02	7.432e-02
1/16	4.192e-02	3.727e-02	3.693e-02	3.698e-02	3.701e-02	3.703e-02	3.703e-02
1/32	2.866e-02	1.983e-02	1.859e-02	1.849e-02	1.849e-02	1.850e-02	1.850e-02
1/64	2.425e-02	1.205e-02	9.573e-03	9.275e-03	9.247e-03	9.247e-03	9.247e-03
1/128	2.301e-02	9.116e-03	5.311e-03	4.703e-03	4.631e-03	4.624e-03	4.624e-03

L_2 error of CG

h \ Δt	1/8	1/16	1/32	1/64	1/128	1/256	1/512
1/2	3.199e-02	3.083e-02	3.054e-02	3.046e-02	3.044e-02	3.043e-02	3.043e-02
1/4	9.663e-03	7.467e-03	6.956e-03	6.827e-03	6.789e-03	6.776e-03	6.772e-03
1/8	5.735e-03	2.700e-03	1.842e-03	1.641e-03	1.589e-03	1.574e-03	1.569e-03
1/16	5.117e-03	1.899e-03	7.976e-04	4.861e-04	4.100e-04	3.905e-04	3.847e-04
1/32	4.995e-03	1.767e-03	6.201e-04	2.464e-04	1.342e-04	1.052e-04	9.793e-05
1/64	4.967e-03	1.739e-03	5.893e-04	2.062e-04	7.890e-05	3.861e-05	2.757e-05
1/128	4.960e-03	1.733e-03	5.827e-04	1.989e-04	6.974e-05	2.600e-05	1.160e-05

H^1 error of DG

h \ Δt	1/8	1/16	1/32	1/64	1/128	1/256	1/512
1/2	2.770e-01	2.844e-01	2.870e-01	2.878e-01	2.881e-01	2.882e-01	2.882e-01
1/4	1.441e-01	1.470e-01	1.484e-01	1.489e-01	1.490e-01	1.491e-01	1.491e-01
1/8	7.401e-02	7.310e-02	7.352e-02	7.374e-02	7.383e-02	7.385e-02	7.386e-02
1/16	4.171e-02	3.701e-02	3.665e-02	3.670e-02	3.673e-02	3.675e-02	3.675e-02
1/32	2.857e-02	1.969e-02	1.843e-02	1.833e-02	1.833e-02	1.834e-02	1.834e-02
1/64	2.422e-02	1.199e-02	9.492e-03	9.190e-03	9.162e-03	9.161e-03	9.162e-03
1/128	2.301e-02	9.095e-03	5.274e-03	4.659e-03	4.587e-03	4.580e-03	4.579e-03

L_2 error of DG

h \ Δt	1/8	1/16	1/32	1/64	1/128	1/256	1/512
1/2	3.196e-02	3.081e-02	3.052e-02	3.043e-02	3.041e-02	3.040e-02	3.040e-02
1/4	9.617e-03	7.419e-03	6.906e-03	6.776e-03	6.736e-03	6.724e-03	6.719e-03
1/8	5.726e-03	2.691e-03	1.831e-03	1.630e-03	1.578e-03	1.562e-03	1.557e-03
1/16	5.115e-03	1.897e-03	7.950e-04	4.830e-04	4.066e-04	3.871e-04	3.813e-04
1/32	4.995e-03	1.766e-03	6.196e-04	2.457e-04	1.333e-04	1.043e-04	9.697e-05
1/64	4.966e-03	1.739e-03	5.892e-04	2.061e-04	7.874e-05	3.839e-05	2.732e-05
1/128	4.960e-03	1.733e-03	5.827e-04	1.988e-04	6.971e-05	2.596e-05	1.155e-05

Table 5.1: Numerical errors; Example 5.1; $k = 1$

H^1 error of CG

h \ Δt	1/8	1/16	1/32	1/64	1/128	1/256	1/512
1/2	8.013e-02	7.963e-02	8.015e-02	8.040e-02	8.049e-02	8.053e-02	8.054e-02
1/4	3.087e-02	2.309e-02	2.209e-02	2.203e-02	2.204e-02	2.205e-02	2.205e-02
1/8	2.323e-02	9.671e-03	6.234e-03	5.730e-03	5.673e-03	5.668e-03	5.668e-03
1/16	2.263e-02	8.025e-03	3.015e-03	1.687e-03	1.460e-03	1.432e-03	1.428e-03
1/32	2.259e-02	7.909e-03	2.683e-03	9.713e-04	4.734e-04	3.734e-04	3.596e-04
1/64	2.259e-02	7.901e-03	2.661e-03	9.077e-04	3.230e-04	1.399e-04	9.699e-05
1/128	2.259e-02	7.901e-03	2.659e-03	9.036e-04	3.111e-04	1.098e-04	4.365e-05

L_2 error of CG

h \ Δt	1/8	1/16	1/32	1/64	1/128	1/256	1/512
1/2	7.027e-03	4.850e-03	4.421e-03	4.338e-03	4.318e-03	4.312e-03	4.310e-03
1/4	5.052e-03	1.899e-03	9.149e-04	7.094e-04	6.761e-04	6.701e-04	6.687e-04
1/8	4.962e-03	1.736e-03	5.907e-04	2.194e-04	1.147e-04	9.419e-05	9.109e-05
1/16	4.958e-03	1.731e-03	5.809e-04	1.973e-04	6.864e-05	2.623e-05	1.426e-05
1/32	4.957e-03	1.731e-03	5.806e-04	1.967e-04	6.748e-05	2.339e-05	8.266e-06
1/64	4.957e-03	1.731e-03	5.806e-04	1.967e-04	6.745e-05	2.334e-05	8.127e-06
1/128	4.957e-03	1.731e-03	5.806e-04	1.967e-04	6.745e-05	2.333e-05	8.124e-06

H^1 error of DG

h \ Δt	1/8	1/16	1/32	1/64	1/128	1/256	1/512
1/2	8.002e-02	7.953e-02	8.006e-02	8.031e-02	8.040e-02	8.044e-02	8.045e-02
1/4	3.068e-02	2.289e-02	2.190e-02	2.184e-02	2.185e-02	2.186e-02	2.186e-02
1/8	2.317e-02	9.595e-03	6.141e-03	5.639e-03	5.584e-03	5.580e-03	5.580e-03
1/16	2.262e-02	8.011e-03	2.993e-03	1.659e-03	1.432e-03	1.404e-03	1.401e-03
1/32	2.259e-02	7.906e-03	2.679e-03	9.662e-04	4.661e-04	3.656e-04	3.520e-04
1/64	2.259e-02	7.901e-03	2.660e-03	9.068e-04	3.218e-04	1.381e-04	9.495e-05
1/128	2.259e-02	7.901e-03	2.659e-03	9.034e-04	3.109e-04	1.096e-04	4.325e-05

L_2 error of DG

h \ Δt	1/8	1/16	1/32	1/64	1/128	1/256	1/512
1/2	7.016e-03	4.838e-03	4.409e-03	4.326e-03	4.305e-03	4.299e-03	4.297e-03
1/4	5.050e-03	1.895e-03	9.075e-04	7.001e-04	6.664e-04	6.603e-04	6.589e-04
1/8	4.961e-03	1.736e-03	5.903e-04	2.183e-04	1.128e-04	9.186e-05	8.868e-05
1/16	4.958e-03	1.731e-03	5.809e-04	1.972e-04	6.857e-05	2.606e-05	1.395e-05
1/32	4.957e-03	1.731e-03	5.806e-04	1.967e-04	6.748e-05	2.339e-05	8.257e-06
1/64	4.957e-03	1.731e-03	5.806e-04	1.967e-04	6.745e-05	2.334e-05	8.127e-06
1/128	4.957e-03	1.731e-03	5.806e-04	1.967e-04	6.745e-05	2.333e-05	8.124e-06

Table 5.2: Numerical errors; Example 5.1; $k = 2$

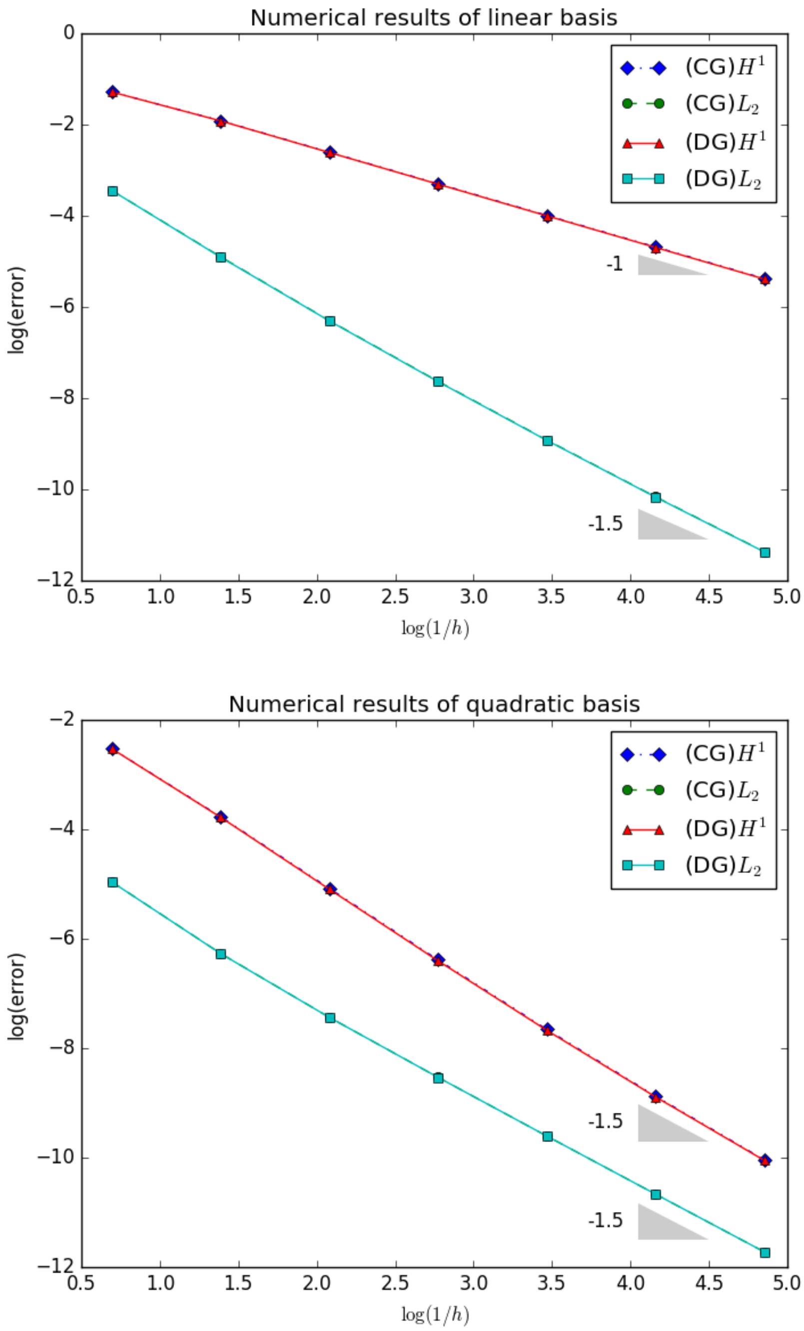

Figure 5.1: Numerical convergent order; Example 5.1

Next, we will consider some example of Corollaries 5.1 and 5.2, i.e. we additionally assume zero initial condition, which allows us to get second order accuracy in time.

Example 5.2.

Let

$$\boldsymbol{w}(x,y,t) = t^{3.5}\begin{bmatrix} \sin(\pi x)\sin(\pi y) \\ xy(1-x)(1-y) \end{bmatrix}$$

be our analytic solution. Obviously, Example 5.2 has higher regularity than Example 5.1. Using (5.1.6), we can obtain source term $\boldsymbol{f}$.

According to Corollaries 5.1 and 5.2, both numerical solutions take a full advantage of the second order finite difference method, since $\boldsymbol{w}(0) = \dot{\boldsymbol{w}}(0) = \boldsymbol{0}$. Thus, the convergent orders are given by $O(h^k + \Delta t^2)$ in energy norms and $O(h^{k+1} + \Delta t^2)$ in L_2 norms, respectively.

With each linear polynomial basis, both numerical approximations show similar errors in Table 5.3. We can observe that

$$\begin{cases} \|\boldsymbol{e}_{\mathrm{CG}}\|_{H^1(\Omega)} = & O(h + \Delta t^2) \\ \|\boldsymbol{e}_{\mathrm{CG}}\|_{L_2(\Omega)} = & O(h^2 + \Delta t^2) \end{cases} \text{and} \begin{cases} \|\boldsymbol{e}_{\mathrm{DG}}\|_{H^1(\Omega)} = & O(h + \Delta t^2) \\ \|\boldsymbol{e}_{\mathrm{DG}}\|_{L_2(\Omega)} = & O(h^2 + \Delta t^2) \end{cases}.$$

Table 5.4 indicates that both numerical solutions have optimal convergent orders for $k = 2$ such as $O(h^2 + \Delta t^2)$ in H^1 norm and $O(h^3 + \Delta t^2)$ in L_2 norm, respectively.

As seen in the above, time convergent order is fixed by 2. More precisely, Figure 5.2 illustrates orders of convergence where $\Delta t \approx h$. As a result, for the linear polynomial basis, the energy error estimates show first order. On the other hand, regardless of a degree of polynomials, L_2 norm of errors has second order accuracy.

To sum up, we have optimal spatial error estimates in theoretical and practical results using not only CGFEM but also DGFEM. However, due to weak singularity in fractional order calculus, there is restriction on second order schemes such as Crank-Nicolson method. Nevertheless, we can obtain optimal second order accuracy in time if certain conditions are satisfied, for example zero initial condition. Furthermore, as we concerned before, fine spatial meshes generally deteriorate solving linear systems due to huge size of matrix if iterative solvers used. Especially, DG requires to solve much bigger matrices than CG so that it is necessary to improve linear solvers to reduce large condition number matters.

Remark Even though using SIPG, DG approximation requires many degrees of freedom as well as ill conditioned matrices to solve for fine spatial meshes. In case of time independent problems, the order of condition numbers depends on penalty parameters, for example $O(h^{-(\beta_0+1)})$. However, SIPG provides optimal L_2 error estimates even if standard penalised. Thus, we can take the benefit of lower condition numbers rather than super-penalised NIPG.

Remark In generalised Maxwell model, hereditary memory terms were dealt with by introducing internal variables rather than numerical integration (e.g. quadrature rules). In other words, we replaced the memory terms with internal variables. In the fully discrete case, the current solution was only used to obtain next time level. For instance, to solve linear system for time step t_{n+1} requires only previous solution t_n. To be simplified, when we consider to solve $Ax_{n+1} = b_n$, b_n consists of only data terms such as the initial condition, the traction and the source term, and current solution corresponding to t_n. In contrast, the numerical schemes of fractional order model were given with numerical integration technique such as linear interpolation for the memory terms. Hence it was necessary to use all history solutions. Thus, b_n contains more previous solutions corresponding to $t_0, \ldots, t_n$. It implies that fractional order model may need much more memory. Roughly speaking, the required physical memory for hereditary terms would be

$$\begin{aligned} \text{Maxwell solid model} &: \#\text{ of degree of freedoms} \times \#\text{ of internal variables} \\ \text{Fractional order model} &: \#\text{ of degree of freedoms} \times \#\text{ of timesteps.} \end{aligned}$$

As a result, for long time period simulations, fractional order model needs huge memory in practice. However, it does not affect the size of linear system (the global matrix). Note that the size of global matrix is (# d.o.f.) $\times$ (# d.o.f.), regardless of model problems.

Summary

In Chapter 5, we have studied fractional order viscoelasticity problems modelled by power law with using two finite element methods. Both spatially finite element approximations give optimal errors with respect to energy norm and L_2 norm. In a similar way with Maxwell model, Crank-Nicolson finite difference scheme is used for time discretisation. However, we no longer introduce auxiliary ODEs for Volterra integral parts in the power law model. Hence it is necessary to use numerical integration. Moreover, due to weak singularity of kernel, we use linear interpolation technique to derive second order accuracy in time for the sake of Crank-Nicolson method. Nevertheless, the weakly singular kernel restricts regularity of solutions so it has effect on loss of benefit of second order schemes. Although we cannot fully take an advantage of second order methods, further suppositions such as zero initial conditions lead our numerical approximations to be optimal. In the end, we can prove stability as well as optimal error estimates theorems.

H^1 error of CG

h \ Δt	1/8	1/16	1/32	1/64	1/128	1/256	1/512
1/2	1.749e-03	1.340e-03	1.253e-03	1.233e-03	1.229e-03	1.227e-03	1.227e-03
1/4	1.199e-03	7.269e-04	6.460e-04	6.311e-04	6.277e-04	6.269e-04	6.267e-04
1/8	9.475e-04	4.063e-04	3.197e-04	3.083e-04	3.063e-04	3.059e-04	3.058e-04
1/16	8.726e-04	2.781e-04	1.656e-04	1.533e-04	1.518e-04	1.515e-04	1.515e-04
1/32	8.530e-04	2.360e-04	9.541e-05	7.736e-05	7.577e-05	7.559e-05	7.556e-05
1/64	8.480e-04	2.243e-04	6.744e-05	4.042e-05	3.797e-05	3.778e-05	3.775e-05
1/128	8.467e-04	2.213e-04	5.840e-05	2.342e-05	1.921e-05	1.890e-05	1.888e-05

L_2 error of CG

h \ Δt	1/8	1/16	1/32	1/64	1/128	1/256	1/512
1/2	1.716e-04	1.223e-04	1.201e-04	1.204e-04	1.205e-04	1.205e-04	1.205e-04
1/4	1.702e-04	4.835e-05	2.600e-05	2.474e-05	2.486e-05	2.491e-05	2.493e-05
1/8	1.808e-04	4.676e-05	1.226e-05	5.822e-06	5.417e-06	5.450e-06	5.466e-06
1/16	1.838e-04	4.767e-05	1.181e-05	3.050e-06	1.400e-06	1.292e-06	1.301e-06
1/32	1.845e-04	4.795e-05	1.193e-05	2.937e-06	7.577e-07	3.458e-07	3.187e-07
1/64	1.847e-04	4.803e-05	1.198e-05	2.960e-06	7.291e-07	1.885e-07	8.613e-08
1/128	1.848e-04	4.804e-05	1.199e-05	2.969e-06	7.344e-07	1.813e-07	4.697e-08

H^1 error of DG

h \ Δt	1/8	1/16	1/32	1/64	1/128	1/256	1/512
1/2	1.751e-03	1.341e-03	1.254e-03	1.235e-03	1.230e-03	1.229e-03	1.229e-03
1/4	1.195e-03	7.233e-04	6.431e-04	6.282e-04	6.250e-04	6.242e-04	6.240e-04
1/8	9.454e-04	4.045e-04	3.182e-04	3.069e-04	3.049e-04	3.044e-04	3.043e-04
1/16	8.720e-04	2.773e-04	1.647e-04	1.524e-04	1.510e-04	1.507e-04	1.507e-04
1/32	8.528e-04	2.358e-04	9.498e-05	7.688e-05	7.530e-05	7.512e-05	7.509e-05
1/64	8.479e-04	2.243e-04	6.728e-05	4.018e-05	3.773e-05	3.753e-05	3.751e-05
1/128	8.467e-04	2.213e-04	5.835e-05	2.331e-05	1.908e-05	1.877e-05	1.875e-05

L_2 error of DG

h \ Δt	1/8	1/16	1/32	1/64	1/128	1/256	1/512
1/2	1.714e-04	1.221e-04	1.198e-04	1.201e-04	1.203e-04	1.203e-04	1.203e-04
1/4	1.703e-04	4.828e-05	2.583e-05	2.456e-05	2.468e-05	2.474e-05	2.475e-05
1/8	1.808e-04	4.676e-05	1.225e-05	5.797e-06	5.388e-06	5.421e-06	5.438e-06
1/16	1.838e-04	4.767e-05	1.182e-05	3.048e-06	1.393e-06	1.285e-06	1.293e-06
1/32	1.845e-04	4.795e-05	1.194e-05	2.937e-06	7.572e-07	3.439e-07	3.165e-07
1/64	1.847e-04	4.803e-05	1.198e-05	2.960e-06	7.292e-07	1.884e-07	8.562e-08
1/128	1.848e-04	4.804e-05	1.199e-05	2.969e-06	7.344e-07	1.813e-07	4.693e-08

Table 5.3: Numerical errors; Example 5.2; $k = 1$

H^1 error of CG

h \ Δt	1/8	1/16	1/32	1/64	1/128	1/256	1/512
1/2	9.604e-04	4.283e-04	3.442e-04	3.328e-04	3.307e-04	3.302e-04	3.301e-04
1/4	8.550e-04	2.421e-04	1.080e-04	9.185e-05	9.037e-05	9.019e-05	9.015e-05
1/8	8.468e-04	2.218e-04	5.999e-05	2.702e-05	2.343e-05	2.316e-05	2.314e-05
1/16	8.464e-04	2.204e-04	5.539e-05	1.487e-05	6.751e-06	5.891e-06	5.832e-06
1/32	8.463e-04	2.203e-04	5.508e-05	1.374e-05	3.692e-06	1.686e-06	1.475e-06
1/64	8.463e-04	2.203e-04	5.506e-05	1.366e-05	3.410e-06	9.189e-07	4.212e-07
1/128	8.463e-04	2.203e-04	5.506e-05	1.366e-05	3.391e-06	8.481e-07	2.290e-07

L_2 error of CG

h \ Δt	1/8	1/16	1/32	1/64	1/128	1/256	1/512
1/2	1.808e-04	4.931e-05	2.034e-05	1.730e-05	1.718e-05	1.719e-05	1.720e-05
1/4	1.844e-04	4.797e-05	1.220e-05	3.943e-06	2.756e-06	2.677e-06	2.674e-06
1/8	1.848e-04	4.804e-05	1.199e-05	2.989e-06	8.192e-07	4.082e-07	3.694e-07
1/16	1.848e-04	4.805e-05	1.199e-05	2.972e-06	7.384e-07	1.889e-07	6.541e-08
1/32	1.848e-04	4.805e-05	1.199e-05	2.972e-06	7.372e-07	1.834e-07	4.600e-08
1/64	1.848e-04	4.805e-05	1.199e-05	2.972e-06	7.372e-07	1.833e-07	4.564e-08
1/128	1.848e-04	4.805e-05	1.199e-05	2.972e-06	7.372e-07	1.833e-07	4.564e-08

H^1 error of DG

h \ Δt	1/8	1/16	1/32	1/64	1/128	1/256	1/512
1/2	9.607e-04	4.283e-04	3.440e-04	3.325e-04	3.304e-04	3.300e-04	3.298e-04
1/4	8.554e-04	2.421e-04	1.075e-04	9.126e-05	8.976e-05	8.957e-05	8.953e-05
1/8	8.470e-04	2.219e-04	5.998e-05	2.683e-05	2.318e-05	2.290e-05	2.288e-05
1/16	8.464e-04	2.204e-04	5.541e-05	1.486e-05	6.696e-06	5.815e-06	5.753e-06
1/32	8.463e-04	2.203e-04	5.509e-05	1.374e-05	3.690e-06	1.671e-06	1.454e-06
1/64	8.463e-04	2.203e-04	5.506e-05	1.366e-05	3.411e-06	9.184e-07	4.173e-07
1/128	8.463e-04	2.203e-04	5.506e-05	1.366e-05	3.392e-06	8.484e-07	2.289e-07

L_2 error of DG

h \ Δt	1/8	1/16	1/32	1/64	1/128	1/256	1/512
1/2	1.809e-04	4.930e-05	2.030e-05	1.725e-05	1.713e-05	1.714e-05	1.714e-05
1/4	1.844e-04	4.798e-05	1.219e-05	3.922e-06	2.725e-06	2.645e-06	2.642e-06
1/8	1.848e-04	4.804e-05	1.199e-05	2.988e-06	8.161e-07	4.017e-07	3.622e-07
1/16	1.848e-04	4.805e-05	1.199e-05	2.972e-06	7.384e-07	1.887e-07	6.461e-08
1/32	1.848e-04	4.805e-05	1.199e-05	2.972e-06	7.372e-07	1.833e-07	4.599e-08
1/64	1.848e-04	4.805e-05	1.199e-05	2.972e-06	7.372e-07	1.833e-07	4.564e-08
1/128	1.848e-04	4.805e-05	1.199e-05	2.972e-06	7.372e-07	1.833e-07	4.564e-08

Table 5.4: Numerical errors; Example 5.2; $k = 2$

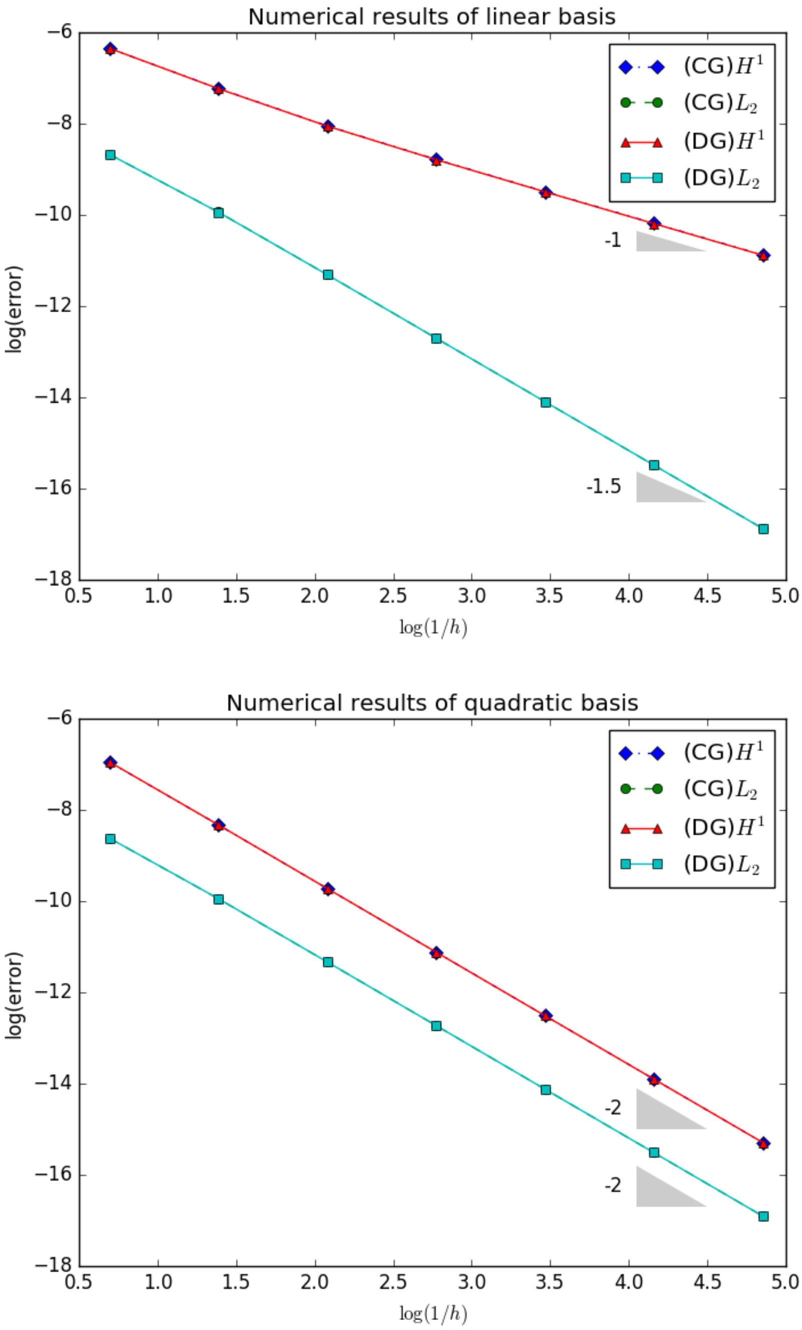

Figure 5.2: Numerical convergent order; Example 5.2

Conclusion

We have studied viscoelastic wave problems with finite element methods. One of our main achievements showed that our numerical solutions are applicable for long time integration. On account of absence of Grönwall's inequality in proofs, the bound constants of stability and error bounds depend on the final time but not exponentially increasing. As a result, we could simulate viscoelastic wave propagation for a long period of time. Not only have well-posedness been proved, but also optimal error estimates. We could obtain fixed second order accuracy in time as well as optimal energy norm error (H^1 norm). At the same time, elliptic regularity estimations allow the numerical approximations to give optimal L_2 error estimates.

When we considered generalised Maxwell solid model, this rheological model gave an idea of internal variables as partial constitutive relations. Hence we could define two types of internal variables and so we solved the model problem in two ways, which governed with integration by parts. In the meantime, we have derived variational problems of scalar/vector-valued wave problems with CG and DG. Regardless of finite element methods, we observed sufficiently good stability analysis and optimal error analysis. However, use of NIPG enforced us to have more restrictions in proofs. For example, since we used the nonsymmetric bilinear form, we had more difficulty in changing order in the bilinear form. Nevertheless, as shown in the proof of coercivity, we could derive the boundedness of interior penalty terms so that we could also deal with skew symmetric parts. As a result, appropriate stability and error bounds were controlled by penalty parameters, in other words, it required sufficiently large α_0 and β_0. Turning back to discussion of forms of internal variables, there was no significant difference between CG and DG. All analysis results needed same conditions such as smoothness of data and regularity of solutions. Hence a choice of internal variables is a matter of taste. However, in a practical sense, global matrices resulted by linear systems vary with internal variables. In this thesis, we do not investigate it in detail but it may become future works to decide which form of internal variables is better.

Not only have we shown theoretical results, but also numerical experiments. A variety of examples have been carried out based on FEniCS since FEniCS provides many useful and powerful tools for finite element methods such as various linear solvers, mesh generators, function spaces, etc. On account of the good environment platform, we could easily check our numerical schemes and theorems. To be specific, in case of symmetric cases, indeed CGFEM, *conjugate gradient method* has been used as a iterative linear solver with *successive over-relaxtion* for preconditioning. A number of examples for CGFEM exhibited optimal convergence rates with respect to time and space. In contrast, the linear systems by DGFEM was defined with nonsymmetric global matrices so linear solvers for symmetric problems were no longer used. Thus, *biconjugate gradient stabilized method* has been introduced with *incomplete LU factorization* for preconditioning. In a similar way with CGFEM, numerical results of DGFEM fulfilled stability and error analysis theorems. Interestingly, regardless of penalisation, numerical solutions had optimal L_2 norm errors for odd degrees of polynomials. Our theorems demonstrated only that DG schemes required super-penalised NIPG for elliptic regularity estimations. Thus, it still remains to prove how standard penalisation works for optimal L_2 error estimates with odd degrees of polynomials. Moreover, solving the linear systems had serious issues for fine spatial meshes. As shown in numerical experiments sections, iterative solvers encountered ill-conditioned linear systems for small h. In particular, super-penalised DG has seriously worse condition numbers than standard DG and CG.

At the final stage, we investigated fractional order viscoelastic model problems in CG and DG. A choice of power law led memory terms to be fractional order calculus. This weakly singular kernel imposed many restrictions on regularity and time integration. In case of generalised Maxwell model, internal variables have been introduced to replace memory terms with auxiliary ODEs, whereas the weakly singular kernels could not have generated auxiliary terms by differentiating. To be more precisely, using the fact that a derivative of exponential function is also exponential, we can derive some auxiliary ODEs by differentiating convolution with exponentially decaying kernels. However, a derivative of power law type convolution has strong singularity, which gives difficulty in dealing with stability analysis. Hence it was necessary to apply numerical integration for time in discrete cases. Linear interpolation in time was employed to replace fractional order integration with second order accuracy. We showed well-posedness and error estimates theorems for semidiscrete problems as well as fully discrete problems. Due to properties of positive definite kernel, we completed the proofs of semidiscrete cases. On the other hand, we proved some analysis theorems for fully discrete formulations by mathematical inductions. As we noted before, loss of regularity of solutions occurred in the fractional model. In spite of second order schemes in time, the model problem could not guarantee second order accuracy. Nevertheless, zero initial conditions or kernel of spatial differential operator allowed our discrete solution to have optimal convergence rates in time as well as space. In comparison with use of internal variables, numerical approximations of Maxwell model involved only current data for next time step but fully discrete schemes of the fractional order model required to store all past history data. In practice, computer memory is limited hence shortage of memory issue may happen in a large number of time steps. Even if long time travel allowed in the theory, the machines encounter huge memory requirements. Keeping this issue in mind, we developed bespoke code implementations in FEniCS and carried out numerous examples. Although orders of convergence with respect to the space domain are the same as the generalised Maxwell solid model, we could not take a full benefit of Crank-Nicolson method. It was only observed that the example of zero initial conditions had second order accuracy in time. Also, it was necessary for us to improve iterative solvers for super-penalised DG.

Future Works

As we concerned, numerical approaches are so useful and good tools to solve complicated model problems in real world. Especially, finite element analysis allows us to solve and simulate various PDE models with many advantages, for example conservation law of mass, complex geometry, local physical effects and the benefit of algebraic expressions. Therefore, we can also apply our numerical schemes to more practical viscoelastic model problems. Beyond this thesis, we are going to consider the following further studies.

- Simulation of viscoelasctic wave propagations: In this thesis, we presented numerical experiments of analytic solutions given in 2D. Hence we could verify our approximate schemes are applicable enough for the certain models. For the next stage, this leads us to simulate a number of viscoelastic wave propagations in industrial areas, even in 3D. For instance, we can understand and simulate viscoelastic behaviours of half elastic and half viscoelastic material. Moreover, we can assume not only simple domains but also complex geometry on demand.

- Fractional order viscoelastic models with fractional order derivatives: Recall (5.2.7). Instead of dealing with fractional integration in the constitutive law as in (5.2.1), use of identity of fractional calculus gives $1 + \alpha$ order differential equations. In a similar way with the fractional integration problems, we can derive variational forms and show existence and uniqueness of solution as well as error estimates theorems with Theorem 5.2. However, the typical approaches as we used before may not work properly such as integration by parts for fractional derivative in L_2 inner product. Therefore, we need more cautions to $1 + \alpha$ order time derivatives.

- Preconditioning and iterative solver for super-penalised DG: As shown in previous numerical experiments of DG, solving linear systems encountered ill-condition problems. To be specific, for spatially optimal L_2 norm error, it is necessary to use super-penalisation so that the linear system becomes ill-conditioned according to condition numbers of global matrices. Therefore, we have to improve its linear solver for fine spatial meshes. In [65, 68, 69], we can find a various linear solver algorithms and preconditioners. For example, *Multigrid method* may be possible remedy to solve fine spatial mesh problems. We are going to implement and apply it to resolve high condition numbers on super-penalised DG.

- Optimal L_2 error with standard penalisation: We noted that NIPG requires super-penalisation for elliptic regularity estimates. However, the standard penalised NIPG with odd degree of polynomial basis also provides optimal L_2 convergence. This is only theoretically shown in rectangular meshes or on 1D in [61, 62, 63]. We still have a big curiosity in case of triangular meshes in higher dimensions. The key is using similar techniques in 1D cases and finding geometric properties of triangular(or tetrahedral) meshes.

- Code improvements in FEniCS: In a practical sense, the number of the degrees of freedom are significantly important to solve linear systems and allocate memory. Many computational issues may occur with huge linear systems, for instance non-negligible round-off error and out of memory. FEniCS is of high quality to assemble variational problems and solve linear systems with a number of solvers. Nevertheless, it has to be improved more for large systems. My aim is to develop iterative solvers and multigrid preconditioning. On the other hand, in terms of running time to simulate, we want to enhance speed-up so that we would like to consider parallel computing. FEniCS is compatible with MPI which is one of parallel computing methods. Therefore, it is able to implement codes applied with MPI.

- Other interests: I believe finite element method is powerful tool to understand and solve many mathematical models and physical phenomena in real world. Particularly, I am so interested in multi-scale FEM (see e.g. [85, 86]). My goal is developing multi-scale DG approach to air pollutions of PM(Particular Matter)[87]. A huge number of people have seriously suffered from PM in air and it deteriorates not only their health conditions but also financial problems. I hope our mathematician's contributions could help to manage this issue. On the other hand, I have studied machine learning as well. From various resources, we are able to learn key ideas of machine learning and use many applications in science. At some point, this machine learning approach is more effective to solve more realistic problems [88] than FEM.

There are a large number of challenging questions to sort out. Even though our contribution is tiny, I wish it makes our world better. I would like to conclude this thesis with the following quotes.

Great things are not done by impulse, but by a series of small things brought together.
- Vincent Van Gogh -

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